Intro to Contemporary Math

Conditional Probability and Independent Events

Department of Mathematics
UK

October 5, 2018

Announcement

- You have a homework assignment due next Monday.

Drawing a Card

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A deck of 10 cards has 2 suits (A-B) and 5 ranks (1-5).

Suit \ Rank	1	2	3	4	5
A	A1	A2	A3	A4	A5
B	B1	B2	B3	B4	B5

Drawing a Card

A deck of 10 cards has 2 suits (A-B) and 5 ranks (1-5).

Suit \ Rank	1	2	3	4	5
A	A1	A2	A3	A4	A5
B	B1	B2	B3	B4	B5

- Let E be the event "A card with suit B is drawn."
- Let F be the event "A card with rank 2 is drawn."

Drawing a Card with a Desired Suit

A deck of 10 cards has 2 suits (A-B) and 5 ranks (1-5).

Suit \ Rank	1	2	3	4	5
A	A1	A2	A3	A4	A5
B	**B1**	**B2**	**B3**	**B4**	**B5**

The probability of drawing a card with suit B is:

$$P(E) = \frac{\text{5 such cards}}{\text{10 cards total}} = \frac{5}{10} = \frac{1}{2}.$$

Drawing a Card with a Desired Rank

A deck of 10 cards has 2 suits (A-B) and 5 ranks (1-5).

Suit \ Rank	1	**2**	3	4	5
A	A1	**A2**	A3	A4	A5
B	B1	**B2**	B3	B4	B5

The probability of drawing a card with rank 2 is:

$$P(F) = \frac{\text{2 such cards}}{\text{10 cards total}} = \frac{2}{10} = \frac{1}{5}.$$

Drawing a Specific Card

A deck of 10 cards has 2 suits (A-B) and 5 ranks (1-5).

Suit \ Rank	1	2	3	4	5
A	A1	A2	A3	A4	A5
B	B1	**B2**	B3	B4	B5

The probability of drawing card B2 (suit B **and** rank 2) is:

$$P(E \bigcap F) = \frac{\text{1 such card B2}}{\text{10 cards total}} = \frac{1}{10}.$$

Observation

Notice that

$$P(E) = \frac{1}{2},\ P(F) = \frac{1}{5},\ P(E \bigcap F) = \frac{1}{10}.$$

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$$P(E) = \frac{1}{2},\ P(F) = \frac{1}{5},\ P(E \bigcap F) = \frac{1}{10}.$$

Coincidentally,

$$P(E) \cdot P(F) = \frac{1}{2} \cdot \frac{1}{5} = \frac{1}{10}$$

as well. When can we multiply the probabilities of two events to find the probability of their intersection?

Drawing a Card: Conditional Probability

Let us compute:

- $P(F|E)$: the probability that the drawn card has rank 2, **given** that it has suit B.
- $P(E|F)$: the probability that the drawn card has suit B, **given** that it has rank 2.

We will compare these to each other, and to $P(E)$ and $P(F)$.

Drawing a Card: $P(F|E)$ (Given Suit)

A deck of 10 cards has 2 suits (A-B) and 5 ranks (1-5).

Suit \ Rank	1	2	3	4	5
A	A1	A2	A3	A4	A5
B	B1	B2	B3	B4	B5

Let's find the probability that the drawn card has rank 2 given that it has suit B.
The given event narrows down the sample space to just the five cards with suit B.

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Let's find the probability that the drawn card has rank 2 given that it has suit B.

The given event narrows down the sample space to just the five cards with suit B.

The probability that the drawn card has rank 2 given that it has suit B is

$$P(F|E) = \frac{\text{1 card among those with suit B that has rank 2}}{\text{5 cards with suit B}} = \frac{1}{5}.$$

Conditional Probability Numerator

Let's find the probability that the drawn card has rank 2 given that it has suit B.

The given event narrows down the sample space to just the five cards with suit B.

