Intro to Contemporary Math

Fair Shares and Fair Arrangements

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Agenda

- Fair Shares and Fair Arrangements
- Picking Winners with the Average Bid
- Constructing Fair Arrangements
- Fairness Interval for Two People

Announcements

- Homework due next Wednesday
- Exam next Friday!

Fair Compensation Arrangements

- A compensation arrangement is **Fair** if **everyone's** compensation is greater than (or equal to) their fair share (bid divided by number of people):

$$x_{person} \geq \frac{b_{person}}{N}$$

A compensation arrangement is **Unfair for a person** if his or her compensation is less than his or her fair share.

Example

	Tim	Shawn	Leo
Bids on item:	\$13,500	\$12,000	\$9,000

Leo wins the item, and he decides to **pay Tim \$4,500 and Shawn \$4,000.**

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$$x_{Leo} = \underbrace{9,000}_{\text{Leo's winning bid}} - \underbrace{4,500}_{\text{Pay Tim}} - \underbrace{4,000}_{\text{Pay Shawn}} = 500$$

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Fair shares: bid divided by **3** people

	T	S	L
Fair shares:	4,500	4,000	3,000

?(6.2) Fair or Unfair?

	Tim	Shawn	Leo
Bids on item:	\$13,500	\$12,000	\$9,000
x Compensation:	4,500	4,000	500
Fair shares:	4,500	4,000	3,000

Who is this arrangement Unfair for? Type his name, or type "no one."

Fair or Unfair?

	Tim	Shawn	Leo
Bids on item:	\$13,500	\$12,000	\$9,000
x Compensation:	4,500	4,000	500
Fair shares:	4,500	4,000	3,000

Tim and Shawn's compensations are greater than or equal to their fair shares. They're okay.

Leo's compensation is below his fair share! The arrangement is **Unfair** for him!

Fair Arrangements and Average Bid

$$\text{Average bid} = m = \frac{\text{Sum of all bids}}{\text{Number of people}}$$

If we want to make a Fair arrangement, we need to know who can win and how much they pay each loser.

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- The winner's compensation is their winning bid minus all payments.

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- Any person whose bid is greater than or equal to m can be chosen as the winner.
- Then the winner should pay each loser their fair share* in compensation.
- The winner's compensation is their winning bid minus all payments.
  WARNING! The winner's compensation may be more than their fair share!

Fair Arrangements and Average Bid (Note)

$$\text{Average bid} = m = \frac{\text{Sum of all bids}}{\text{Number of people}}$$

If we want to make a Fair arrangement, we need to know who can win and how much they pay each loser.

- Any person whose bid is greater than or equal to m can be chosen as the winner.
- Then the winner should pay each loser their fair share* in compensation.
- The winner's compensation is their winning bid minus all payments.
  WARNING! The winner's compensation may be more than their fair share! This is likely to happen if the winner's bid is greater than the average bid.

*Note: The winner can pay each loser more than their fair share, but must be careful not to pay too much and fall below their own fair share.

Failing to be Fair

What happens if the winner bids less than m, the average bid? Either:

- The winner cannot pay someone their fair share in compensation, or
- The winner pays too much and their compensation falls below their own fair share.

?(6.3) Making a Fair Arrangement: Choosing Winner

	Tim	Shawn	Leo
Bids on item:	\$13,500	\$12,000	\$9,000

We want a Fair arrangement. For each person, indicate if we can choose them as a winner or not. Type three letters such as "AAA" and send.
Hint: Compute the average bid first.

1) Tim
2) Shawn
3) Leo

A) Can win
B) Cannot win

Making a Fair Arrangement: Choosing Winner

	Tim	Shawn	Leo
Bids on item:	\$13,500	\$12,000	\$9,000

$$m = \frac{13,500 + 12,000 + 9,000}{3} = \frac{34,500}{3} = \boxed{11,500}$$

Two people, Tim and Shawn, have bids higher than m, so we can choose one of them as the winner. Let's see how Shawn can make a Fair arrangement first.

?(6.4) Making a Fair Arrangement: Payments

	Tim	Shawn	Leo
Bids on item:	\$13,500	\$12,000	\$9,000

What is the **minimum** that Shawn should pay:

1) Tim?
2) Leo?

A) 3,000
B) 4,000
C) 4,500
D) 9,000
E) 12,000
F) 13,500

Type and send two letters.

Making a Fair Arrangement: Payments

Use fair shares:

$$\text{Tim: } \frac{13,500}{3} = 4,500 \qquad \text{Shawn: } \frac{12,000}{3} = 4,000 \qquad \text{Leo: } \frac{9,000}{3} = 3,000$$

Shawn must pay Tim at least 4,500, and pay Leo at least 3,000.

