# Intro to Contemporary Math

## Probability Theory Slides

Department of Mathematics
UK

Fall 2017

# Topic Idea: Probability

## Definition

The **probability** of an event is a measurement of its likelihood to occur.

There are two interpretations of probability: **experimental** and **theoretical**.

# Fractions

In both interpretations, we use fractions to express portions of quantities, and in general, quantities which are not whole numbers.

## Example

The fraction

$$\frac{2}{5}, \text{ or } 2/5$$

means two parts out of five total.

# Fractions

In both interpretations, we use fractions to express portions of quantities, and in general, quantities which are not whole numbers.

## Example

The fraction

$$\frac{2}{5}, \text{ or } 2/5$$

means two parts out of five total.
Fractions can be converted to decimals by dividing:

$$2/5 = 0.4,$$

but we will use fractions for better accuracy.

# Experimental Probability

Perform an experiment over and over, and divide the number of times a desired event occurs by the total number of times the experiment was performed.

## Example

# Experimental Probability

Perform an experiment over and over, and divide the number of times a desired event occurs by the total number of times the experiment was performed.

## Example

We could toss a coin 100 times. If it came up heads 47 times, we would have measured that the probability of flipping this coin and getting heads is

$$\frac{\text{number of heads}}{\text{total tosses}} = \frac{47}{100}.$$

# Theoretical Probability

Suppose an experiment can end in $n$ ways. If the results cannot be told apart except by name, we assume they are equally-likely, and assign each result a probability of

$$\frac{1}{n}.$$

## Example

# Theoretical Probability

Suppose an experiment can end in $n$ ways. If the results cannot be told apart except by name, we assume they are equally-likely, and assign each result a probability of

$$\frac{1}{n}.$$

## Example

The coin can land in 1 of 2 ways: heads or tails. If we assume the coin is equally-likely to land heads or tails, then the probability of each result is

$$\frac{\textbf{One}\text{ side is heads}}{\textbf{Two}\text{ sides total}} = \frac{1}{2}.$$

# Outcomes and Events

## Definitions

A (chance) experiment is a procedure whose result can be one out of many possibilities.

- Each possible result is called an **outcome**.
-
-

# Outcomes and Events

## Definitions

A (chance) experiment is a procedure whose result can be one out of many possibilities.

- Each possible result is called an **outcome**.
- An **event** is any particular outcome or group of outcomes.
- The **sample space** is a list of all possible outcomes.

# Six-Sided Die

If we roll a six-sided die, one numbered side will be on top of the die when it lands.

- There are a total of 6 outcomes, one for each side.
-
-

# Six-Sided Die

If we roll a six-sided die, one numbered side will be on top of the die when it lands.

- There are a total of 6 outcomes, one for each side.
- Example of an event: "We roll an odd number." The outcomes "1," "3," and "5" are described by this event.
-

# Six-Sided Die

If we roll a six-sided die, one numbered side will be on top of the die when it lands.

- There are a total of 6 outcomes, one for each side.
- Example of an event: "We roll an odd number." The outcomes "1," "3," and "5" are described by this event.
- The sample space is a list of all sides:

$$\{1, 2, 3, 4, 5, 6\}.$$

# Computing Probability

Given that all outcomes are equally-likely, the probability of an event, $P(\text{"Event"})$, is

$$P(\text{"Event"}) = \frac{\text{Number of outcomes described by the event}}{\text{Total number of outcomes}}$$

In other words, it is the ratio of outcomes in the event compared to the total amount of outcomes.

# Computing Probability

Given that all outcomes are equally-likely, the probability of an event, $P(\text{"Event"})$, is

$$P(\text{"Event"}) = \frac{\text{Number of outcomes described by the event}}{\text{Total number of outcomes}}$$

In other words, it is the ratio of outcomes in the event compared to the total amount of outcomes.

**Note**: The probability of an event must be a number between 0 and 1.

An event with a probability of 0 is impossible.

An event with a probability of 1 is certain.

# Six-Sided Die

- The event "We roll a 5" describes one outcome (5 on top), so

$$P(\text{"We roll a 5"}) = \frac{\text{One side with a 5}}{\text{Six sides total}} = \boxed{\frac{1}{6}}.$$

# Six-Sided Die

- The event “We roll an odd number” describes three outcomes: 1, 3, or 5 on top. Thus

$$\begin{aligned} P(\text{"We roll an odd number"}) &= \frac{\text{Three sides in event}}{\text{Six sides total}} \\ &= \boxed{\frac{3}{6}} = \frac{1}{2}. \end{aligned}$$

# Six-Sided Die

- The event "We roll an odd number" describes three outcomes: 1, 3, or 5 on top. Thus

$$
\begin{aligned}
P(\text{"We roll an odd number"}) &= \frac{\text{Three sides in event}}{\text{Six sides total}} \\
&= \boxed{\frac{3}{6}} = \frac{1}{2}.
\end{aligned}
$$

The answer 3/6 can be reduced to 1/2, or to the decimal expression 0.5, but the original answer makes it easier to see the number of outcomes in the event and sample space.

# Six-Sided Die

- The event "We roll a 7" describes no outcomes (7 is not on the die), so

$$P(\text{"We roll a 7"}) = \frac{\text{No outcomes in event}}{\text{Six outcomes total}} = \boxed{\frac{0}{6}}, \text{ or } 0.$$

This event is impossible.

# Six-Sided Die

- The event "We roll a number" describes all six outcomes (all six sides have a number on them), so

$$P(\text{"We roll a number"}) = \frac{\text{Six outcomes in event}}{\text{Six outcomes total}} = \boxed{\frac{6}{6}}, \text{ or } 1.$$

This event is certain to occur.

# Six-Sided Die

On a six-sided die, what is the probability of the event "We roll a number strictly greater than 2?"
Give a fraction as your answer.
You do not need to reduce your answer.

# Six-Sided Die

The event "We roll a number strictly greater than 2" describes four outcomes: 3, 4, 5, or 6 on top. Thus

$$\frac{\text{Four outcomes in event}}{\text{Six outcomes total}} = \boxed{\frac{4}{6}} = \frac{2}{3}.$$

# Marbles

A jar has 11 red marbles and 16 blue ones. If a marble is drawn at random, what is the probability of drawing a blue marble?

# Marbles

A jar has 11 red marbles and 16 blue ones. If a marble is drawn at random, what is the probability of drawing a blue marble? Our sample space is all the marbles in the jar:

$$\text{There are } 11 + 16 = 27 \text{ marbles total.}$$

Hence the probability of drawing a blue marble is

$$\frac{16 \text{ blue marbles}}{27 \text{ marbles total}} = \boxed{\frac{16}{27}}.$$

# Checking Answers

The probability of an event is a number between 0 and 1.

- The number of outcomes in an event (numerator) cannot be negative and it cannot be more than the total number of outcomes (denominator).
- 

# Checking Answers

The probability of an event is a number between 0 and 1.

- The number of outcomes in an event (numerator) cannot be negative and it cannot be more than the total number of outcomes (denominator).
- You can convert to a decimal to see how big a fraction is.

Example: if you are computing the probability of an event and get an answer that converts to 1.11, it must be wrong: it is too big.

# Numbers which can be Probabilities

Which of these numbers can be a probability of an event?
$3/2$
$4.6$
$-6/7$
$8/9$
$1.2$
$101\%$

# Numbers which can be Probabilities

Which of these numbers can be a probability of an event?

$$8/9 = \frac{\text{8 outcomes in the event}}{\text{9 outcomes total}}$$

The probability of any event must be a number between 0 and 1.

A number like $3/2$ cannot be a probability. No event can have 3 outcomes in a sample space with only 2.

# Names in a Table

Here is a list of six names:

| | |
|---|---|
| Zoe | Ben |
| Abe | Amy |
| Sue | Max |

If a name is chosen at random, what is the probability that the name starts with A?
Give a fraction as your answer.
Hint: The outcomes are the names. The sample space is a set of six names.

# Names in a Table

Here is a list of six names:

| | |
|---|---|
| Zoe | Ben |
| **Abe** | **Amy** |
| Sue | Max |

If a name is chosen at random, what is the probability that the name starts with A?

There are two names that start with A. Hence the probability of choosing a name that starts with A is:

$$\frac{\text{Two names: Abe and Amy}}{\text{Six names total}} = \boxed{\frac{2}{6}}.$$

# Names in a Table

Here is a list of six names:

| | |
|---|---|
| Zoe | Ben |
| Abe | Amy |
| Sue | Max |

If a name is chosen at random, what is the probability that the name **does not** start with A?

# Names in a Table

Here is a list of six names:

| | |
|---|---|
| **Zoe** | **Ben** |
| Abe | Amy |
| **Sue** | **Max** |

If a name is chosen at random, what is the probability that the name **does not** start with A?
There are four names that don't start with A. Hence the probability of choosing one of them is:

$$\frac{\text{Four names: Zoe, Ben, Sue, Max}}{\text{Six names total}} = \boxed{\frac{4}{6}}.$$

# Names in a Table

| | |
|---|---|
| Zoe | Ben |
| Abe | Amy |
| Sue | Max |

Number of names that start with A: 2
Probability: 2/6
Number of names that don't start with A:

$$4 = 6 - 2 = \text{Total number minus number who do}$$

| | |
|---|---|
| Zoe | Ben |
| Abe | Amy |
| Sue | Max |

Number of names that start with A: 2
Probability: 2/6
Number of names that don't start with A:

$$4 = 6 - 2 = \text{Total number minus number who do}$$

Probability:

$$\frac{4}{6} = \frac{6-2}{6} = \frac{6}{6} - \frac{2}{6}$$

| | |
|---|---|
| Zoe | Ben |
| Abe | Amy |
| Sue | Max |

Number of names that start with A: 2
Probability: 2/6
Number of names that don't start with A:

$$4 = 6 - 2 = \text{Total number minus number who do}$$

Probability:

$$\frac{4}{6} = \frac{6-2}{6} = \frac{6}{6} - \frac{2}{6} = 1 - \frac{2}{6}$$

$1 -$ Probability of picking a name that starts with A

# Complements

- The complement of an event $E$ is the event "$E$ does not happen."
- The notation $\bar{E}$ means the complement of $E$.
- If we know the probability of $E$, $P(E)$, then

$$P(\bar{E}) = 1 - P(E).$$

# Complements Practice

An event $E$ in a sample space with seven objects has probability $4/7$ of occurring. What fraction is $P(\bar{E})$?

# Complements Practice

An event $E$ in a sample space with seven objects has probability $4/7$ of occurring. What fraction is $P(\bar{E})$?

$$P(\bar{E}) = 1 - P(E)$$
$$= 1 - \frac{4}{7}$$

Use common denominators: $1 = 7/7$:

$$P(\bar{E}) = \frac{7}{7} - \frac{4}{7}$$
$$= \frac{7-4}{7} = \boxed{\frac{3}{7}}.$$

# Unions and Intersections (Or and And)

Let $E$ and $F$ be two events in a sample space.

- The event that has outcomes in $E$, **or** in $F$, **or** in both, is called the **union** of $E$ and $F$ and is denoted $E \bigcup F$.
-

# Unions and Intersections (Or and And)

Let $E$ and $F$ be two events in a sample space.

- The event that has outcomes in $E$, **or** in $F$, **or** in both, is called the **union** of $E$ and $F$ and is denoted $E \bigcup F$.
- The event that has outcomes in **both** $E$ **and** $F$, is called the **intersection** of $E$ and $F$ and is denoted $E \bigcap F$.

# Names in a Table

| | |
|---|---|
| Zoe | Ben |
| Abe | Amy |
| Sue | Max |

Let $E$ be the event "The name ends with E," and
$F$ be the event "The name begins with A."

- $E \bigcup F$ is the event "The name ends with E or begins with A."
- 

# Names in a Table

| | |
|---|---|
| Zoe | Ben |
| Abe | Amy |
| Sue | Max |

Let $E$ be the event "The name ends with E," and $F$ be the event "The name begins with A."

