Intro to Contemporary Math

Conditional Probability Intro

Department of Mathematics
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Announcement

- You have a homework assignment due tonight.
- Mini-exam 2 is this Wednesday.

Back to Discrete Probability

Let Ω be a sample space and E be an event. The probability of E is:

Number of outcomes (objects) in E

divided by...

Number of outcomes in Ω

Cards with Suits and Ranks

Each card in a deck can be identified by a **suit** (symbol) *and* **rank** (number). Suppose there are n suits and m ranks, and each suit has m cards of each rank, and each rank has n cards of each suit. Then there are

$$n \cdot m \text{ cards total.}$$

Cards with Suits and Ranks

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Example: the standard 52-card deck has 4 suits (Clubs, Diamonds, Hearts, and Spades) and 13 ranks (Ace, 2-10, Jack, Queen, King). There are 13 Club cards (one of each rank), and there are 4 Aces (one of each suit).

Notation

From now on, suits will be labeled with letters (starting with A), and ranks will be labeled with numbers exclusively. We may also work with made-up decks.
A deck of 15 cards has 3 suits (A-C) and 5 ranks (1-5).

Suit \ Rank	1	2	3	4	5
A	A1	A2	A3	A4	A5
B	B1	**B2**	B3	B4	B5
C	C1	C2	C3	C4	C5

Drawing a Certain Card

A deck of 15 cards has 3 suits (A-C) and 5 ranks (1-5).

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B	B1	**B2**	B3	B4	B5
C	C1	C2	C3	C4	C5

Notice the table helps us see why the total number of cards is the number of suits times the number of ranks.
The probability of drawing card B2 is:

$$\frac{\text{1 desired card, B2}}{\text{15 cards total}} = \frac{1}{15}.$$

Drawing a Card with a Desired Suit

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A	A1	A2	A3	A4	A5
B	**B1**	**B2**	**B3**	**B4**	**B5**
C	C1	C2	C3	C4	C5

The probability of drawing a card with suit B is:

$$\frac{\text{5 desired cards}}{\text{15 cards total}} = \frac{5}{15} = \frac{1}{3},$$

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B	**B1**	**B2**	**B3**	**B4**	**B5**
C	C1	C2	C3	C4	C5

The probability of drawing a card with suit B is:

$$\frac{\text{5 desired cards}}{\text{15 cards total}} = \frac{5}{15} = \frac{1}{3},$$

but if we think of picking the suit (letter) B from a sample space of 3 suits, we get:

$$\frac{\text{1 desired suit, B}}{\text{3 suits total}} = \frac{1}{3} \text{ as well.}$$

Drawing a Card with Even Rank

A deck of 15 cards has 3 suits (A-C) and 5 ranks (1-5).

Suit \ Rank	1	**2**	3	**4**	5
A	A1	**A2**	A3	**A4**	A5
B	B1	**B2**	B3	**B4**	B5
C	C1	**C2**	C3	**C4**	C5

The probability of drawing a card with even rank is:

$$\frac{\text{6 desired cards}}{\text{15 cards total}} = \frac{6}{15} = \frac{2}{5},$$

Drawing a Card with Even Rank

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B	B1	**B2**	B3	**B4**	B5
C	C1	**C2**	C3	**C4**	C5

The probability of drawing a card with even rank is:

$$\frac{\text{6 desired cards}}{\text{15 cards total}} = \frac{6}{15} = \frac{2}{5},$$

but if we think of picking the the ranks (numbers) 2 or 4 from a sample space of 5 ranks, we get:

$$\frac{\text{2 desired ranks, 2 or 4}}{\text{5 ranks total}} = \frac{2}{5} \text{ as well.}$$

?(5.1) Drawing a Card (Intersection)

A deck of 15 cards has 3 suits (A-C) and 5 ranks (1-5).

Suit \ Rank	1	2	3	4	5
A	A1	A2	A3	A4	A5
B	B1	B2	B3	B4	B5
C	C1	C2	C3	C4	C5

What is the probability of drawing a card with suit B and even rank?

Drawing a Card (Intersection)

A deck of 15 cards has 3 suits (A-C) and 5 ranks (1-5).

Suit \ Rank	1	*2*	3	*4*	5
A	A1	*A2*	A3	*A4*	A5
B	*B1*	**B2**	*B3*	**B4**	*B5*
C	C1	*C2*	C3	*C4*	C5

What is the probability of drawing a card with suit B and even rank?

Among the 15 cards, two of them, B2 and B4, have rank B and even (2 or 4) rank.

$$\frac{2 \text{ desired cards}}{15 \text{ cards total}} = \frac{2}{15}.$$

?(5.2) Drawing a Card with a Desired Suit

A deck of cards has 3 suits (A-C) and 5 ranks (1-5), but cards A2 and B4 are missing! There are only 13 cards now.

Suit \ Rank	1	2	3	4	5
A	A1		A3	A4	A5
B	B1	B2	B3		B5
C	C1	C2	C3	C4	C5

Now what is the probability of drawing a card with suit B?

Drawing a Card with a Desired Suit

A deck of cards has 3 suits (A-C) and 5 ranks (1-5), but cards A2 and B4 are missing! There are only 13 cards now.

