Intro to Contemporary Math

Equitable and Envy-Free Arrangements

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Agenda

- XB Ratios and Equitable Arrangements
- Creating Envy-Free Arrangements
  - Picking Winners using Bids
  - Paying Losers using Fair Shares

Announcements

- Homework due this Wednesday
- Exam on Friday

- Project (all parts) due
  November 20

Equitable Arrangements

A person's XB ratio is their compensation divided by their bid:

$$\frac{x_{Person}}{b_{Person}}$$

A compensation arrangement is **equitable** if everyone's XB ratios are equal.

Example

	Tim	Shawn	Leo
Bids on item:	$13,500	$12,000	$9,000

Tim wins the item, and he decides to **pay Shawn $5,000 and Leo $3,000.**

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$$x_{Shawn} = \underbrace{0}_{\text{No Item}} + \underbrace{5,000}_{\text{From Tim}} = 5,000$$

$$x_{Leo} = \underbrace{0}_{\text{No Item}} + \underbrace{3,000}_{\text{From Tim}} = 3,000$$

?(7.2) Is It Equitable?

	Tim	Shawn	Leo
Bids on item:	$13,500	$12,000	$9,000
Compensation:	5,500	5,000	3,000

Is this an equitable arrangement? If not, then whose XB ratio is the highest?
Type "**Yes**" if the arrangement is equitable.
Otherwise, type the **name of the person whose XB ratio is the largest**.

Is It Equitable? No.

- Tim's XB ratio is
$$\frac{x_{Tim}}{b_{Tim}} = \frac{5,500}{13,500} \approx 0.407$$
- Shawn's XB ratio is
$$\frac{x_{Shawn}}{b_{Shawn}} = \frac{5,000}{12,000} \approx 0.417$$
- Leo's XB ratio is
$$\frac{x_{Leo}}{b_{Leo}} = \frac{3,000}{9,000} \approx 0.333$$

- The arrangement is not equitable, because the XB ratios are not the same. Shawn's XB ratio is the highest.

Equitable Arrangements for Two People

When two people bid with $b_1 \leq b_2$, then the arrangement is equitable if the higher bidder Person 2 wins and pays Person 1

$$x_1 = \frac{b_1 \times b_2}{b_1 + b_2}$$

Then Person 2's compensation is winning bid minus payment:

$$x_2 = b_2 - x_1$$

Example with Two People

	Bob	Alice
Bids	50	12

To make an equitable arrangement, Bob, the higher bidder should win, and he should pay Alice

$$x_{Alice} = \frac{12 \times 50}{12 + 50} = \frac{600}{62} \approx 9.68$$

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$$x_{Bob} = 50 - 9.68 = 40.32$$

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$$x_{Bob} = 50 - 9.68 = 40.32$$

XB ratios are the same:

$$\text{Alice: } \frac{9.68}{12} \approx 0.81, \text{ and Bob: } \frac{40.32}{50} \approx 0.81$$

Equitable Formula Warning

Be careful with the formula: make sure to multiply and add bids first:

$$12 \times 50 = 600, \text{ and } 12 + 50 = 62$$

before dividing in your calculator:

$$x_{Pam} = \frac{12 \times 50}{12 + 50} = \frac{600}{62} \approx 9.68$$

Do not divide by the first bid and then add the second one:

$$12 \times 50 = 600, \; 600 \div 12 = 50, \; 50 + 50 = 100$$

Envy

A person will **envy** another person if he or she thinks the other person's compensation is higher than his or her own. An arrangement is **envy-free** if no one envies anyone.

Envy-Free Compensation Theorem

To make an Envy-Free compensation arrangement with N people, we need to know **who wins** and **how much the winner pays each loser**. Let h be the **highest bid** and b_2 be the **second-highest bid**.