- $E \bigcup F$ is the event "The name ends with E or begins with A."
  The outcomes (names) in the union are:

  $$\text{Zoe, Abe, Amy, Sue}$$

-

| | |
|---|---|
| Zoe | Ben |
| Abe | Amy |
| Sue | Max |

Let $E$ be the event "The name ends with E," and
$F$ be the event "The name begins with A."

- $E \bigcup F$ is the event "The name ends with E or begins with A."
  The outcomes (names) in the union are:

  $$\text{Zoe, Abe, Amy, Sue}$$

- Hence $P(E \bigcup F)$ equals

  $$\frac{4 \text{ names in } E \bigcup F}{6 \text{ names total}} = \frac{4}{6}.$$

# Names in a Table

| | |
|---|---|
| Zoe | Ben |
| Abe | Amy |
| Sue | Max |

Let $E$ be the event "The name ends with E," and
$F$ be the event "The name begins with A."

- $E \bigcap F$ is the event "The name ends with E and begins with A."
-

# Names in a Table

| | |
|---|---|
| Zoe | Ben |
| Abe | Amy |
| Sue | Max |

Let $E$ be the event "The name ends with E," and
$F$ be the event "The name begins with A."

- $E \bigcap F$ is the event "The name ends with E and begins with A."
  The outcome in the intersection is:

  Abe

-

| | |
|---|---|
| Zoe | Ben |
| Abe | Amy |
| Sue | Max |

Let $E$ be the event "The name ends with E," and $F$ be the event "The name begins with A."

- $E \bigcap F$ is the event "The name ends with E and begins with A."
  The outcome in the intersection is:

  $$\text{Abe}$$

- Hence $P(E \bigcap F)$ equals

  $$\frac{1 \text{ name in } E \bigcap F}{6 \text{ names total}} = \frac{1}{6}.$$

# Numbers in a Table

Here is a list of 123 beads sorted by color and shape. Suppose we draw a bead.

| | $\triangle$ | $\square$ | Total |
|---|---|---|---|
| Red | 1 | 8 | 9 |
| Green | 64 | 16 | 80 |
| Blue | 2 | 32 | 34 |
| Total | 67 | 56 | 123 |

# Numbers in a Table

Here is a list of 123 beads sorted by color and shape. Suppose we draw a bead.

| | △ | □ | Total |
|---|---|---|---|
| Red | 1 | 8 | 9 |
| Green | 64 | 16 | 80 |
| Blue | 2 | 32 | 34 |
| Total | 67 | 56 | 123 |

Let $E$ be the event "The bead is green," and $F$ be the event "The bead is a square."

# Numbers in a Table

Here is a list of 123 beads sorted by color and shape. Suppose we draw a bead.

| | $\triangle$ | $\square$ | Total |
|---|---|---|---|
| Red | 1 | 8 | 9 |
| Green | 64 | 16 | 80 |
| Blue | 2 | 32 | 34 |
| Total | 67 | 56 | 123 |

Let $E$ be the event "The bead is green," and $F$ be the event "The bead is a square."

$64 + 16 = 80$ beads are green out of 123 total, so

$$P(E) = \boxed{\frac{80}{123}}.$$

# Numbers in a Table (Event F)

Here is a list of 123 beads sorted by color and shape. Suppose we draw a bead.

| | $\triangle$ | $\square$ | Total |
|---|---|---|---|
| Red | 1 | 8 | 9 |
| Green | 64 | 16 | 80 |
| Blue | 2 | 32 | 34 |
| Total | 67 | 56 | 123 |

Let $E$ be the event "The bead is green," and $F$ be the event "The bead is a square."
What is $P(F)$, the probability of drawing a square bead?

# Numbers in a Table (Event F)

Here is a list of 123 beads sorted by color and shape. Suppose we draw a bead.

| | △ | □ | Total |
|---|---|---|---|
| Red | 1 | 8 | 9 |
| Green | 64 | 16 | 80 |
| Blue | 2 | 32 | 34 |
| Total | 67 | 56 | 123 |

Let $E$ be the event "The bead is green," and $F$ be the event "The bead is a square."

There are $8+16+32=56$ square beads out of 123 total, so

$$P(F)=\boxed{\frac{56}{123}}.$$

# Numbers in a Table (Complement)

Here is a list of 123 beads sorted by color and shape. Suppose we draw a bead.

| | $\triangle$ | $\square$ | Total |
|---|---|---|---|
| Red | 1 | 8 | 9 |
| Green | 64 | 16 | 80 |
| Blue | 2 | 32 | 34 |
| Total | 67 | 56 | 123 |

Let $E$ be the event "The bead is green," and $F$ be the event "The bead is a square."
Find $P(\bar{E})$.

# Numbers in a Table (Complement)

| | △ | □ | Total |
|---|---|---|---|
| Red | 1 | 8 | 9 |
| Green | 64 | 16 | 80 |
| Blue | 2 | 32 | 34 |
| Total | 67 | 56 | 123 |

Let $E$ be the event "The bead is green," and $F$ be the event "The bead is a square."
Then $\bar{E}$ is the event "The bead is not green," so it is red or blue:

$$9 + 34 = 43 \text{ beads are not green.}$$

$$P(\bar{E}) = \boxed{\frac{43}{123}}.$$

# Numbers in a Table (Complement)

| | △ | □ | Total |
|---|---|---|---|
| Red | 1 | 8 | 9 |
| Green | 64 | 16 | 80 |
| Blue | 2 | 32 | 34 |
| Total | 67 | 56 | 123 |

Let $E$ be the event "The bead is green," and $F$ be the event "The bead is a square."
Using the complement, we see that 80 beads are green out of 123 total, so

$$P(\bar{E}) = 1 - P(E) = 1 - \frac{80}{123} = \frac{123}{123} - \frac{80}{123} = \boxed{\frac{43}{123}}.$$

# Numbers in a Table (Or)

| | $\triangle$ | $\square$ | Total |
|---|---|---|---|
| Red | 1 | 8 | 9 |
| Green | 64 | 16 | 80 |
| Blue | 2 | 32 | 34 |
| Total | 67 | 56 | 123 |

Let $E$ be the event "The bead is green," and $F$ be the event "The bead is a square."
To find $P(E \bigcup F)$, we need to count the beads which are green, or square, or both.

| | $\triangle$ | $\square$ | Total |
|---|---|---|---|
| Red | 1 | 8 | 9 |
| Green | 64 | 16 | 80 |
| Blue | 2 | 32 | 34 |
| Total | 67 | 56 | 123 |

Let $E$ be the event "The bead is green," and $F$ be the event "The bead is a square."

To find $P(E \bigcup F)$, we need to count the beads which are green, or square, or both.

- The 64 **green** beads, even though they are triangular
-
-
-

| | $\triangle$ | $\square$ | Total |
|---|---|---|---|
| Red | 1 | 8 | 9 |
| Green | 64 | 16 | 80 |
| Blue | 2 | 32 | 34 |
| Total | 67 | 56 | 123 |

Let $E$ be the event "The bead is green," and $F$ be the event "The bead is a square."
To find $P(E \bigcup F)$, we need to count the beads which are green, or square, or both.

- The 64 **green** beads, even though they are triangular
- The 8 square beads, even though they are red

| | △ | □ | Total |
|---|---|---|---|
| Red | 1 | 8 | 9 |
| Green | 64 | 16 | 80 |
| Blue | 2 | 32 | 34 |
| Total | 67 | 56 | 123 |

Let $E$ be the event "The bead is green," and $F$ be the event "The bead is a square."
To find $P(E \bigcup F)$, we need to count the beads which are green, or square, or both.

- The 64 **green** beads, even though they are triangular
- The 8 square beads, even though they are red
- The 32 square beads, even though they are blue
-

| | △ | □ | Total |
|---|---|---|---|
| Red | 1 | 8 | 9 |
| Green | 64 | 16 | 80 |
| Blue | 2 | 32 | 34 |
| Total | 67 | 56 | 123 |

Let $E$ be the event "The bead is green," and $F$ be the event "The bead is a square."
To find $P(E \bigcup F)$, we need to count the beads which are green, or square, or both.

- The 64 **green** beads, even though they are triangular
- The 8 square beads, even though they are red
- The 32 square beads, even though they are blue
- The 16 **green** and *square* beads.

# Numbers in a Table

| | △ | □ | Total |
|---|---|---|---|
| Red | 1 | 8 | 9 |
| Green | 64 | 16 | 80 |
| Blue | 2 | 32 | 34 |
| Total | 67 | 56 | 123 |

Let $E$ be the event "The bead is green," and $F$ be the event "The bead is a square."
To find $P(E \bigcup F)$, we need to count the beads which are green, or square, or both.

$$64 + 16 + 8 + 32 = 120 \text{ beads in } E \bigcup F.$$

Hence

$$P(E \bigcup F) = \frac{120}{123}.$$

# Numbers in a Table (And)

| | △ | □ | Total |
|---|---|---|---|
| Red | 1 | 8 | 9 |
| Green | 64 | 16 | 80 |
| Blue | 2 | 32 | 34 |
| Total | 67 | 56 | 123 |

Let $E$ be the event "The bead is green," and $F$ be the event "The bead is a square."
Find $P(E \bigcap F)$.

# Numbers in a Table (And)

| | △ | □ | Total |
|---|---|---|---|
| Red | 1 | 8 | 9 |
| Green | 64 | **16** | 80 |
| Blue | 2 | 32 | 34 |
| Total | 67 | 56 | 123 |

Let $E$ be the event "The bead is green," and $F$ be the event "The bead is a square."
The event $E \bigcap F$ is "The bead is green **and** square:"

$$16 \text{ beads in } E \bigcap F.$$

Hence

$$P(E \bigcap F) = \boxed{\frac{16}{123}}.$$

# Unions and Intersections Formula

The probability of event $E$ or event $F$ occurring (or both) is

$$P(E \bigcup F) = P(E) + P(F) - P(E \bigcap F).$$

This formula is useful for checking your answers, and if you do not know what outcomes are in the events.

# Unions and Intersections Formula

| | △ | □ | Total |
|---|---|---|---|
| Red | 1 | 8 | 9 |
| Green | 64 | 16 | 80 |
| Blue | 2 | 32 | 34 |
| Total | 67 | 56 | 123 |

Let $E$ be the event "The bead is green," and $F$ be the event "The bead is a square."

The event $E \bigcup F$ is "The bead is green **or** square." The event $E \bigcap F$ is "The bead is green **and** square."

$$P(E) = \frac{80}{123}$$
$$P(F) = \frac{56}{123}$$
$$P(E \bigcap F) = \frac{16}{123}$$

# Unions and Intersections Formula

$$P(E) = \frac{80}{123}$$
$$P(F) = \frac{56}{123}$$
$$P(E \bigcap F) = \frac{16}{123}$$

Thus

$$\begin{aligned} P(E \bigcup F) &= P(E) + P(F) - P(E \bigcap F) \\ &= \frac{80}{123} + \frac{56}{123} - \frac{16}{123} \\ &= \frac{80+56-16}{123} = \boxed{\frac{120}{123}} \end{aligned}$$

$$
\begin{aligned}
P(E \bigcup F) &= P(E) + P(F) - P(E \bigcap F) \\
&= \frac{80}{123} + \frac{56}{123} - \frac{16}{123} \\
&= \frac{80 + 56 - 16}{123} = \boxed{\frac{120}{123}}
\end{aligned}
$$

If we just add the number of green beads and the number of square beads,

$$80 + 56 = 136,$$

we get an answer that is too big. It double counts the 16 beads that are green and square, so that is why the formula subtracts the probability of the intersection. In the numerator, it subtracts the number of beads that are green and square:

$$136 - 16 = 120,$$

the correct numerator and the actual number of beads that are green and square.

# Discrete Probability

At this point, we have been doing probability on discrete sets: sets of separate, individual objects like marbles, people, or whole numbers.

For example, if we have these 7 marbles:

then the probability of drawing a green marble is

$$\frac{1 \text{ green marble}}{7 \text{ marbles total}} = \frac{1}{7},$$

At this point, we have been doing probability on discrete sets: sets of separate, individual objects like marbles, people, or whole numbers.