Making a Fair Arrangement: Payments

Use fair shares:

Tim: $\frac{13,500}{3} = 4,500$ Shawn: $\frac{12,000}{3} = 4,000$ Leo: $\frac{9,000}{3} = 3,000$

Shawn must pay Tim at least 4,500, and pay Leo at least 3,000.

Let's have him pay exactly those amounts, so Tim and Leo's compensations are those payments:

$$x_{Tim} = 4,500, \quad x_{Leo} = 3,000$$

Reminder: in a dispute with one item, the people who do not win the item only get cash as their compensation.

?(6.5) Making a Fair Arrangement: Winner's Comp

	Tim	Shawn	Leo
Bids on item:	\$13,500	\$12,000	\$9,000

Shawn must pay Tim at least 4,500, and pay Leo at least 3,000.
Let's have him pay exactly those amounts, so Tim and Leo's compensations are those payments:

$$x_{Tim} = 4,500, \quad x_{Leo} = 3,000$$

What is Shawn's compensation? Type and send a number.

Making a Fair Arrangement: Winner's Comp

	Tim	Shawn	Leo
Bids on item:	\$13,500	\$12,000	\$9,000

Shawn must pay Tim at least 4,500, and pay Leo at least 3,000.

Let's have him pay exactly those amounts, so Tim and Leo's compensations are those payments:

$$x_{Tim} = 4,500, \quad x_{Leo} = 3,000$$

Shawn's compensation, as always, is his winning bid minus the payments!

$$x_{Shawn} = \underbrace{12,000}_{\text{Shawn's winning bid}} - \underbrace{4,500}_{\text{Pay Tim}} - \underbrace{3,000}_{\text{Pay Leo}} = \boxed{4,500}$$

Notice Shawn's compensation is more than his fair share (4,000), so the arrangement is Fair (for all).

Making a Fair Arrangement: Summary

	Tim	Shawn	Leo
Bids on item:	\$13,500	\$12,000	\$9,000

Winner: Shawn (one of two bidders whose bid is higher than the average bid)
Shawn pays Tim 4,500 and Leo 3,000:

$$x_{Tim} = 4,500, \quad x_{Leo} = 3,000$$

$$x_{Shawn} = \underbrace{12,000}_{\text{Shawn's winning bid}} - \underbrace{4,500}_{\text{Pay Tim}} - \underbrace{3,000}_{\text{Pay Leo}} = \boxed{4,500}$$

Making a Fair Arrangement: Paying More

	Tim	Shawn	Leo
Bids on item:	\$13,500	\$12,000	\$9,000

Shawn must pay Tim at least 4,500, and pay Leo at least 3,000. Let's have him pay a little more than that:

Pay Tim 4,650 and pay Leo 3,100

so Tim and Leo's compensations are those payments:

$$x_{Tim} = 4,650, \quad x_{Leo} = 3,100$$

$$x_{Shawn} = \underbrace{12,000}_{\text{Shawn's winning bid}} - \underbrace{4,650}_{\text{Pay Tim}} - \underbrace{3,100}_{\text{Pay Leo}} = 4,250$$

Notice Shawn's compensation is still more than his fair share (4,000), so the arrangement is still Fair.

Paying Too Much

	Tim	Shawn	Leo
Bids on item:	\$13,500	\$12,000	\$9,000

Shawn must pay Tim at least 4,500, and pay Leo at least 3,000. Let's have him pay more than that:

Pay Tim 4,800 and pay Leo 3,250

so Tim and Leo's compensations are those payments:

$$x_{Tim} = 4,800, \quad x_{Leo} = 3,250$$

$$x_{Shawn} = \underbrace{12,000}_{\text{Shawn's winning bid}} - \underbrace{4,800}_{\text{Pay Tim}} - \underbrace{3,250}_{\text{Pay Leo}} = 3,950$$

This time, Shawn's compensation is less than his fair share (4,000), so the arrangement is Unfair for him.

Paying Too Little

	Tim	Shawn	Leo
Bids on item:	\$13,500	\$12,000	\$9,000

Shawn must pay Tim at least 4,500, and pay Leo at least 3,000. Let's have him pay less than that:

Pay Tim 4,450 and pay Leo 2,900

so Tim and Leo's compensations are those payments:

$$x_{Tim} = 4,450, \quad x_{Leo} = 2,900$$

$$x_{Shawn} = \underbrace{12,000}_{\text{Shawn's winning bid}} - \underbrace{4,450}_{\text{Pay Tim}} - \underbrace{2,900}_{\text{Pay Leo}} = 4,650$$

This time, Shawn's compensation is more than his fair share (4,000), but Tim and Leo got paid less than their fair shares, so the arrangement is Unfair for them.