The probability that the drawn card has rank 2 given that it has suit B is

$$P(F|E) = \frac{\text{1 card among those with suit B that has rank 2}}{\text{5 cards with suit B}} = \frac{1}{5}.$$

We can rephrase the numerator:

$$P(F|E) = \frac{\text{1 card with suit B \textbf{and} rank 2}}{\text{5 cards with suit B}} = \frac{1}{5}.$$

Conditional Probability Computation

Let E and F be events in a sample space Ω. Then $P(F|E)$, the probability of getting an outcome in event F given that the outcome is in event E, is

$$P(F|E) = \frac{\text{Number of outcomes in } E \bigcap F}{\text{Number of outcomes in } E}.$$

Hint: The given event appears in the denominator.

Drawing a Card: $P(E|F)$ (Given Rank)

A deck of 10 cards has 2 suits (A-B) and 5 ranks (1-5).

Suit \ Rank	1	2	3	4	5
A	A1	A2	A3	A4	A5
B	B1	B2	B3	B4	B5

Let's find the probability that the drawn card has suit B given that it has rank 2.
The given event narrows down the sample space to just the two cards with rank 2.

Drawing a Card: $P(E|F)$ (Given Rank)

A deck of 10 cards has 2 suits (A-B) and 5 ranks (1-5).

Suit \ Rank		2			
A		A2			
B		B2			

Let's find the probability that the drawn card has suit B given that it has rank 2.
The given event narrows down the sample space to just the two cards with rank 2.

Drawing a Card: $P(E|F)$ (Given Rank)

A deck of 10 cards has 2 suits (A-B) and 5 ranks (1-5).

Suit \ Rank		2			
A		A2			
B		**B2**			

Let's find the probability that the drawn card has suit B given that it has rank 2.

The given event narrows down the sample space to just the two cards with rank 2.

The probability that the drawn card has suit B given that it has rank 2 is

$$P(E|F) = \frac{\text{1 card with rank 2 } \textbf{and} \text{ suit B}}{\text{2 cards with rank 2}} = \frac{1}{2}.$$

Observations

- Notice that

$$P(E) = \frac{1}{2}, \text{ and } P(E|F) = \frac{1}{2} \text{ as well.}$$

  Knowing the card's rank did not affect its chances of having suit B.

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Observations

- Notice that

$$P(E) = \frac{1}{2}, \text{ and } P(E|F) = \frac{1}{2} \text{ as well.}$$

  Knowing the card's rank did not affect its chances of having suit B.

- Similarly,

$$P(F) = \frac{1}{5}, \text{ and } P(F|E) = \frac{1}{5} \text{ as well.}$$

  Knowing the card's suit did not affect its chances of having rank 2.

Warning: Order Matters

Beware: $P(E|F)$ and $P(F|E)$ are not equal in general!

- In this example, $P(E|F) = \dfrac{1}{2}$ while $P(F|E) = \dfrac{1}{5}$.
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- In this example, $P(E|F) = \frac{1}{2}$ while $P(F|E) = \frac{1}{5}$.
- For $P(E|F)$, the given event restricted us to the cards with rank 2.
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Warning: Order Matters

Beware: $P(E|F)$ and $P(F|E)$ are not equal in general!

- In this example, $P(E|F) = \frac{1}{2}$ while $P(F|E) = \frac{1}{5}$.
- For $P(E|F)$, the given event restricted us to the cards with rank 2.
- On the other hand, for $P(F|E)$, the given event restricted us to the cards with suit B.

Independent Events

Let E and F be events in a sample space Ω. Then E and F are said to be **independent events** if

$$P(F|E) = P(F).$$

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Let E and F be events in a sample space Ω. Then E and F are said to be **independent events** if

$$P(F|E) = P(F).$$

That is, the conditional probability of F given that E occurred is the same as the (regular) probability of F.
The probability of F occurring is the same whether or not E occurs.