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- Make the **highest bidder win.**
- Make the winner pay each loser the **same amount**

  $$x_{Loser}$$

  between the second-highest and highest bidders' fair shares:

  $$\frac{b_2}{N} \leq x_{Loser} \leq \frac{h}{N}$$

-

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between the second-highest and highest bidders' fair shares:

$$\frac{b_2}{N} \leq x_{Loser} \leq \frac{h}{N}$$

- Each loser's compensation is their payment, and the winner's compensation is winning bid minus all payments.

?(7.3) Envy-Free: Choose Winner

	Tim	Shawn	Leo
Bids on item:	$13,500	$12,000	$9,000

We want an Envy-free arrangement. Who could win the item? Type his name, and if there is more than one person, type a list of names.

Envy-Free: Choose Winner

	Tim	Shawn	Leo
Bids on item:	\$13,500	\$12,000	\$9,000

In an envy-free arrangement, the **highest bidder must win**.
Hence only **Tim** can be chosen as the winner.
Tim must pay each loser the same amount x_{Loser}.

?(7.4) Envy-Free: Payment Range

	Tim	Shawn	Leo
Bids on item:	\$13,500	\$12,000	\$9,000

How much can the winner pay each loser?
A) Between 12,000 to 13,500
B) Between 3,000 to 4,000
C) Between 9,000 to 12,000
D) Between 6,000 to 6,750
E) Between 4,500 to 6,000
F) Between 4,000 to 4,500

Envy-Free: Payment Range

	Tim	Shawn	Leo
Bids on item:	$13,500	$12,000	$9,000

Use fair shares:

Tim: $\dfrac{13,500}{3} = 4,500$

Shawn: $\dfrac{12,000}{3} = 4,000$

Leo: $\dfrac{9,000}{3} = 3,000$

Envy-Free: Payment Range

	Tim	Shawn	Leo
Bids on item:	$13,500	$12,000	$9,000

Use fair shares:

Tim: $\frac{13,500}{3} = 4,500$ Shawn: $\frac{12,000}{3} = 4,000$ Leo: $\frac{9,000}{3} = 3,000$

Tim must pay between the second highest bidder's (Shawn's) fair share of 4,000 and Tim's own highest fair share of 4,500:

$$4,000 \leq x_{Loser} \leq 4,500$$

Loser's Compensations

Tim must pay between the second highest bidder's (Shawn's) fair share of 4,000 and Tim's own highest fair share of 4,500:

$$4,000 \leq x_{Loser} \leq 4,500$$

Tim can pay each loser the same amount between these two numbers.

For example, let's have Tim go in the middle and pay $x_{Loser} = 4,250$ each to Shawn and Leo:

$$x_{Shawn} \text{ and } x_{Leo} \text{ is } x_{Loser} = 4,250$$

?(7.5) Winner's Compensation

For example, let's have Tim go in the middle and pay $x_{Loser} = 4,250$ each to Shawn and Leo:

$$x_{Shawn} \text{ and } x_{Leo} \text{ is } x_{Loser} = 4,250$$

	Tim	Shawn	Leo
Bids on item:	\$13,500	\$12,000	\$9,000

What is the winner's compensation (remember he is paying two people)?
Type and send a number.

Winner's Compensation

For example, let's have Tim go in the middle and pay $x_{Loser} = 4,250$ each to Shawn and Leo:

$$x_{Shawn} \text{ and } x_{Leo} \text{ is } x_{Loser} = 4,250$$

	Tim	Shawn	Leo
Bids on item:	\$13,500	\$12,000	\$9,000

$$x_{Tim} = \underbrace{13,500}_{\text{T's winning bid}} - \underbrace{4,250}_{\text{Pay Shawn}} - \underbrace{4,250}_{\text{Pay Leo}} = 5,000$$

What the Losers Think

Shawn thinks Tim got

$$\underbrace{12,000}_{\text{Shawn's bid}} - \underbrace{4,250}_{\text{Pay Shawn}} - \underbrace{4,250}_{\text{Pay Leo}} = 3,500$$

Leo thinks Tim got

$$\underbrace{9,000}_{\text{Leo's bid}} - \underbrace{4,250}_{\text{Pay Shawn}} - \underbrace{4,250}_{\text{Pay Leo}} = 500$$

Both losers think Tim got a smaller compensation than they did (4,250 each).

