Intro to Contemporary Math

Planar Graphs

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Announcements

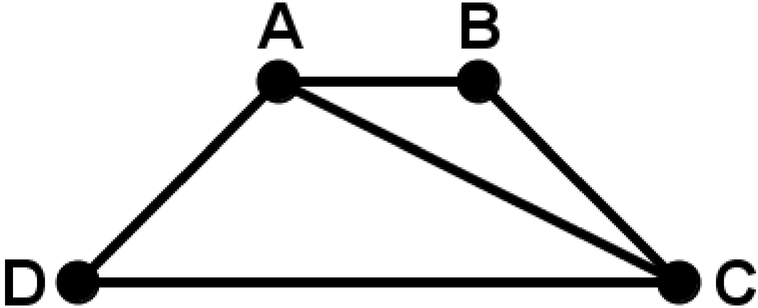
- Your project (all parts) must be uploaded on Canvas by Tuesday, November 20th.

- There will be a homework assignment on WebWork
  It will be due Monday, November 26th.

- Mini-Exam 4 is Wednesday, November 28th.

Graphs with no Overlaps

Definition: A graph is **planar** if it can be **drawn so that its edges do not cross.**



Faces of a Graph

In any planar graph, drawn with no intersections, the edges divide the planes into different regions.

- The regions enclosed by the planar graph are called **interior faces** of the graph.
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Faces of a Graph

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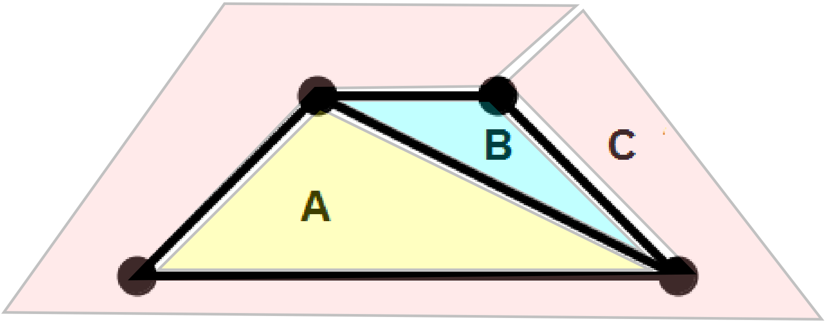
- The regions enclosed by the planar graph are called **interior faces** of the graph.
- The region surrounding (outside) the planar graph is called the **exterior face** of the graph.
- When we say **faces of the graph** we mean BOTH the interior AND the exterior faces. We usually denote the number of faces of a planar graph by f.

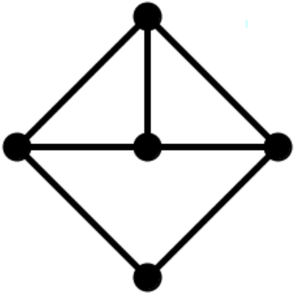
Face Count Example



This graph has a total of three faces: $f = 3$

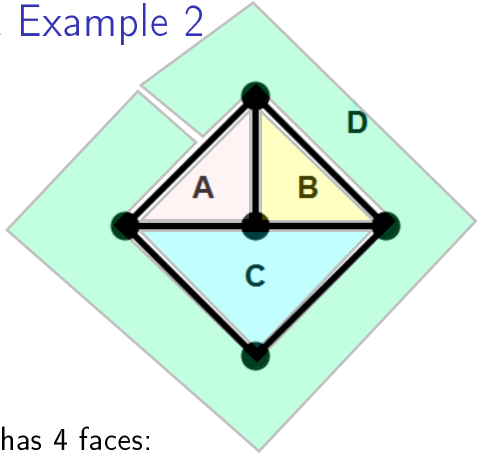
- Two interior faces (A, B)
- One exterior face (C)

?(3.2) Face Count Example 2



How many faces does this graph have (what is f)?
Type and send a number.

Face Count Example 2



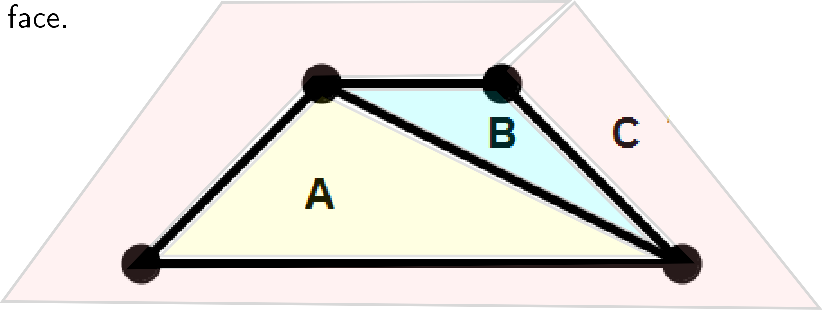
This graph has 4 faces:

A, B, and C are interior faces

D is the exterior face

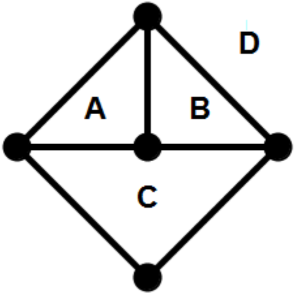
Degree of a Face

For a planar graph drawn without edges crossing, the number of edges bordering a particular face is called the degree of the face.



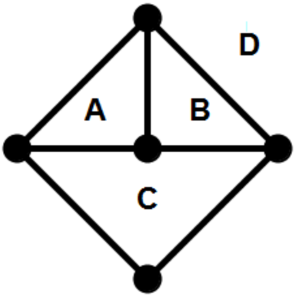
The two interior faces A and B have degree 3, while the exterior face C has degree 4.

?(3.3) Degree of a Face Example 2



- The interior faces A and B each have degree 3.
- The degree of face C is what number?

Degree of a Face Example 2



- The interior faces A and B each have degree 3.
- The interior face C has degree 4.
- The exterior face D has degree 4.

Sum of Degrees

In a planar graph,

- If you **add up the degrees** of every vertex and **divide by 2**, you get the **number of edges.**
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In a planar graph,

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Sum of Degrees

In a planar graph,

- If you **add up the degrees** of every vertex and **divide by 2**, you get the **number of edges.**
- If you **add up the degrees** of every face and **divide by 2**, you get the **number of edges.**
- Conversely, if you take the **number of edges and multiply by 2**, you get the **sum of degrees of the vertices or faces.**

Sum of Degrees Picture (Vertices)

One edge is attached to **TWO** vertices, and gets counted in degree of left vertex and right vertex.

Sum of Degrees Picture (Faces)

One edge borders **TWO** faces, and gets counted in degree of upper face and lower face.

?(3.4) Sum of Degrees Example

A graph has degree list 2, 2, 3, 3, 4, 4, 5, 5. How many edges does it have?
Type and send a number.

Sum of Degrees Example

A graph has degree list 2, 2, 3, 3, 4, 4, 5, 5.
Add up degrees of vertices:

$$2 + 2 + 3 + 3 + 4 + 4 + 5 + 5 = 28$$

then divide by 2 to get the number of edges e:

$$\frac{28}{2} = \boxed{14}$$

Note: Since the degree list has 8 entries, the graph must have 8 vertices.

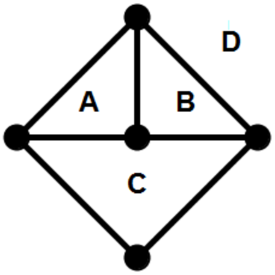
Euler's Formula

There is another relationship between the number of vertices, edges, and faces:

- For a connected (one-piece) planar graph with v vertices, e edges, and f faces,

$$v - e + f = 2$$

Euler's Formula Showcase



$v = 5,\ e = 7,\ f = 4$:

$$v - e + f = 5 - 7 + 4 = 2.$$

?(3.5) Euler's Formula Example

A connected planar graph has 24 vertices and 30 faces. How many edges does the graph have?
Type and send a number.

Euler's Formula Example

$v = 24$ and $f = 30$, so in Euler's formula,

$$\begin{aligned} v - e + f &= 2 \\ 24 - e + 30 &= 2 \\ -e + 54 &= 2 \\ -e &= 2 - 54 = -52 \\ e &= 52 \end{aligned}$$

so there are 52 edges: $e = 52$.

?(3.6) Euler's Formula Example

$$
\begin{aligned}
v - e + f &= 2 \\
24 - e + 30 &= 2 \\
-e + 54 &= 2 \\
-e &= 2 - 54 = -52 \\
e &= 52
\end{aligned}
$$

so there are 52 edges: $e = 52$.

- What number would you get if you add up the degrees of the vertices?

$v = 24$ and $f = 30$, so in Euler's formula,

$$\begin{aligned} v - e + f &= 2 \\ 24 - e + 30 &= 2 \\ -e + 54 &= 2 \\ -e &= 2 - 54 = -52 \\ e &= 52 \end{aligned}$$

so there are 52 edges: $e = 52$.

- Thus, the sum of the degrees of every vertex is 2 times the number of edges:

$$2e = 2 \times 52 = \boxed{104}$$

Next Time

- Graphs with Labeled Edges