Checking Independence

Let E and F be events in a sample space Ω.
To check if E and F are independent events,

1. Compute $P(E)$, $P(F)$, and $P(F|E)$.
2. Compare $P(F|E)$ and $P(F)$. Are they **equal** or **not**?

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Checking Independence

Let E and F be events in a sample space Ω.
To check if E and F are independent events,

1. Compute $P(E)$, $P(F)$, and $P(F|E)$.
2. Compare $P(F|E)$ and $P(F)$. Are they **equal** or **not**?

- If $P(F|E)$ and $P(F)$ are **equal**, then E and F **are independent** events.
- If $P(F|E)$ and $P(F)$ are **different** numbers, then E and F **are not independent** events.

Independence with a Full Deck

A deck of 10 cards has 2 suits (A-B) and 5 ranks (1-5).

Suit \ Rank	1	2	3	4	5
A	A1	A2	A3	A4	A5
B	B1	B2	B3	B4	B5

- Let E be the event "A card with suit B is drawn."
- Let F be the event "A card with rank 2 is drawn."

We saw that

$$P(F|E) = \frac{1}{5}, \text{ and } P(F) = \frac{1}{5} \text{ (both equal 0.2).}$$

Hence E and F **are independent** events.

Drawing a Card 2

A deck of 8 cards has 2 suits and 5 ranks, but cards B4 and B5 are not included:

Suit \ Rank	1	2	3	4	5
A	A1	A2	A3	A4	A5
B	B1	B2	B3		

?(6.1): Drawing a Card 2

A deck of 8 cards has 2 suits and 5 ranks, but cards B4 and B5 are not included:

Suit \ Rank	1	2	3	4	5
A	A1	A2	A3	A4	A5
B	B1	B2	B3		

- Let E be the event "A card with suit B is drawn." Find $P(E)$.

  $$A)\ 1/2\ \ B)\ 3/10\ \ C)\ 3/5\ \ D)\ 3/8\ \ E)\ 1/3$$

- Let F be the event "A card with rank 2 is drawn." Find $P(F)$.

  $$A)\ 1/5\ \ B)\ 1/8\ \ C)\ 2/5\ \ D)\ 1/2\ \ E)\ 2/8$$

- Then $E \bigcap F$ is the event "Card B2 is drawn." Find $P(E \bigcap F)$.

  $$A)\ 1/2\ \ B)\ 1/10\ \ C)\ 1/8\ \ D)\ 1/5\ \ E)\ 6/64$$

Drawing a Card with a Desired Suit 2

Suit \ Rank	1	2	3	4	5
A	A1	A2	A3	A4	A5
B	**B1**	**B2**	**B3**		

The probability of drawing a card with suit B is:

$$P(E) = \frac{3 \text{ such cards}}{8 \text{ cards total}} = \frac{3}{8}.$$

Drawing a Card with a Desired Rank 2

Suit \ Rank	1	**2**	3	4	5
A	A1	**A2**	A3	A4	A5
B	B1	**B2**	B3		

The probability of drawing a card with rank 2 is:

$$P(F) = \frac{\text{2 such cards}}{\text{8 cards total}} = \frac{2}{8} = \frac{1}{4}.$$

Drawing a Specific Card 2

Suit \ Rank	1	2	3	4	5
A	A1	A2	A3	A4	A5
B	B1	**B2**	B3		

The probability of drawing card B2 (suit B **and** rank 2) is:

$$P(E \bigcap F) = \frac{\text{1 such card B2}}{\text{8 cards total}} = \frac{1}{8}.$$

Different Observation

Notice that

$$P(E) = \frac{3}{8}, \; P(F) = \frac{1}{4}, \; P(E \bigcap F) = \frac{1}{8}.$$

Different Observation

Notice that

$$P(E) = \frac{3}{8},\ P(F) = \frac{1}{4},\ P(E \bigcap F) = \frac{1}{8}.$$

This time,

$$P(E) \cdot P(F) = \frac{3}{8} \cdot \frac{1}{4} = \frac{3}{32} \neq \frac{1}{8}.$$

Multiplying the probabilities of two events did not equal the probability of their intersection.