Suit \ Rank	1	2	3	4	5
A	A1		A3	A4	A5
B	**B1**	**B2**	**B3**		**B5**
C	C1	C2	C3	C4	C5

Now what is the probability of drawing a card with suit B?

$$\frac{\text{4 desired cards}}{\text{13 cards total}} = \frac{4}{13}.$$

This is not the same answer as earlier. The numerator is different because there are fewer cards with rank B (due to B4 being gone), and the denominator changed as well due to the missing cards.

Suits, Ranks, and Missing Cards

- In the examples with no cards missing, the proportion of desired cards to total cards equaled the proportion of desired suits/ranks to total suits/ranks. Each suit/rank had the same number of cards as well.
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Suits, Ranks, and Missing Cards

- In the examples with no cards missing, the proportion of desired cards to total cards equaled the proportion of desired suits/ranks to total suits/ranks. Each suit/rank had the same number of cards as well.
- However, in the example with the missing cards, the proportion of desired cards to total cards did not equal the proportion of desired suits to total suits. The suits had different numbers of cards (suit B had 4 cards left while suit C still had 5 cards).

?(5.3) Drawing a Card (Intersection) with Missing Cards

A deck of 15 cards has 3 suits (A-C) and 5 ranks (1-5), but cards A2 and B4 are missing!

Suit \ Rank	1	2	3	4	5
A	A1		A3	A4	A5
B	B1	B2	B3		B5
C	C1	C2	C3	C4	C5

What is the probability of drawing a card with suit B and even rank?

Drawing a Card (Intersection) with Missing Cards

A deck of 15 cards has 3 suits (A-C) and 5 ranks (1-5), but cards A2 and B4 are missing!

Suit \ Rank	1	*2*	3	*4*	5
A	A1		A3	*A4*	A5
B	*B1*	**B2**	*B3*		*B5*
C	C1	*C2*	C3	*C4*	C5

What is the probability of drawing a card with suit B and even rank?

Among the 13 cards, one of them, B2, has rank B and even (2 or 4) rank.

$$\frac{1 \text{ desired card}}{13 \text{ cards total}} = \frac{1}{13}.$$

Drawing a Card (Union)

A deck of 15 cards has 3 suits (A-C) and 5 ranks (1-5).

Suit \ Rank	1	2	3	4	5
A	A1	A2	A3	A4	A5
B	B1	B2	B3	B4	B5
C	C1	C2	C3	C4	C5

What is the probability of drawing a card with suit B or even rank?

There are 5 cards with suit B and 6 cards with even rank.
Why is the answer not 11/15?

Drawing a Card (Union)

A deck of 15 cards has 3 suits (A-C) and 5 ranks (1-5).

Suit \ Rank	1	**2**	3	**4**	5
A	A1	**A2**	A3	**A4**	A5
B	**B1**	**B2**	**B3**	**B4**	**B5**
C	C1	**C2**	C3	**C4**	C5

What is the probability of drawing a card with suit B or even rank?

A total of 9 cards have suit B, or even rank (or both):

$$\frac{9 \text{ desired cards}}{15 \text{ cards total}} = \frac{9}{15}.$$

There are 5 cards with suit B and 6 cards with even rank. Why is the answer NOT 11/15?

Drawing a Card (Union)

If we use the formula for unions and intersections, we get:

$$\begin{aligned} P(\text{Card has suit B}) & & 5/15 \\ +P(\text{Card has even rank}) & & +6/15 \\ -P(\text{Card has suit B and even rank}) & & -2/15 \\ & = & 9/15. \end{aligned}$$

Suit B:

Even rank:

?(5.5) Drawing from a Big Deck

A deck of cards has 11 suits (A-K) and 25 ranks (1-25), with no missing cards. What is the probability of drawing a card whose suit is a vowel (A, E, I, O, or U)?
Hint: use the suits as the sample space since no cards are missing.

Drawing from a Big Deck

A deck of cards has 11 suits (A-K) and 25 ranks (1-25), with no missing cards. What is the probability of drawing a card whose suit is a vowel (A, E, I, O, or U)?
There are 11 suits, and among them, 3 of them, A, E, and I, are vowels, so the probability is

$$\frac{3 \text{ desired suits}}{11 \text{ suits total}} = \frac{3}{11}.$$

Drawing from a Big Deck

Since each suit has 25 cards each (one per rank), we can see that there are

$$3 \cdot 25 = 75 \text{ cards whose suit is a vowel,}$$

and since there are

$$11 \cdot 25 = 275 \text{ cards total,}$$

the probability of drawing a card whose suit is a vowel is also

$$\frac{75}{275}$$

which reduces to our earlier answer. As we will see, our answer of $3/11$ will come in handy later.

?(5.6) Drawing from a Big Deck

A deck of cards has 11 suits (A-K) and 25 ranks (1-25), with no missing cards. What is the probability of drawing a card whose rank is a multiple of 6 (6, 12, ??,...)?

Drawing from a Big Deck

A deck of cards has 11 suits (A-K) and 25 ranks (1-25), with no missing cards. What is the probability of drawing a card whose rank is a multiple of 6 (6, 12, 18, 24)?

There are 25 ranks, and 4 of them are multiples of 6, so the probability is

$$\frac{4 \text{ desired ranks}}{25 \text{ ranks total}} = \frac{4}{25}.$$