No Envy

Person at left of row thinks Person at top of column gets____

Each person's view of his own compensation is **in blue**.

	Tim	Shawn	Leo
Tim	**5,000**	4,250	4,250
Shawn	3,500	**4,250**	4,250
Leo	500	4,250	**4,250**

Each person thinks his compensation is no smaller than anyone else's compensation. There is no envy. The arrangement is envy-free.

Paying the Maximum

	Tim	Shawn	Leo
Bids on item:	\$13,500	\$12,000	\$9,000

Tim must pay between the second highest bidder's (Shawn's) fair share of 4,000 and Tim's own fair share of 4,500:

$$4,000 \leq x_{Loser} \leq 4,500$$

For this example, let's have Tim pay $x_{Loser} = 4,500$ each to Shawn and Leo:

$$x_{Shawn} \text{ and } x_{Leo} \text{ is } x_{Loser} = 4,500$$

$$x_{Tim} = \underbrace{13,500}_{\text{T's winning bid}} - \underbrace{4,500}_{\text{Pay Shawn}} - \underbrace{4,500}_{\text{Pay Leo}} = 4,500$$

What the Losers Think

Shawn thinks Tim got

$$\underbrace{12,000}_{\text{Shawn's bid}} - \underbrace{4,500}_{\text{Pay Shawn}} - \underbrace{4,500}_{\text{Pay Leo}} = 3,000$$

Leo thinks Tim got

$$\underbrace{9,000}_{\text{Leo's bid}} - \underbrace{4,500}_{\text{Pay Shawn}} - \underbrace{4,500}_{\text{Pay Leo}} = 0$$

Both losers think Tim got a smaller compensation than they did (4,500 each). Leo even thinks Tim got nothing: Leo thinks Tim lost all of his earnings (from winning the item) on both payments due to Leo's smaller bid.

No Envy

Person at left of row thinks Person at top of column gets____

Each person's view of his own compensation is **in blue**.

	Tim	Shawn	Leo
Tim	**4,500**	4,500	4,500
Shawn	3,000	**4,500**	4,500
Leo	0	4,500	**4,500**

Each person thinks his compensation is no smaller than anyone else's compensation. There is no envy. The arrangement is envy-free.

Paying the Minimum

	Tim	Shawn	Leo
Bids on item:	\$13,500	\$12,000	\$9,000

Tim must pay between the second highest bidder's (Shawn's) fair share of 4,000 and Tim's own fair share of 4,500:

$$4,000 \leq x_{Loser} \leq 4,500$$

For this example, let's have Tim pay $x_{Loser} = 4,000$ each to Shawn and Leo:

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What the Losers Think

Shawn thinks Tim got

$$\underbrace{12,000}_{\text{Shawn's bid}} - \underbrace{4,000}_{\text{Pay Shawn}} - \underbrace{4,000}_{\text{Pay Leo}} = 4,000$$

Leo thinks Tim got

$$\underbrace{9,000}_{\text{Leo's bid}} - \underbrace{4,000}_{\text{Pay Shawn}} - \underbrace{4,000}_{\text{Pay Leo}} = 1,000$$

Both losers think Tim got a compensation that is no bigger than theirs' (4,000 each).

No Envy

Person at left of row thinks Person at top of column gets____

Each person's view of his own compensation is **in blue**.

	Tim	Shawn	Leo
Tim	**5,500**	4,000	4,000
Shawn	4,000	**4,000**	4,000
Leo	1,000	4,000	**4,000**

Each person thinks his compensation is no smaller than anyone else's compensation. There is no envy. The arrangement is envy-free.

Next time

- Review