?(6.2): Drawing a Card: Conditional Probability

Suit \ Rank	1	2	3	4	5
A	A1	A2	A3	A4	A5
B	B1	B2	B3		

Now compute:

- $P(F|E)$: the probability that the drawn card has rank 2, **given** that it has suit B.

$$A)\ 3/8\ B)\ 1/8\ C)\ 1/3\ D)\ 2/8\ E)\ 2/3$$

- $P(E|F)$: the probability that the drawn card has suit B, **given** that it has rank 2.

$$A)\ 2/8\ B)\ 1/2\ C)\ 3/2\ D)\ 1/8\ E)\ 3/8$$

We will compare these to each other, and to $P(E)$ and $P(F)$.

Drawing a Card: $P(F|E)$ (Given Suit) 2

Suit \ Rank	1	2	3	4	5
A	A1	A2	A3	A4	A5
B	B1	B2	B3		

Let's find the probability that the drawn card has rank 2 given that it has suit B.
The given event narrows down the sample space to just the three cards with suit B.

Drawing a Card: $P(F|E)$ (Given Suit) 2

Suit \ Rank	1	2	3	4	5
A					
B	B1	B2	B3		

Let's find the probability that the drawn card has rank 2 given that it has suit B.

The given event narrows down the sample space to just the three cards with suit B.

Drawing a Card: $P(F|E)$ (Given Suit) 2

Suit \ Rank	1	2	3	4	5
A					
B	B1	**B2**	B3		

Let's find the probability that the drawn card has rank 2 given that it has suit B.
The given event narrows down the sample space to just the three cards with suit B.
The probability that the drawn card has rank 2 given that it has suit B is

$$P(F|E) = \frac{\text{1 card with suit B } \textbf{and} \text{ rank 2}}{\text{3 cards with suit B}} = \frac{1}{3}.$$

Drawing a Card: $P(E|F)$ (Given Rank) 2

Suit \ Rank	1	2	3	4	5
A	A1	A2	A3	A4	A5
B	B1	B2	B3		

Let's find the probability that the drawn card has suit B given that it has rank 2.
The given event narrows down the sample space to just the two cards with rank 2.

Drawing a Card: $P(E|F)$ (Given Rank) 2

Suit \ Rank	1	2	3	4	5
A		A2			
B		B2			

Let's find the probability that the drawn card has suit B given that it has rank 2.
The given event narrows down the sample space to just the two cards with rank 2.

Drawing a Card: $P(E|F)$ (Given Rank) 2

Suit \ Rank	1	2	3	4	5
A		A2			
B		**B2**			

Let's find the probability that the drawn card has suit B given that it has rank 2.
The given event narrows down the sample space to just the two cards with rank 2.
The probability that the drawn card has suit B given that it has rank 2 is

$$P(E|F) = \frac{\text{1 card with rank 2 \textbf{and} suit B}}{\text{2 cards with rank 2}} = \frac{1}{2}.$$

?(6.3) Independent?

Our calculations so far:

- $P(E) = 3/8$
- $P(F) = 2/8$
- $P(F|E) = 1/3$
- $P(E|F) = 1/2$

Are the events E and F independent? Yes or no?

Observation: $P(E)$ vs. $P(E|F)$

- Notice that

$$P(E) = \frac{3}{8}, \text{ but } P(E|F) = \frac{1}{2}.$$

Normally, we have a 3 in 8 chance (0.375) of drawing a card with suit B, but if we are given that the drawn card has rank 2, that raises the chances of drawing a card with suit B to 1/2 (0.5).

Observation: $P(F)$ vs. $P(F|E)$

- Notice that

$$P(F) = \frac{1}{4}, \text{ but } P(F|E) = \frac{1}{3}.$$

Normally, we have a 1 in 4 chance (0.25) of drawing a card with rank 2, but if we are given that the drawn card has suit B, that raises the chances of drawing a card with rank 2 to 1/3 (about 0.3333).

Independence with Missing Cards

Suit \ Rank	1	2	3	4	5
A	A1	A2	A3	A4	A5
B	B1	B2	B3		

Since

$$P(F) = \frac{1}{4}, \text{ but } P(F|E) = \frac{1}{3},$$

the events E and F are not independent.

Next Time

We will do some more practice with conditional probability using a table.