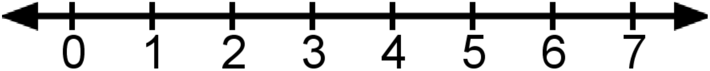
For example, if we have these 7 marbles:

then the probability of drawing a green marble is

$$\frac{1 \text{ green marble}}{7 \text{ marbles total}} = \frac{1}{7},$$

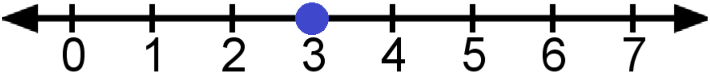
the **ratio** of the *amount* of outcomes (marbles) in the event "we draw a green marble" over the *total amount* of outcomes. The green marbles take up 1/7th of the total amount of marbles.

# Long Jump




An athlete is standing at the 0 feet mark. They perform a long jump, and land on a random location between the 0 and 7 feet mark.

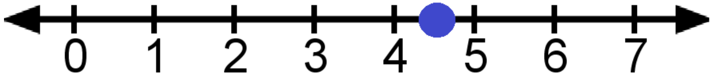
# Long Jump




An athlete is standing at the 0 feet mark. They perform a long jump, and land on a random location between the 0 and 7 feet mark.

- They could land right on a whole number like 3.

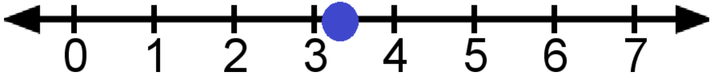
# Long Jump




An athlete is standing at the 0 feet mark. They perform a long jump, and land on a random location between the 0 and 7 feet mark.

- But they might land somewhere in between the whole number marks, such as at a fraction (rational number) like $9/2 = 4.5$.

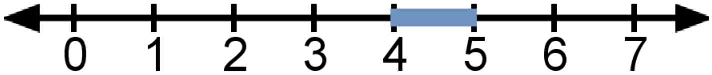
# Long Jump




An athlete is standing at the 0 feet mark. They perform a long jump, and land on a random location between the 0 and 7 feet mark.

- But they might land somewhere in between the whole number marks, such as at a fraction (rational number) like $9/2 = 4.5$.
- They could even land in between rational numbers on an irrational number like $\pi \approx 3.14...$!
- Even between 0 and 7, there are an infinite amount of possible landing spots.

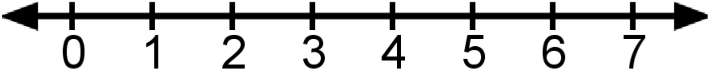
# Long Jump




An athlete is standing at the 0 feet mark. They perform a long jump, and land on a random location between the 0 and 7 feet mark.

What is the probability that the athlete lands between the 4 and 5 feet marks?

# Picking a Random Real Number




This problem boils down to picking a random real number between 0 and 7 and finding the probability of picking a number between 4 and 5. Unlike earlier problems, our sample space, the set of real numbers between 0 and 7, not only has an infinite amount of outcomes, but these outcomes lie in **a continuous range**, a line, instead of being separate objects.

# Intervals

Let $c$ and $d$ be any real numbers with $c$ less than (left of) $d$.

- An *interval* between $c$ and $d$ is a set of *all real numbers between* $c$ *and* $d$.
  A number is inside an interval if it is between $c$ and $d$.
-
-
-
-
-

# Intervals

Let $c$ and $d$ be any real numbers with $c$ less than (left of) $d$.

- An *interval* between $c$ and $d$ is a set of *all real numbers between c and d*.
  A number is inside an interval if it is between $c$ and $d$.
- The numbers $c$ and $d$ are the *endpoints* of the interval. *They may or may not be included in the interval.*
- $[c, d]$: include $c$ and $d$
-
-
-

# Intervals

Let $c$ and $d$ be any real numbers with $c$ less than (left of) $d$.

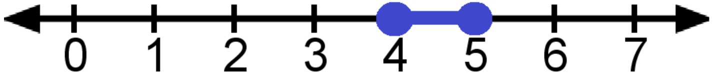
- An *interval* between $c$ and $d$ is a set of *all real numbers between $c$ and $d$.*
  A number is inside an interval if it is between $c$ and $d$.
- The numbers $c$ and $d$ are the *endpoints* of the interval. *They may or may not be included in the interval.*
- $[c, d]$: include $c$ and $d$
- $[c, d)$: include $c$ only
- $(c, d]$: include $d$ only
- $(c, d)$: don't include endpoints

# Drawing Intervals

Use **dots** to indicate endpoints:

- Use a **closed dot** to **include** the endpoint.
- Use an **open dot** to **not include** the endpoint.
- Then draw a **line between dots** to include the **numbers between the endpoints**.

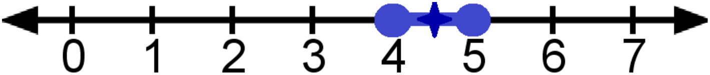
# Interval Example 1




This interval has endpoints 4 and 5.
It includes 4, and 5, so it is written as

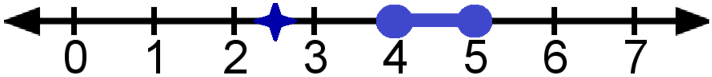
$$[4, 5].$$

# Interval Example 1




This interval is $[4, 5]$.
For example, the real number 4.68 is in this interval, because 4.68 is between 4 and 5.

# Interval Example 1

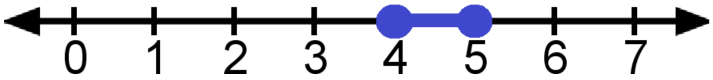



This interval is $[4, 5]$.
For example, the real number 2.5 is not in this interval, because it is to the left of 4.

# Interval Example 1

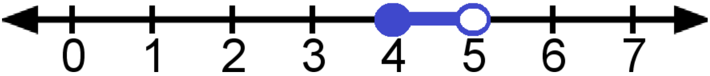

This interval is $[4, 5]$.
For example, the real number 5.7 is not in this interval, because it is to the right of 5.

# Interval Example 1




This interval has endpoints 4 and 5.
Some other numbers inside this interval are 4, 5, and also 4.68, $9/2 = 4.5$, and $\sqrt{17} \approx 4.123$.
There are infinitely many real numbers in this interval.

# Interval Example 2




This interval has endpoints 4 and 5.
It includes 4, but not 5, so it is written as

$$[4,5).$$

Some numbers inside this interval are 4, and also 4.68, $9/2 = 4.5$, and $\sqrt{17} \approx 4.123$.
There are infinitely many real numbers in this interval.

# Picking a Random Real Number

When picking a random real number, our sample spaces and events will be intervals.

- 
- 

# Picking a Random Real Number

When picking a random real number, our sample spaces and events will be intervals.

- Back in discrete probability, we *found the sizes of our events and sample spaces by counting the amount of outcomes* in them.
- 

# Picking a Random Real Number

When picking a random real number, our sample spaces and events will be intervals.

- Back in discrete probability, we *found the sizes of our events and sample spaces by counting the amount of outcomes* in them.
- Since we cannot count every real number in an interval, we need a way of measuring the size of an interval.

# Length of an Interval

The **length** of an interval with endpoints $c$ and $d$, $c \leq d$, is

$$d - c.$$

**Right Endpoint** minus **Left Endpoint**

The length should be a nonnegative number.

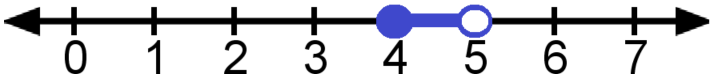
# Length Examples


The interval $[4, 5]$ has length

$$5 - 4 = \boxed{1}.$$

The interval is one unit long.

# Length Examples




The interval $[4,5)$ has length

$$5 - 4 = \boxed{1}.$$

The interval is one unit long.
Removing an endpoint is like shaving a small bit off the end of a stick, and the bit is so tiny that the remaining length is the same.

# Length Practice

Find the length of the following intervals:

a) $[2, 8]$

b) $[23, 91]$

c) $[12.6,\ 57.6]$

d) $\left[\frac{62}{3}, \frac{104}{3}\right]$

# Length Practice

Find the length of the following intervals:

a) $[2, 8]$

$$8 - 2 = \boxed{6}$$

b) $[23, 91]$

$$91 - 23 = \boxed{58}$$

c) $[12.6,\ 57.6]$

$$57.6 - 12.6 = \boxed{45}$$

d) $\left[\frac{62}{3}, \frac{104}{3}\right]$

$$\frac{104}{3} - \frac{62}{3} = \frac{104 - 62}{3} = \frac{42}{3} = \boxed{14}$$

# Picking a Random Real Number

When picking a random real number, our sample spaces and events will be intervals.
The probability of any event is

$$\frac{\text{Length of the event interval}}{\text{Length of the sample space interval}}$$

Throughout the rest of this chapter, all intervals will include endpoints.

# Probability of an Event Interval


We are picking a random real number between 0 and 7. What is the probability that the chosen number is between 4 and 5?

-
-
-

# Probability of an Event Interval


We are picking a random real number between 0 and 7. What is the probability that the chosen number is between 4 and 5?

- Sample space: the interval $[0,7]$.
- Event: (we pick a real number inside) the interval $[4,5]$.
-


We are picking a random real number between 0 and 7. What is the probability that the chosen number is between 4 and 5?

- Sample space: the interval $[0, 7]$.
- Event: (we pick a real number inside) the interval $[4, 5]$.
- Probability:

$$\frac{\text{Length of } [4,5]}{\text{Length of } [0,7]} = \frac{5-4}{7-0} = \frac{1}{7}.$$

Notice that the event interval takes up 1/7th of the total length of the sample space.

# Probability of an Event Interval

We are picking a random real number between 23 and 91. What is the probability that the chosen number is between 34 and 47?

# Probability of an Event Interval

We are picking a random real number between 23 and 91. What is the probability that the chosen number is between 34 and 47?

- Sample space: the interval $[23, 91]$.
- Event: (we pick a real number inside) the interval $[34, 47]$.
- Probability:

$$\frac{\text{Length of } [34, 47]}{\text{Length of } [23, 91]} = \frac{47 - 34}{91 - 23} = \frac{13}{68}.$$

# Comparing Discrete and Continuous Probability

- Discrete probability: used when the sample space is a set of separate, individual objects
- Continuous probability: used when the sample space is an interval of real numbers (we are picking a random real number)

# Comparing Discrete and Continuous Probability

Let $\Omega$ be a sample space and $E$ be an event.

- Discrete probability: The probability of $E$ is:

  Number of objects in $E$

  divided by...

  Number of objects in $\Omega$

- Continuous probability: The probability of an interval event $E$ is:

  Length of $E$

  Length of $E$, or sum of lengths of each piece for unions,

  divided by...

  Length of $\Omega$

# Unions and Intersections of Intervals

Let $\Omega$ be an interval of real numbers, and let $E$ and $F$ be event intervals in $\Omega$.

- A real number outcome is in $E$ if it is between the endpoints of $E$.
-
    -
-
    -

# Unions and Intersections of Intervals

Let $\Omega$ be an interval of real numbers, and let $E$ and $F$ be event intervals in $\Omega$.

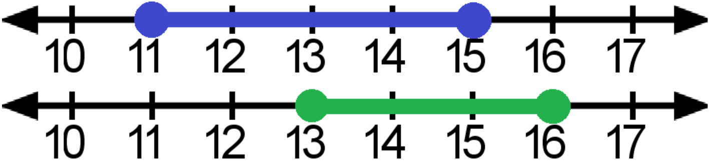
- A real number outcome is in $E$ if it is between the endpoints of $E$.
- The event $E \bigcup F$ is the **union** of $E$ and $F$. It is the set of real number outcomes in $E$ **or** in $F$ (or in both).
  - The union could be a larger interval, or two separate intervals, depending on whether $E$ and $F$ overlap.
-
  -

# Unions and Intersections of Intervals

Let $\Omega$ be an interval of real numbers, and let $E$ and $F$ be event intervals in $\Omega$.

- A real number outcome is in $E$ if it is between the endpoints of $E$.
- The event $E \bigcup F$ is the **union** of $E$ and $F$. It is the set of real number outcomes in $E$ **or** in $F$ (or in both).
  - The union could be a larger interval, or two separate intervals, depending on whether $E$ and $F$ overlap.
- The event $E \bigcap F$ is the **intersection** of $E$ and $F$. It is the set of real number outcomes in $E$, **and** in $F$.
  - If $E$ and $F$ overlap, then the intersection is the interval formed by the overlap.