Making a Fair Arrangement: Higher Bidder Wins

	Tim	Shawn	Leo
Bids on item:	\$13,500	\$12,000	\$9,000

If Tim is the winner, he will have an easier time paying the losers and keeping enough for his fair share than Shawn did. Tim must pay Shawn at least 4,000 (his fair share) and Leo at least 3,000 (his fair share). Let's have him pay exactly those amounts, so Shawn and Leo's compensations are those payments:

$$x_{Shawn} = 4,000, \quad x_{Leo} = 3,000$$

$$x_{Tim} = \underbrace{13,500}_{\text{T's winning bid}} - \underbrace{4,000}_{\text{Pay Shawn}} - \underbrace{3,000}_{\text{Pay Leo}} = 6,500$$

Tim's compensation is way more than his fair share (4,500).

Fairness Interval for Two People

When $N = 2$, and two people bid with $b_1 \leq b_2$, then **the arrangement is Fair** only if the higher bidder wins and pays Person 1 x_1, where

$$\frac{b_1}{2} \leq x_1 \leq \frac{b_2}{2}.$$

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We say that x_1 is inside the **Fairness Interval:**

it is between the fair shares $\frac{b_1}{2}$ and $\frac{b_2}{2}$.

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We say that x_1 is inside the **Fairness Interval:**

it is between the fair shares $\frac{b_1}{2}$ and $\frac{b_2}{2}$.

- Here, only the higher bidder bids more than the average bid, so there is only one choice for the winner.
- If $x_1 < \frac{b_1}{2}$, then Person 1 will not get his or her fair share.
- If $x_1 > \frac{b_2}{2}$, then Person 2 will not get his or her fair share.

Making a Fair Arrangement: Two People

	Bob	Alice
Bids	50	12

If we want a Fair arrangement, then Bob, the higher bidder, must win. He can pay Alice any amount x_{Alice} between

$$\frac{12}{2} \leq x_{Alice} \leq \frac{50}{2},$$

$$6 \leq x_{Alice} \leq 25$$

and his compensation x_{Bob} would be his winning bid minus what he paid Alice:

$$x_{Bob} = 50 - x_{Alice}$$

Making a Fair Arrangement: Two People (Minimum)

	Bob	Alice
Bids	50	12

Bob must win.

$$6 \leq x_{Alice} \leq 25$$

$$x_{Bob} = 50 - x_{Alice}$$

For example, if Bob pays Alice 6, her fair share (and the bare minimum), then

$$\begin{aligned} x_{Alice} &= 6 \text{ (payment from Bob)} \\ x_{Bob} &= 50 - 6 = 44 \text{ (winning bid minus payment)} \end{aligned}$$

Alice's compensation equals her fair share, while Bob's compensation is higher than his fair share. The arrangement is Fair.

Making a Fair Arrangement: Two People (Maximum)

	Bob	Alice
Bids	50	12

Bob must win.

$$6 \leq x_{Alice} \leq 25$$

$$x_{Bob} = 50 - x_{Alice}$$

For example, if Bob pays Alice 25, his fair share (and the maximum), then

$$\begin{aligned} x_{Alice} &= 25 \text{ (payment from Bob)} \\ x_{Bob} &= 50 - 25 = 25 \text{ (winning bid minus payment)} \end{aligned}$$

Alice's compensation is over her fair share, while Bob's compensation equals his fair share. The arrangement is Fair.

Paying Too Little

	Bob	Alice
Bids	50	12

Bob must win.

$$6 \leq x_{Alice} \leq 25$$

$$x_{Bob} = 50 - x_{Alice}$$

For example, if Bob pays Alice 5, less than her fair share (below the minimum), then

$$\begin{aligned} x_{Alice} &= 5 \text{ (payment from Bob)} \\ x_{Bob} &= 50 - 5 = 45 \text{ (winning bid minus payment)} \end{aligned}$$

Alice's **compensation is less than her fair share**, while Bob's compensation is higher than his fair share. The arrangement is Unfair for Alice.

Paying Too Much

	Bob	Alice
Bids	50	12

Bob must win.

$$6 \leq x_{Alice} \leq 25$$

$$x_{Bob} = 50 - x_{Alice}$$

For example, if Bob pays Alice 26, over his fair share (above the maximum), then

$$\begin{aligned} x_{Alice} &= 26 \text{ (payment from Bob)} \\ x_{Bob} &= 50 - 26 = 24 \text{ (winning bid minus payment)} \end{aligned}$$

Alice's compensation is over her fair share, but Bob's **compensation is less than his fair share**. The arrangement is Unfair for Bob.

Next time

- XB-ratios and Equitable Arrangements
- Envy-Free Arrangements