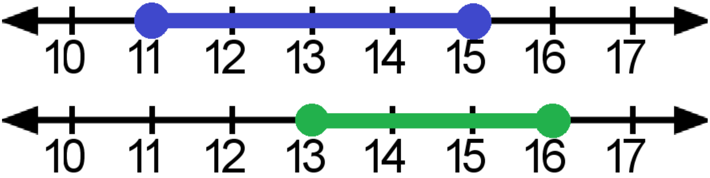
# Union of Two Overlapping Intervals




Let $\Omega = [10, 17]$, $E = [11, 15]$ and $F = [13, 16]$. The event $E \bigcup F$ consists of real numbers between 11 and 15, **or** between 13 and 16:

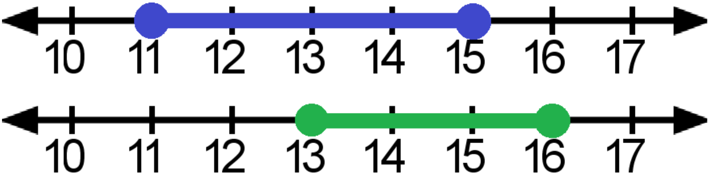
- The real number 12 is in $E \bigcup F$, because 12 is in $E$ (between 11 and 15).
- The real number 15.5 is in $E \bigcup F$, because 15.5 is in $F$ (between 13 and 16).
- The real number 14 is in $E \bigcup F$, because 14 is in $E$ and in $F$ (at least one of them).
- The real number 17 is not in $E \bigcup F$, because 17 is not in $E$ nor in $F$.

# Union of Two Overlapping Intervals




Let $\Omega = [10, 17]$, $E = [11, 15]$ and $F = [13, 16]$. The event $E \bigcup F$ consists of real numbers between 11 and 15, **or** between 13 and 16:

# Union of Two Overlapping Intervals




Let $\Omega = [10, 17]$, $E = [11, 15]$ and $F = [13, 16]$. The event $E \bigcup F$ consists of real numbers between 11 and 15, **or** between 13 and 16:

# Union of Two Overlapping Intervals

Let $\Omega = [10, 17]$, $E = [11, 15]$ and $F = [13, 16]$. The event $E \bigcup F$ consists of real numbers between 11 and 15, **or** between 13 and 16:

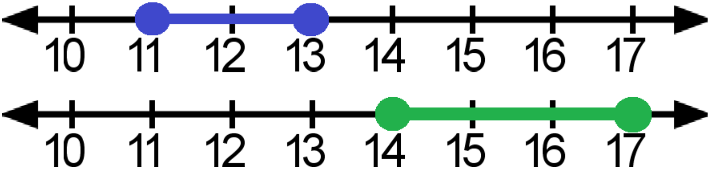

Thus, $E \bigcup F$ is actually the interval $[11, 16]$. Its length is

$$16 - 11 = 5.$$

Thus,

$$P(E \bigcup F) = \frac{\textit{Length of } E \bigcup F}{\textit{Length of } \Omega} = \frac{16 - 11}{17 - 10} = \boxed{\frac{5}{7}}$$

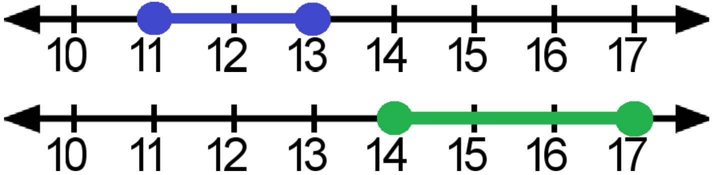
# Union of Two Separate Intervals




Let $\Omega = [10, 17]$, $E = [11, 13]$ and $F = [14, 17]$. The event $E \bigcup F$ consists of real numbers between 11 and 13, **or** between 14 and 17:

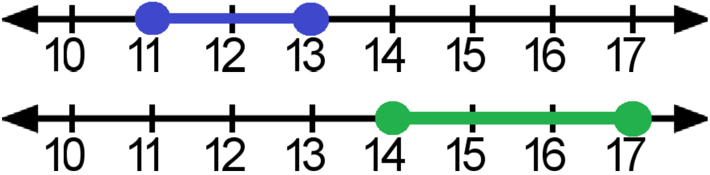
- The real number 12 is in $E \bigcup F$, because 12 is in $E$.
- The real number 15.5 is in $E \bigcup F$, because 15.5 is in $F$.
- The real number 13.5 is not in $E \bigcup F$, because 13.5 is not in $E$ nor in $F$.

# Union of Two Separate Intervals




Let $\Omega = [10, 17]$, $E = [11, 13]$ and $F = [14, 17]$. The event $E \bigcup F$ consists of real numbers between 11 and 13, **or** between 14 and 17:

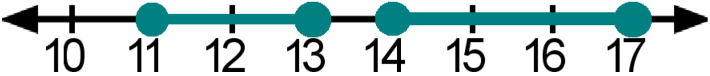
# Union of Two Separate Intervals




Let $\Omega = [10, 17]$, $E = [11, 13]$ and $F = [14, 17]$. The event $E \bigcup F$ consists of real numbers between 11 and 13, **or** between 14 and 17:

# Union of Two Separate Intervals

Let $\Omega = [10, 17]$, $E = [11, 13]$ and $F = [14, 17]$. The event $E \bigcup F$ consists of real numbers between 11 and 13, **or** between 14 and 17:




Thus, $E \bigcup F$ is actually two intervals. It has a total length found by adding (combining) the lengths of $E$ and $F$:

$$\text{Length of } E: \; 3 - 1 = 2$$

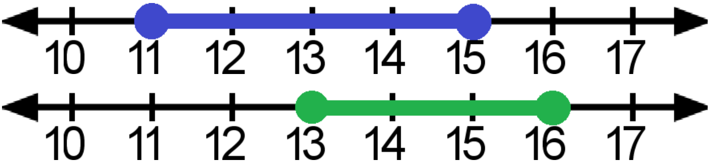
$$\text{Length of } F: \; 7 - 4 = 3$$

$$\text{Total length of } E \bigcup F: \; 2 + 3 = 5.$$

Hence

$$P(E \bigcup F) = \frac{\textit{Total length of } E \bigcup F}{\textit{Length of } \Omega} = \boxed{\frac{5}{7}}.$$

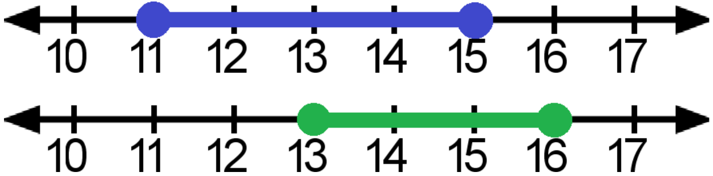
## Intersection of Two Intervals




Let $\Omega = [10, 17]$, $E = [11, 15]$ and $F = [13, 16]$. The event $E \bigcap F$ consists of real numbers between 11 and 15, **and** between 13 and 16:

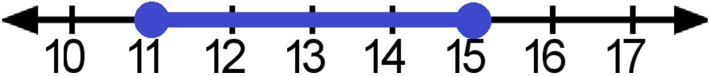
- The real number 12 is not in $E \bigcap F$, because 12 is in $E$, but not in $F$.
- The real number 15.5 is not in $E \bigcap F$, because 15.5 is in $F$, but not in $E$.
- The real number 14 is in $E \bigcap F$, because 14 is in $E$ and in $F$ (both of them).

# Intersection of Two Intervals




Let $\Omega = [10, 17]$, $E = [11, 15]$ and $F = [13, 16]$. The event $E \bigcap F$ consists of real numbers between 11 and 15, **and** between 13 and 16:

# Intersection of Two Intervals





Let $\Omega = [10, 17]$, $E = [11, 15]$ and $F = [13, 16]$. The event $E \bigcap F$ consists of real numbers between 11 and 15, **and** between 13 and 16:

# Intersection of Two Intervals

Let $\Omega = [10,17]$, $E = [11,15]$ and $F = [13,16]$. The event $E \bigcap F$ consists of real numbers between 11 and 15, **and** between 13 and 16:


Thus, $E \bigcap F$ is actually the interval $[13,15]$. Its length is

$$5 - 3 = 2.$$

Hence

$$P(E \bigcap F) = \frac{Length\ of\ \ E \bigcap F}{Length\ of\ \ \Omega} = \boxed{\frac{2}{7}}.$$

## ?(4.1) Union/Intersection Practice 1

Let $\Omega$ be the interval $[8,24]$, $E$ be the interval $[12,16]$, and $F$ be the interval $[15,19]$:


If we pick a random real number between 8 and 24, find the **probability** of the event $E \bigcup F$.

Hints:

1. Identify the event $[12,16] \bigcup [15,19]$ as an interval.
2. What is the length of the union?
3. What is the length of the sample space?
4. Find $P([12,16] \bigcup [15,19])$. Type and send a fraction.

# Union/Intersection Practice 1

Let $\Omega$ be the interval $[8, 24]$, $E$ be the interval $[12, 16]$, and $F$ be the interval $[15, 19]$:


If we pick a random real number between 8 and 24, find the **probability** of the event $E \bigcup F$.

Hints:

1. Identify the event $[12, 16] \bigcup [15, 19]$ as an interval.
2. What is the length of the union?
3. What is the length of the sample space?
4. Find $P([12, 16] \bigcup [15, 19])$. Type and send a fraction.

# Union/Intersection Practice 1

Let $\Omega$ be the interval $[8, 24]$, $E$ be the interval $[12, 16]$, and $F$ be the interval $[15, 19]$:


If we pick a random real number between 8 and 24, find the **probability** of the event $E \bigcup F$.

$$E \bigcup F = [12, 19].$$

$$P(E \bigcup F) = \frac{\text{Length of } E \bigcup F}{\text{Length of } \Omega} = \frac{19 - 12}{24 - 8} = \boxed{\frac{7}{16}}$$

## ?(4.2) Union/Intersection Practice 2

Let $\Omega$ be the interval $[24, 47]$, $E$ be the interval $[29, 34]$, and $F$ be the interval $[36, 43]$:

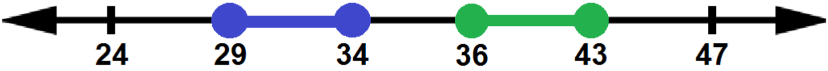

If we pick a random real number between 24 and 47, find the **probability** of the event $E \bigcup F$.

Hints:

1. Identify the event $[29, 34] \bigcup [36, 43]$ as a union of two separate intervals.
2. What is the total length of the union?
3. What is the length of the sample space?
4. Find $P([29, 34] \bigcup [36, 43])$. Type and send a fraction.

# Union/Intersection Practice 2

Let $\Omega$ be the interval $[24, 47]$, $E$ be the interval $[29, 34]$, and $F$ be the interval $[36, 43]$:



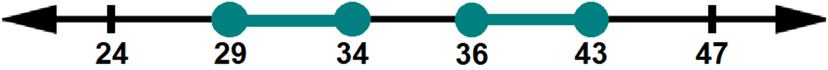

If we pick a random real number between 24 and 47, find the **probability** of the event $E \bigcup F$.

Hints:

1. Identify the event $[29, 34] \bigcup [36, 43]$ as a union of two separate intervals.
2. What is the total length of the union?
3. What is the length of the sample space?
4. Find $P([29, 34] \bigcup [36, 43])$. Type and send a fraction.

# Union/Intersection Practice 2

Let $\Omega$ be the interval $[24, 47]$, $E$ be the interval $[29, 34]$, and $F$ be the interval $[36, 43]$:




If we pick a random real number between 24 and 47, find the **probability** of the event $E \bigcup F$.

$$E \bigcup F = [29, 34] \bigcup [36, 43]$$

$$\begin{aligned} P(E \bigcup F) &= \frac{\text{Total Length of } E \bigcup F}{\text{Length of } \Omega} \\ &= \frac{(34 - 29) + (43 - 36)}{47 - 24} \\ &= \frac{5 + 7}{23} = \boxed{\frac{12}{23}} \end{aligned}$$

# ?(4.3) Union/Intersection Practice 3

Let $\Omega$ be the interval $[47, 78]$, $E$ be the interval $[51, 60]$, and $F$ be the interval $[54, 63]$:


If we pick a random real number between 47 and 78, find the **probability** of the event $E \bigcap F$.

Hints:

1. Identify the event $[51, 60] \bigcap [54, 63]$ as a smaller interval.
2. What is the length of the intersection?
3. What is the length of the sample space?
4. Find $P([51, 60] \bigcap [54, 63])$. Type and send a fraction.

# Union/Intersection Practice 3

Let $\Omega$ be the interval $[47, 78]$, $E$ be the interval $[51, 60]$, and $F$ be the interval $[54, 63]$:


If we pick a random real number between 47 and 78, find the **probability** of the event $E \bigcap F$.

Hints:

1. Identify the event $[51, 60] \bigcap [54, 63]$ as a smaller interval.
2. What is the length of the intersection?
3. What is the length of the sample space?
4. Find $P([51, 60] \bigcap [54, 63])$. Type and send a fraction.

# Union/Intersection Practice 3

Let $\Omega$ be the interval $[47, 78]$, $E$ be the interval $[51, 60]$, and $F$ be the interval $[54, 63]$:


If we pick a random real number between 47 and 78, find the **probability** of the event $E \bigcap F$.

$$E \bigcap F = [54, 60].$$

$$P(E \bigcap F) = \frac{\text{Length of } E \bigcap F}{\text{Length of } \Omega} = \frac{60 - 54}{78 - 47} = \boxed{\frac{6}{31}}$$

# Back to Discrete Probability

Let $\Omega$ be a sample space and $E$ be an event. The probability of $E$ is:

Number of outcomes (objects) in $E$

divided by...

Number of outcomes in $\Omega$

# Cards with Suits and Ranks

Each card in a deck can be identified by a **suit** (symbol) *and* **rank** (number). Suppose there are $n$ suits and $m$ ranks, and each suit has $m$ cards of each rank, and each rank has $n$ cards of each suit. Then there are

$$n \cdot m \text{ cards total.}$$

# Cards with Suits and Ranks

Each card in a deck can be identified by a **suit** (symbol) *and* **rank** (number). Suppose there are $n$ suits and $m$ ranks, and each suit has $m$ cards of each rank, and each rank has $n$ cards of each suit. Then there are

$$n \cdot m \text{ cards total.}$$

Example: the standard 52-card deck has 4 suits (Clubs, Diamonds, Hearts, and Spades) and 13 ranks (Ace, 2-10, Jack, Queen, King). There are 13 Club cards (one of each rank), and there are 4 Aces (one of each suit).

# Notation

From now on, suits will be labeled with letters (starting with A), and ranks will be labeled with numbers exclusively. We may also work with made-up decks.

# Drawing a Certain Card

A deck of 15 cards has 3 suits (A-C) and 5 ranks (1-5).

| Suit \ Rank | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|
| A | A1 | A2 | A3 | A4 | A5 |
| B | B1 | **B2** | B3 | B4 | B5 |
| C | C1 | C2 | C3 | C4 | C5 |

Notice the table helps us see why the total number of cards is the number of suits times the number of ranks.
The probability of drawing card B2 is:

$$\frac{\text{1 desired card, B2}}{\text{15 cards total}} = \frac{1}{15}.$$

# Drawing a Card with a Desired Suit

A deck of 15 cards has 3 suits (A-C) and 5 ranks (1-5).

| Suit \ Rank | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|
| A | A1 | A2 | A3 | A4 | A5 |
| **B** | **B1** | **B2** | **B3** | **B4** | **B5** |
| C | C1 | C2 | C3 | C4 | C5 |

The probability of drawing a card with suit B is:

$$\frac{5 \text{ desired cards}}{15 \text{ cards total}} = \frac{5}{15} = \frac{1}{3},$$

# Drawing a Card with a Desired Suit

A deck of 15 cards has 3 suits (A-C) and 5 ranks (1-5).

| Suit \ Rank | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|
| A | A1 | A2 | A3 | A4 | A5 |
| **B** | **B1** | **B2** | **B3** | **B4** | **B5** |
| C | C1 | C2 | C3 | C4 | C5 |

The probability of drawing a card with suit B is:

$$\frac{5 \text{ desired cards}}{15 \text{ cards total}} = \frac{5}{15} = \frac{1}{3},$$

but if we think of picking the suit (letter) B from a sample space of 3 suits, we get:

$$\frac{1 \text{ desired suit, B}}{3 \text{ suits total}} = \frac{1}{3} \text{ as well.}$$

# Drawing a Card with Even Rank

A deck of 15 cards has 3 suits (A-C) and 5 ranks (1-5).

| Suit \ Rank | 1 | **2** | 3 | **4** | 5 |
|---|---|---|---|---|---|
| A | A1 | **A2** | A3 | **A4** | A5 |
| B | B1 | **B2** | B3 | **B4** | B5 |
| C | C1 | **C2** | C3 | **C4** | C5 |

The probability of drawing a card with even rank is:

$$\frac{6 \text{ desired cards}}{15 \text{ cards total}} = \frac{6}{15} = \frac{2}{5},$$

# Drawing a Card with Even Rank

A deck of 15 cards has 3 suits (A-C) and 5 ranks (1-5).

| Suit \ Rank | 1 | **2** | 3 | **4** | 5 |
|---|---|---|---|---|---|
| A | A1 | **A2** | A3 | **A4** | A5 |
| B | B1 | **B2** | B3 | **B4** | B5 |
| C | C1 | **C2** | C3 | **C4** | C5 |

The probability of drawing a card with even rank is:

$$\frac{\text{6 desired cards}}{\text{15 cards total}} = \frac{6}{15} = \frac{2}{5},$$

but if we think of picking the the ranks (numbers) 2 or 4 from a sample space of 5 ranks, we get:

$$\frac{\text{2 desired ranks, 2 or 4}}{\text{5 ranks total}} = \frac{2}{5} \text{ as well.}$$

# Drawing a Card with a Desired Suit

A deck of cards has 3 suits (A-C) and 5 ranks (1-5), but cards A2 and B4 are missing! There are only 13 cards now.

| Suit \ Rank | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|
| A | A1 | | A3 | A4 | A5 |
| B | B1 | B2 | B3 | | B5 |
| C | C1 | C2 | C3 | C4 | C5 |

Now what is the probability of drawing a card with suit B?

# Drawing a Card with a Desired Suit

A deck of cards has 3 suits (A-C) and 5 ranks (1-5), but cards A2 and B4 are missing! There are only 13 cards now.

| Suit \ Rank | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|
| A | A1 | | A3 | A4 | A5 |
| **B** | **B1** | **B2** | **B3** | | **B5** |
| C | C1 | C2 | C3 | C4 | C5 |

Now what is the probability of drawing a card with suit B?

$$\frac{4 \text{ desired cards}}{13 \text{ cards total}} = \frac{4}{13}.$$

This is not the same answer as earlier. The numerator is different because there are fewer cards with rank B (due to B4 being gone), and the denominator changed as well due to the missing cards.

# Suits, Ranks, and Missing Cards

- In the examples with no cards missing, the proportion of desired cards to total cards equaled the proportion of desired suits/ranks to total suits/ranks. Each suit/rank had the same number of cards as well.
-

# Suits, Ranks, and Missing Cards

- In the examples with no cards missing, the proportion of desired cards to total cards equaled the proportion of desired suits/ranks to total suits/ranks. Each suit/rank had the same number of cards as well.
- However, in the example with the missing cards, the proportion of desired cards to total cards did not equal the proportion of desired suits to total suits. The suits had different numbers of cards (suit B had 4 cards left while suit C still had 5 cards).

# Drawing a Card (Intersection)

A deck of 15 cards has 3 suits (A-C) and 5 ranks (1-5).

| Suit \ Rank | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|
| A | A1 | A2 | A3 | A4 | A5 |
| B | B1 | B2 | B3 | B4 | B5 |
| C | C1 | C2 | C3 | C4 | C5 |

What is the probability of drawing a card with suit B and even rank?

# Drawing a Card (Intersection)

A deck of 15 cards has 3 suits (A-C) and 5 ranks (1-5).

| Suit \ Rank | 1 | *2* | 3 | *4* | 5 |
|---|---|---|---|---|---|
| A | A1 | *A2* | A3 | *A4* | A5 |
| B | *B1* | **B2** | *B3* | **B4** | *B5* |
| C | C1 | *C2* | C3 | *C4* | C5 |

What is the probability of drawing a card with suit B and even rank?

Among the 15 cards, two of them, B2 and B4, have rank B and even (2 or 4) rank.

$$\frac{\text{2 desired cards}}{\text{15 cards total}} = \frac{2}{15}.$$

# Drawing a Card (Intersection)

Notice the probability of choosing B from 3 suits and the probability of choosing an even number between 1 and 5 is

$$\frac{\text{1 desired suit}}{\text{3 suits total}} = \frac{1}{3}, \quad \frac{\text{2 desired ranks}}{\text{5 ranks total}} = \frac{2}{5},$$

and if these are multiplied, we get

$$\frac{1}{3} \cdot \frac{2}{5} = \frac{2}{15}, \text{ our earlier answer.}$$

# Drawing a Card (Intersection)

Notice the probability of choosing B from 3 suits and the probability of choosing an even number between 1 and 5 is

$$\frac{1 \text{ desired suit}}{3 \text{ suits total}} = \frac{1}{3}, \quad \frac{2 \text{ desired ranks}}{5 \text{ ranks total}} = \frac{2}{5},$$

and if these are multiplied, we get

$$\frac{1}{3} \cdot \frac{2}{5} = \frac{2}{15}, \text{ our earlier answer.}$$

If multiplying the numbers of suits and ranks gives the number of cards, perhaps probabilities for suits and ranks can multiply to get probabilities for cards?

# Drawing a Card (Intersection) with Missing Cards

A deck of 15 cards has 3 suits (A-C) and 5 ranks (1-5), but cards A2 and B4 are missing!

| Suit \ Rank | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|
| A | A1 | | A3 | A4 | A5 |
| B | B1 | B2 | B3 | | B5 |
| C | C1 | C2 | C3 | C4 | C5 |

What is the probability of drawing a card with suit B and even rank?

# Drawing a Card (Intersection) with Missing Cards

A deck of 15 cards has 3 suits (A-C) and 5 ranks (1-5), but cards A2 and B4 are missing!

| Suit \ Rank | 1 | *2* | 3 | *4* | 5 |
|---|---|---|---|---|---|
| A | A1 | | A3 | *A4* | A5 |
| *B* | *B1* | **B2** | *B3* | | *B5* |
| C | C1 | *C2* | C3 | *C4* | C5 |

What is the probability of drawing a card with suit B and even rank?

Among the 13 cards, one of them, B2, has rank B and even (2 or 4) rank.

$$\frac{\text{1 desired card}}{\text{13 cards total}} = \frac{1}{13}.$$

# Drawing a Card (Union)

A deck of 15 cards has 3 suits (A-C) and 5 ranks (1-5).

| Suit \ Rank | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|
| A | A1 | A2 | A3 | A4 | A5 |
| B | B1 | B2 | B3 | B4 | B5 |
| C | C1 | C2 | C3 | C4 | C5 |

What is the probability of drawing a card with suit B or even rank?

# Drawing a Card (Union)

A deck of 15 cards has 3 suits (A-C) and 5 ranks (1-5).

| Suit \ Rank | 1 | **2** | 3 | **4** | 5 |
|---|---|---|---|---|---|
| A | A1 | **A2** | A3 | **A4** | A5 |
| **B** | **B1** | **B2** | **B3** | **B4** | **B5** |
| C | C1 | **C2** | C3 | **C4** | C5 |

What is the probability of drawing a card with suit B or even rank?

A total of 9 cards have suit B, or even rank (or both):

$$\frac{9 \text{ desired cards}}{15 \text{ cards total}} = \frac{9}{15}.$$

# Drawing a Card (Union)

If we use the formula for unions and intersections, we get:

$$\begin{aligned} & P(\text{Card has suit B}) & & 5/15 \\ & +P(\text{Card has even rank}) & & +6/15 \\ & -P(\text{Card has suit B and even rank}) & & -2/15 \\ & & = \quad & 9/15. \end{aligned}$$

If we did not subtract the probability of the intersection, we would get an answer that is too large, because it counts cards B2 and B4 twice.

# Drawing from a Big Deck

A deck of cards has 11 suits (A-K) and 25 ranks (1-25), with no missing cards. What is the probability of drawing a card whose suit is a vowel (A, E, I, O, or U)?
Hint: use the suits as the sample space since no cards are missing.

# Drawing from a Big Deck

A deck of cards has 11 suits (A-K) and 25 ranks (1-25), with no missing cards. What is the probability of drawing a card whose suit is a vowel (A, E, I, O, or U)?
There are 11 suits, and among them, 3 of them, A, E, and I, are vowels, so the probability is

$$\frac{3 \text{ desired suits}}{11 \text{ suits total}} = \frac{3}{11}.$$

# Drawing from a Big Deck

Since each suit has 25 cards each (one per rank), we can see that there are

$$3 \cdot 25 = 75 \text{ cards whose suit is a vowel,}$$

and since there are

$$11 \cdot 25 = 275 \text{ cards total,}$$

the probability of drawing a card whose suit is a vowel is also

$$\frac{75}{275}$$

which reduces to our earlier answer. As we will see, our answer of $3/11$ will come in handy later.

# Drawing from a Big Deck

A deck of cards has 11 suits (A-K) and 25 ranks (1-25), with no missing cards. What is the probability of drawing a card whose rank is a multiple of 6 (6, 12, ??,...)?

# Drawing from a Big Deck

A deck of cards has 11 suits (A-K) and 25 ranks (1-25), with no missing cards. What is the probability of drawing a card whose rank is a multiple of 6 (6, 12, 18, 24)?

There are 25 ranks, and 4 of them are multiples of 6, so the probability is

$$\frac{\text{4 desired ranks}}{\text{25 ranks total}} = \frac{4}{25}.$$

# Drawing from a Big Deck

A deck of cards has 11 suits (A-K) and 25 ranks (1-25), with no missing cards. What is the probability of drawing a card whose suit is a vowel **and** whose rank is a multiple of 6? Hint: use the previous two exercises.

# Drawing from a Big Deck

A deck of cards has 11 suits (A-K) and 25 ranks (1-25), with no missing cards. What is the probability of drawing a card whose suit is a vowel **and** whose rank is a multiple of 6?

There are 11 suits, and among them, 3 of them, A, E, and I, are vowels. There are 25 ranks, and 4 of them are multiples of 6. Hence the probability of drawing a card whose suit is a vowel **and** whose rank is a multiple of 6 is

$$\frac{3}{11} \cdot \frac{4}{25} = \frac{12}{275}.$$

The 12 cards in the intersection are:

| Suit \ Rank | 6 | 12 | 18 | 24 |
|---|---|---|---|---|
| A | A6 | A12 | A18 | A24 |
| E | E6 | E12 | E18 | E24 |
| I | I6 | I12 | I18 | I24 |

# Sequences of Experiments

Suppose two experiments are performed, one after the other. We will need to be careful if the results of the first experiment affect the second one.
In the next examples, we will work with a deck of 15 cards with 3 suits (A-C) and 5 ranks (1-5).

| Suit \ Rank | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|
| A | A1 | A2 | A3 | A4 | A5 |
| B | B1 | B2 | B3 | B4 | B5 |
| C | C1 | C2 | C3 | C4 | C5 |

# Drawing Two Cards with Replacement

We draw a card from a deck, **put it back (and shuffle)**, and then draw a second card. If the first card has suit B, would the chances of drawing a card with suit B on the second draw be:

a) Equal to $\frac{5}{15}$

b) **Not** equal to $\frac{5}{15}$

If the first card does not have suit B, would the chances of drawing a card with suit B on the second draw be:

a) Equal to $\frac{5}{15}$

b) **Not** equal to $\frac{5}{15}$

Hint: how many cards are in the deck on the second draw? How many of them have suit B?

# Drawing Two Cards with Replacement

We draw a card from a deck, **put it back (and shuffle)**, and then draw a second card. If the first card has suit B, would the chances of drawing a card with suit B on the second draw be:

$$\text{Equal to } \frac{5}{15} \text{ in both cases}$$

Since the first card is replaced, on the second draw, there are still 15 cards, and there are still 5 cards with suit B.

# Drawing Two Cards without Replacement

We draw a card from a deck, and then draw a second card **without replacing the first one**. If the first card has suit B, would the chances of drawing a card with suit B from the remaining cards be:

a) Equal to $\frac{5}{15}$

b) **Not** equal to $\frac{5}{15}$

If the first card did not have suit B, would the chances of drawing a card with suit B from the remaining cards be:

a) Equal to $\frac{5}{15}$

b) **Not** equal to $\frac{5}{15}$

# Drawing Two Cards without Replacement

We draw a card from a deck, and then draw a second card **without replacing the first one**.

Then the second draw will come from **14 cards**.

If the first card has suit B, then the probability of the second card having suit B is

$$\frac{\text{4 cards with suit B left}}{\text{only 14 cards left}} = \frac{4}{14} \neq \frac{5}{15}.$$

If the first card does not have suit B, then the probability of the second card having suit B is

$$\frac{\text{still have 5 cards with suit B}}{\text{only 14 cards left}} = \frac{5}{14} \neq \frac{5}{15}.$$

The first draw affects the sample space and probabilities for the second draw. We will need a new tool to study this.

# Conditional Probability

The probability that event $F$ occurs, **given** that event $E$ (has already) occurred, is denoted as $P(F|E)$.
This is read as "the (conditional) probability of $F$ **given** $E$."
Computing $P(F|E)$ involves counting outcomes just like for usual probability, but you need to watch out for changes in:

- 
- 

# Conditional Probability

The probability that event $F$ occurs, **given** that event $E$ (has already) occurred, is denoted as $P(F|E)$.
This is read as "the (conditional) probability of $F$ **given** $E$."
Computing $P(F|E)$ involves counting outcomes just like for usual probability, but you need to watch out for changes in:

- the total number of outcomes, and
- the number of remaining outcomes described by $F$

caused by event $E$.

# Drawing Two Cards (Conditional Probability)

We can rephrase our answers from the previous problems in terms of given events and conditional probability.
The (conditional) probability of the second card having suit B given that the first card had suit B is

$$\frac{\text{4 cards with suit B left}}{\text{only 14 cards left}} = \frac{4}{14}.$$

# Drawing Two Cards (Conditional Probability)

We can rephrase our answers from the previous problems in terms of given events and conditional probability.

The (conditional) probability of the second card having suit B given that the first card had suit B is

$$\frac{\text{4 cards with suit B left}}{\text{only 14 cards left}} = \frac{4}{14}.$$

The (conditional) probability of the second card having suit B given that the first card did not have suit B is

$$\frac{\text{still have 5 cards with suit B}}{\text{only 14 cards left}} = \frac{5}{14}.$$

# Drawing Two Marbles

A jar has ten marbles - 7 are red and 3 are blue. If we draw just one marble, then the probability of it being blue is

$$\frac{\text{3 blue marbles}}{\text{10 marbles total}} = \frac{3}{10}.$$

If two marbles are drawn (without replacement), what is the probability of drawing a blue marble on the second draw given that the first marble drawn was blue?

Since the first blue marble was not replaced,

-
-

# Drawing Two Marbles

A jar has ten marbles - 7 are red and 3 are blue. If we draw just one marble, then the probability of it being blue is

$$\frac{3 \text{ blue marbles}}{10 \text{ marbles total}} = \frac{3}{10}.$$

If two marbles are drawn (without replacement), what is the probability of drawing a blue marble on the second draw given that the first marble drawn was blue?

Since the first blue marble was not replaced,

- There are only 9 marbles left for the second draw.
- There are only 2 blue marbles left.

Hence the probability of drawing a blue marble on the second draw given that the first marble drawn was blue is

$$\frac{2 \text{ blue marbles left}}{9 \text{ marbles left}} = \frac{2}{9}.$$

# Second Draw Probabilities

What does it mean to find the probability of drawing a blue marble on the second draw (without a given event)?

The problem is that without a given event that helps us identify the marbles in the jar on the second draw, we have to consider all possible ways to draw two marbles. Since there are 10 marbles for the first draw, and 9 left for the second draw, the total number of pairs of marbles is

$$10 \cdot 9 = 90 \text{ pairs of marbles.}$$

# Drawing Two Marbles

A jar has ten marbles - 7 are red and 3 are blue. If two marbles are drawn (without replacement), what is the probability of drawing a blue marble on the second draw given that the first marble drawn was red?

# Drawing Two Marbles

A jar has ten marbles - 7 are red and 3 are blue. If two marbles are drawn (without replacement), what is the probability of drawing a blue marble on the second draw given that the first marble drawn was red?

Since the first red marble was not replaced,

- There are only 9 marbles left for the second draw.
- However, there are still 3 blue marbles left.

Hence the probability of drawing a blue marble on the second draw given that the first marble drawn was red is

$$\frac{\text{3 blue marbles left}}{\text{9 marbles left}} = \frac{3}{9}.$$

# Drawing a Card

A deck of 10 cards has 2 suits (A-B) and 5 ranks (1-5).

| Suit \ Rank | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|
| A | A1 | A2 | A3 | A4 | A5 |
| B | B1 | B2 | B3 | B4 | B5 |

# Drawing a Card

A deck of 10 cards has 2 suits (A-B) and 5 ranks (1-5).

| Suit \ Rank | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|
| A | A1 | A2 | A3 | A4 | A5 |
| B | B1 | B2 | B3 | B4 | B5 |

- Let $E$ be the event "A card with suit B is drawn."
- Let $F$ be the event "A card with rank 2 is drawn."

# Drawing a Card with a Desired Suit

A deck of 10 cards has 2 suits (A-B) and 5 ranks (1-5).

| Suit \ Rank | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|
| A | A1 | A2 | A3 | A4 | A5 |
| **B** | **B1** | **B2** | **B3** | **B4** | **B5** |

The probability of drawing a card with suit B is:

$$P(E) = \frac{\text{5 such cards}}{\text{10 cards total}} = \frac{5}{10} = \frac{1}{2}.$$

# Drawing a Card with a Desired Rank

A deck of 10 cards has 2 suits (A-B) and 5 ranks (1-5).

| Suit \ Rank | 1 | **2** | 3 | 4 | 5 |
|---|---|---|---|---|---|
| A | A1 | **A2** | A3 | A4 | A5 |
| B | B1 | **B2** | B3 | B4 | B5 |

The probability of drawing a card with rank 2 is:

$$P(F) = \frac{\text{2 such cards}}{\text{10 cards total}} = \frac{2}{10} = \frac{1}{5}.$$

# Drawing a Specific Card

A deck of 10 cards has 2 suits (A-B) and 5 ranks (1-5).

| Suit \ Rank | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|
| A | A1 | A2 | A3 | A4 | A5 |
| B | B1 | **B2** | B3 | B4 | B5 |

The probability of drawing card B2 (suit B **and** rank 2) is:

$$P(E \bigcap F) = \frac{\text{1 such card B2}}{\text{10 cards total}} = \frac{1}{10}.$$

# Observation

Notice that

$$P(E) = \frac{1}{2},\ P(F) = \frac{1}{5},\ P(E \bigcap F) = \frac{1}{10}.$$

# Observation

Notice that

$$P(E) = \frac{1}{2},\ P(F) = \frac{1}{5},\ P(E \bigcap F) = \frac{1}{10}.$$

Coincidentally,

$$P(E) \cdot P(F) = \frac{1}{2} \cdot \frac{1}{5} = \frac{1}{10}$$

as well. When can we multiply the probabilities of two events to find the probability of their intersection?

# Drawing a Card: Conditional Probability

Let us compute:

- $P(F|E)$: the probability that the drawn card has rank 2, **given** that it has suit B.
- $P(E|F)$: the probability that the drawn card has suit B, **given** that it has rank 2.

We will compare these to each other, and to $P(E)$ and $P(F)$.

# Drawing a Card: $P(F|E)$ (Given Suit)

A deck of 10 cards has 2 suits (A-B) and 5 ranks (1-5).

| Suit \ Rank | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|
| A | A1 | A2 | A3 | A4 | A5 |
| B | B1 | B2 | B3 | B4 | B5 |

Let's find the probability that the drawn card has rank 2 given that it has suit B.
The given event narrows down the sample space to just the five cards with rank B.

# Drawing a Card: $P(F|E)$ (Given Suit)

A deck of 10 cards has 2 suits (A-B) and 5 ranks (1-5).

| Suit \ Rank | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|
| | | | | | |
| B | B1 | B2 | B3 | B4 | B5 |

Let's find the probability that the drawn card has rank 2 given that it has suit B.
The given event narrows down the sample space to just the five cards with suit B.

# Drawing a Card: $P(F|E)$ (Given Suit)

A deck of 10 cards has 2 suits (A-B) and 5 ranks (1-5).

| Suit \ Rank | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|
| | | | | | |
| B | B1 | **B2** | B3 | B4 | B5 |

Let's find the probability that the drawn card has rank 2 given that it has suit B.

The given event narrows down the sample space to just the five cards with suit B.

The probability that the drawn card has rank 2 given that it has suit B is

$$P(F|E) = \frac{\text{1 card among those with suit B that has rank 2}}{\text{5 cards with suit B}} = \frac{1}{5}.$$

# Conditional Probability Numerator

Let's find the probability that the drawn card has rank 2 given that it has suit B.

The given event narrows down the sample space to just the five cards with suit B.

The probability that the drawn card has rank 2 given that it has suit B is

$$P(F|E) = \frac{\text{1 card among those with suit B that has rank 2}}{\text{5 cards with suit B}} = \frac{1}{5}.$$

We can rephrase the numerator:

$$P(F|E) = \frac{\text{1 card with suit B } \textbf{and} \text{ rank 2}}{\text{5 cards with suit B}} = \frac{1}{5}.$$

# Conditional Probability Computation

Let $E$ and $F$ be events in a sample space $\Omega$. Then $P(F|E)$, the probability of getting an outcome in event $F$ given that the outcome is in event $E$, is

$$P(F|E) = \frac{\text{Number of outcomes in } E \bigcap F}{\text{Number of outcomes in } E}.$$

Hint: The given event appears in the denominator.

# Drawing a Card: $P(E|F)$ (Given Rank)

A deck of 10 cards has 2 suits (A-B) and 5 ranks (1-5).

| Suit \ Rank | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|
| A | A1 | A2 | A3 | A4 | A5 |
| B | B1 | B2 | B3 | B4 | B5 |

Let's find the probability that the drawn card has suit B given that it has rank 2.
The given event narrows down the sample space to just the two cards with rank 2.

# Drawing a Card: $P(E|F)$ (Given Rank)

A deck of 10 cards has 2 suits (A-B) and 5 ranks (1-5).

| Suit \ Rank | | 2 | | | |
|---|---|---|---|---|---|
| A | | A2 | | | |
| B | | B2 | | | |

Let's find the probability that the drawn card has suit B given that it has rank 2.
The given event narrows down the sample space to just the two cards with rank 2.

# Drawing a Card: $P(E|F)$ (Given Rank)

A deck of 10 cards has 2 suits (A-B) and 5 ranks (1-5).

| Suit \ Rank | | 2 | | | |
|---|---|---|---|---|---|
| A | | A2 | | | |
| B | | **B2** | | | |

Let's find the probability that the drawn card has suit B given that it has rank 2.

The given event narrows down the sample space to just the two cards with rank 2.

The probability that the drawn card has suit B given that it has rank 2 is

$$P(E|F) = \frac{\text{1 card with rank 2 } \textbf{and} \text{ suit B}}{\text{2 cards with rank 2}} = \frac{1}{2}.$$

# Observations

- Notice that

$$P(E) = \frac{1}{2}, \text{ and } P(E|F) = \frac{1}{2} \text{ as well.}$$

  Knowing the card's rank did not affect its chances of having suit B.
- 

# Observations

- Notice that

$$P(E) = \frac{1}{2}, \text{ and } P(E|F) = \frac{1}{2} \text{ as well.}$$

  Knowing the card's rank did not affect its chances of having suit B.

- Similarly,

$$P(F) = \frac{1}{5}, \text{ and } P(F|E) = \frac{1}{5} \text{ as well.}$$

  Knowing the card's suit did not affect its chances of having rank 2.

# Warning: Order Matters

Beware: $P(E|F)$ and $P(F|E)$ are not equal in general!

- In this example, $P(E|F) = \frac{1}{2}$ while $P(F|E) = \frac{1}{5}$.
- 
- 

# Warning: Order Matters

Beware: $P(E|F)$ and $P(F|E)$ are not equal in general!

- In this example, $P(E|F) = \frac{1}{2}$ while $P(F|E) = \frac{1}{5}$.
- For $P(E|F)$, the given event restricted us to the cards with rank 2.
-

# Warning: Order Matters

Beware: $P(E|F)$ and $P(F|E)$ are not equal in general!

- In this example, $P(E|F) = \dfrac{1}{2}$ while $P(F|E) = \dfrac{1}{5}$.
- For $P(E|F)$, the given event restricted us to the cards with rank 2.
- On the other hand, for $P(F|E)$, the given event restricted us to the cards with suit B.

# Independent Events

Let $E$ and $F$ be events in a sample space $\Omega$. Then $E$ and $F$ are said to be **independent events** if

$$P(F|E) = P(F).$$

That is, the conditional probability of $F$ given that $E$ occurred is the same as the (regular) probability of $F$.
The probability of $F$ occurring is the same whether or not $E$ occurs.

# Checking Independence

Let $E$ and $F$ be events in a sample space $\Omega$.
To check if $E$ and $F$ are independent events,

1. Compute $P(E)$, $P(F)$, and $P(F|E)$.
2. Compare $P(F|E)$ and $P(F)$. Are they **equal** or **not**?

-
-

# Checking Independence

Let $E$ and $F$ be events in a sample space $\Omega$.
To check if $E$ and $F$ are independent events,

1. Compute $P(E)$, $P(F)$, and $P(F|E)$.
2. Compare $P(F|E)$ and $P(F)$. Are they **equal** or **not**?

- If $P(F|E)$ and $P(F)$ are **equal**, then $E$ and $F$ **are independent** events.
- If $P(F|E)$ and $P(F)$ are **different** numbers, then $E$ and $F$ **are not independent** events.

# Independence with a Full Deck

A deck of 10 cards has 2 suits (A-B) and 5 ranks (1-5).

| Suit \ Rank | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|
| A | A1 | A2 | A3 | A4 | A5 |
| B | B1 | B2 | B3 | B4 | B5 |

- Let $E$ be the event "A card with suit B is drawn."
- Let $F$ be the event "A card with rank 2 is drawn."

We saw that

$$P(F|E) = \frac{1}{5}, \text{ and } P(F) = \frac{1}{5} \text{ (both equal 0.2).}$$

Hence $E$ and $F$ **are independent** events.

# Drawing a Card 2

A deck of 8 cards has 2 suits and 5 ranks, but cards B4 and B5 are not included:

| Suit \ Rank | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|
| A | A1 | A2 | A3 | A4 | A5 |
| B | B1 | B2 | B3 | | |

# Drawing a Card 2

A deck of 8 cards has 2 suits and 5 ranks, but cards B4 and B5 are not included:

| Suit \ Rank | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|
| A | A1 | A2 | A3 | A4 | A5 |
| B | B1 | B2 | B3 | | |

- Let $E$ be the event "A card with suit B is drawn." Find $P(E)$.
- Let $F$ be the event "A card with rank 2 is drawn." Find $P(F)$.
- Then $E \bigcap F$ is the event "Card B2 is drawn." Find $P(E \bigcap F)$.

# Drawing a Card with a Desired Suit 2

| Suit \ Rank | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|
| A | A1 | A2 | A3 | A4 | A5 |
| **B** | **B1** | **B2** | **B3** | | |

The probability of drawing a card with suit B is:

$$P(E) = \frac{3 \text{ such cards}}{8 \text{ cards total}} = \frac{3}{8}.$$

# Drawing a Card with a Desired Rank 2

| Suit \ Rank | 1 | **2** | 3 | 4 | 5 |
|---|---|---|---|---|---|
| A | A1 | **A2** | A3 | A4 | A5 |
| B | B1 | **B2** | B3 | | |

The probability of drawing a card with rank 2 is:

$$P(F) = \frac{\text{2 such cards}}{\text{8 cards total}} = \frac{2}{8} = \frac{1}{4}.$$

# Drawing a Specific Card 2

| Suit \ Rank | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|
| A | A1 | A2 | A3 | A4 | A5 |
| B | B1 | **B2** | B3 | | |

The probability of drawing card B2 (suit B **and** rank 2) is:

$$P(E \bigcap F) = \frac{1 \text{ such card B2}}{8 \text{ cards total}} = \frac{1}{8}.$$

# Different Observation

Notice that

$$P(E) = \frac{3}{8},\ P(F) = \frac{1}{4},\ P(E \bigcap F) = \frac{1}{8}.$$

# Different Observation

Notice that

$$P(E) = \frac{3}{8},\ P(F) = \frac{1}{4},\ P(E \bigcap F) = \frac{1}{8}.$$

This time,

$$P(E) \cdot P(F) = \frac{3}{8} \cdot \frac{1}{4} = \frac{3}{32} \neq \frac{1}{8}.$$

Multiplying the probabilities of two events did not equal the probability of their intersection.

# Drawing a Card: Conditional Probability

Now compute:

- $P(F|E)$: the probability that the drawn card has rank 2, **given** that it has suit B.
- $P(E|F)$: the probability that the drawn card has suit B, **given** that it has rank 2.

We will compare these to each other, and to $P(E)$ and $P(F)$.

# Drawing a Card: $P(F|E)$ (Given Suit) 2

| Suit \ Rank | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|
| A | A1 | A2 | A3 | A4 | A5 |
| B | B1 | B2 | B3 | | |

Let's find the probability that the drawn card has rank 2 given that it has suit B.
The given event narrows down the sample space to just the three cards with rank B.
Find $P(F|E)$.

# Drawing a Card: $P(F|E)$ (Given Suit) 2

| Suit \ Rank | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|
| A | | | | | |
| B | B1 | B2 | B3 | | |

Let's find the probability that the drawn card has rank 2 given that it has suit B.
The given event narrows down the sample space to just the three cards with rank B.

# Drawing a Card: $P(F|E)$ (Given Suit) 2

| Suit \ Rank | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|
| A | | | | | |
| B | B1 | **B2** | B3 | | |

Let's find the probability that the drawn card has rank 2 given that it has suit B.
The given event narrows down the sample space to just the three cards with rank B.
The probability that the drawn card has rank 2 given that it has suit B is

$$P(F|E) = \frac{\text{1 card with suit B \textbf{and} rank 2}}{\text{3 cards with suit B}} = \frac{1}{3}.$$

# Drawing a Card: $P(E|F)$ (Given Rank) 2

| Suit \ Rank | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|
| A | A1 | A2 | A3 | A4 | A5 |
| B | B1 | B2 | B3 | | |

Let's find the probability that the drawn card has suit B given that it has rank 2.
The given event narrows down the sample space to just the two cards with rank 2.
Find $P(E|F)$.

# Drawing a Card: $P(E|F)$ (Given Rank) 2

| Suit \ Rank | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|
| A | | A2 | | | |
| B | | B2 | | | |

Let's find the probability that the drawn card has suit B given that it has rank 2.
The given event narrows down the sample space to just the two cards with rank 2.

# Drawing a Card: $P(E|F)$ (Given Rank) 2

| Suit \ Rank | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|
| A | | A2 | | | |
| B | | **B2** | | | |

Let's find the probability that the drawn card has suit B given that it has rank 2.
The given event narrows down the sample space to just the two cards with rank 2.
The probability that the drawn card has suit B given that it has rank 2 is

$$P(E|F) = \frac{\text{1 card with rank 2 \textbf{and} suit B}}{\text{2 cards with rank 2}} = \frac{1}{2}.$$

# Observation: $P(E)$ vs. $P(E|F)$

- Notice that

$$P(E) = \frac{3}{8}, \text{ but } P(E|F) = \frac{1}{2}.$$

  Normally, we have a 3 in 8 chance (0.375) of drawing a card with suit B, but if we are given that the drawn card has rank 2, that raises the chances of drawing a card with suit B to 1/2 (0.5).

# Observation: $P(F)$ vs. $P(F|E)$

- Notice that

  $$P(F) = \frac{1}{4}, \text{ but } P(F|E) = \frac{1}{3}.$$

  Normally, we have a 1 in 4 chance (0.25) of drawing a card with rank 2, but if we are given that the drawn card has suit B, that raises the chances of drawing a card with rank 2 to 1/3 (about 0.3333).

# Independence with Missing Cards

| Suit \ Rank | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|
| A | A1 | A2 | A3 | A4 | A5 |
| B | B1 | B2 | B3 | | |

Since

$$P(F) = \frac{1}{4}, \text{ but } P(F|E) = \frac{1}{3},$$

the events $E$ and $F$ are not independent.

# Conditional Probability Computation

Let $E$ and $F$ be events in a sample space $\Omega$. Then $P(F|E)$, the probability of getting an outcome in event $F$ given that the outcome is in event $E$, is

$$P(F|E) = \frac{\text{Number of outcomes in } E \bigcap F}{\text{Number of outcomes in } E}.$$

Hint: The given event appears in the denominator.
This formula is very useful when working with tables.

# Independent Events

Let $E$ and $F$ be events in a sample space $\Omega$. Then $E$ and $F$ are said to be **independent events** if

$$P(F|E) = P(F).$$

That is, the conditional probability of $F$ given that $E$ occurred is the same as the (regular) probability of $F$.
The probability of $F$ occurring is the same whether or not $E$ occurs.

# Checking Independence

Let $E$ and $F$ be events in a sample space $\Omega$.
To check if $E$ and $F$ are independent events,

1. Compute $P(E)$, $P(F)$, and $P(F|E)$.
2. Compare $P(F|E)$ and $P(F)$. Are they **equal** or **not**?

- 

- 

# Checking Independence

Let $E$ and $F$ be events in a sample space $\Omega$.
To check if $E$ and $F$ are independent events,

1. Compute $P(E)$, $P(F)$, and $P(F|E)$.
2. Compare $P(F|E)$ and $P(F)$. Are they **equal** or **not**?

- If $P(F|E)$ and $P(F)$ are **equal**, then $E$ and $F$ **are independent** events.
- If $P(F|E)$ and $P(F)$ are **different** numbers, then $E$ and $F$ **are not independent** events.

# Home and Away Games

The stats for a sports team's season are given in the table. It records wins and losses in home and away games:

| | Wins | Losses | Total |
|---|---|---|---|
| Home | 21 | 3 | 24 |
| Away | 11 | 5 | 16 |
| Total | 32 | 8 | 40 |

# Home and Away Games (Counts)

The stats for a sports team's season are given in the table. It records wins and losses in home and away games:

| | Wins | Losses | Total |
|---|---|---|---|
| Home | 21 | 3 | 24 |
| Away | 11 | 5 | 16 |
| Total | 32 | 8 | 40 |

For example,

- The team played 21 games that were winning home games.
-
-
-

# Home and Away Games (Counts)

The stats for a sports team's season are given in the table. It records wins and losses in home and away games:

| | Wins | Losses | Total |
|---|---|---|---|
| Home | 21 | 3 | 24 |
| Away | 11 | 5 | 16 |
| Total | 32 | 8 | 40 |

For example,

- The team played 21 games that were winning home games.
- There were a total of 24 home games.
-
-

# Home and Away Games (Counts)

The stats for a sports team's season are given in the table. It records wins and losses in home and away games:

| | Wins | Losses | Total |
|---|---|---|---|
| Home | 21 | 3 | 24 |
| Away | 11 | 5 | 16 |
| Total | 32 | 8 | 40 |

For example,

- The team played 21 games that were winning home games.
- There were a total of 24 home games.
- The team won 32 games.
- 

# Home and Away Games (Counts)

The stats for a sports team's season are given in the table. It records wins and losses in home and away games:

| | Wins | Losses | Total |
|---|---|---|---|
| Home | 21 | 3 | 24 |
| Away | 11 | 5 | 16 |
| Total | 32 | 8 | 40 |

For example,

- The team played 21 games that were winning home games.
- There were a total of 24 home games.
- The team won 32 games.
- The team played 40 games in total.

# Home and Away Games (Events)

The stats for a sports team's season are given in the table. It records wins and losses in home and away games:

| | Wins | Losses | Total |
|---|---|---|---|
| Home | 21 | 3 | 24 |
| Away | 11 | 5 | 16 |
| Total | 32 | 8 | 40 |

Suppose a game is chosen at random.

- Let $E$ be the event "the game was a home game."
- Let $F$ be the event "the team won the game."

# Home and Away Games ($E$)

The stats for a sports team's season are given in the table. It records wins and losses in home and away games:

| | Wins | Losses | Total |
|---|---|---|---|
| Home | 21 | 3 | 24 |
| Away | 11 | 5 | 16 |
| Total | 32 | 8 | 40 |

The probability that the team played at home is

$$P(E) = \frac{24 \text{ home games}}{40 \text{ games total}} = \frac{24}{40}.$$

# Home and Away Games ($F$)

The stats for a sports team's season are given in the table. It records wins and losses in home and away games:

| | Wins | Losses | Total |
|---|---|---|---|
| Home | 21 | 3 | 24 |
| Away | 11 | 5 | 16 |
| Total | 32 | 8 | 40 |

The probability that the team won is

$$P(F) = \frac{32 \text{ winning games}}{40 \text{ games total}} = \frac{32}{40}.$$

# Home and Away Games $(E \bigcap F)$

The stats for a sports team's season are given in the table. It records wins and losses in home and away games:

| | Wins | Losses | Total |
|---|---|---|---|
| Home | 21 | 3 | 24 |
| Away | 11 | 5 | 16 |
| Total | 32 | 8 | 40 |

The probability that the game took place at home **and** was a winning game is

$$P(E \bigcap F) = \frac{21 \text{ games won at home}}{40 \text{ games total}} = \frac{21}{40}.$$

# Home and Away Games $(F|E)$

The stats for a sports team's season are given in the table. It records wins and losses in home and away games:

| | Wins | Losses | Total |
|---|---|---|---|
| Home | 21 | 3 | 24 |
| Away | 11 | 5 | 16 |
| Total | 32 | 8 | 40 |

The probability that the team won **given** that it was a home game is

$$P(F|E) = \frac{\text{21 games won at home}}{\text{24 home games}}$$

# Home and Away Games $(F|E)$

The stats for a sports team's season are given in the table. It records wins and losses in home and away games:

| | Wins | Losses | Total |
|---|---|---|---|
| Home | 21 | 3 | 24 |
| Away | 11 | 5 | 16 |
| Total | 32 | 8 | 40 |

The probability that the team won **given** that it was a home game is

$$P(F|E) = \frac{\text{21 games won at home}}{\text{24 home games}} = \frac{21}{24}.$$

The given event tells us to restrict to the row of home games.

# Home and Away Games $(E|F)$

The stats for a sports team's season are given in the table. It records wins and losses in home and away games:

| | Wins | Losses | Total |
|---|---|---|---|
| Home | 21 | 3 | 24 |
| Away | 11 | 5 | 16 |
| Total | 32 | 8 | 40 |

The probability that the game was at home **given** that the team won is

$$P(E|F) = \frac{\text{21 games won at home}}{\text{32 winning games}}$$

# Home and Away Games $(E|F)$

The stats for a sports team's season are given in the table. It records wins and losses in home and away games:

| | Wins | Losses | Total |
|---|---|---|---|
| Home | 21 | 3 | 24 |
| Away | 11 | 5 | 16 |
| Total | 32 | 8 | 40 |

The probability that the game was at home **given** that the team won is

$$P(E|F) = \frac{\text{21 games won at home}}{\text{32 winning games}} = \frac{21}{32}.$$

The given event tells us to restrict to the column of wins.

# Home and Away Games: Independence

To check if the events $E$, "the game was a home game," and $F$, "the team won the game," are independent, we compare $P(F)$ and $P(F|E)$.

$$P(F) = \frac{32}{40} = 0.8; \text{ but } P(F|E) = \frac{21}{24} = 0.875.$$

# Home and Away Games: Independence

To check if the events $E$, "the game was a home game," and $F$, "the team won the game," are independent, we compare $P(F)$ and $P(F|E)$.

$$P(F) = \frac{32}{40} = 0.8; \text{ but } P(F|E) = \frac{21}{24} = 0.875.$$

Since $P(F)$ and $P(F|E)$ are **not equal**, the events are not independent.

To check if the events $E$, “the game was a home game,” and $F$, “the team won the game,” are independent, we compare $P(F)$ and $P(F|E)$.

$$P(F) = \frac{32}{40} = 0.8; \text{ but } P(F|E) = \frac{21}{24} = 0.875.$$

Since $P(F)$ and $P(F|E)$ are **not equal**, the events are not independent.
Notice that the team is more likely to win their home games: $P(F|E)$ is a little larger than $P(F)$.

# Home and Away Games ($E \cup F$)

The stats for a sports team's season are given in the table. It records wins and losses in home and away games:

| | Wins | Losses | Total |
|---|---|---|---|
| Home | 21 | 3 | 24 |
| Away | 11 | 5 | 16 |
| Total | 32 | 8 | 40 |

To find the probability that the game was at home **or** a winning game, we count:

# Home and Away Games ($E \cup F$)

The stats for a sports team's season are given in the table. It records wins and losses in home and away games:

| | Wins | Losses | Total |
|---|---|---|---|
| Home | 21 | 3 | 24 |
| Away | 11 | 5 | 16 |
| Total | 32 | 8 | 40 |

To find the probability that the game was at home **or** a winning game, we count:
the 21 games which were winning home games,

# Home and Away Games ($E \cup F$)

The stats for a sports team's season are given in the table. It records wins and losses in home and away games:

| | Wins | Losses | Total |
|---|---|---|---|
| Home | 21 | 3 | 24 |
| Away | 11 | 5 | 16 |
| Total | 32 | 8 | 40 |

To find the probability that the game was at home **or** a winning game, we count:
the 21 games which were winning home games,
the 3 other games that took place at home, and

# Home and Away Games ($E \cup F$)

The stats for a sports team's season are given in the table. It records wins and losses in home and away games:

| | Wins | Losses | Total |
|---|---|---|---|
| Home | 21 | 3 | 24 |
| Away | 11 | 5 | 16 |
| Total | 32 | 8 | 40 |

To find the probability that the game was at home **or** a winning game, we count:
the 21 games which were winning home games,
the 3 other games that took place at home, and
the 11 other winning games.

# Home and Away Games ($E \cup F$)

The stats for a sports team's season are given in the table. It records wins and losses in home and away games:

| | Wins | Losses | Total |
|---|---|---|---|
| Home | 21 | 3 | 24 |
| Away | 11 | 5 | 16 |
| Total | 32 | 8 | 40 |

To find the probability that the game was at home **or** a winning game, we count:
the 21 games which were winning home games,
the 3 other games that took place at home, and
the 11 other winning games.
The union is an event in the sample space of all 40 games.

| | Wins | Losses | Total |
|---|---|---|---|
| Home | 21 | 3 | 24 |
| Away | 11 | 5 | 16 |
| Total | 32 | 8 | 40 |

To find the probability that the game was at home **or** a winning game, we count:
the 21 games which were winning home games,
the 3 other games that took place at home, and
the 11 other winning games.
The union is an event in the sample space of all 40 games.

$$P(E \bigcup F) = \frac{21 + 3 + 11}{40 \text{ games total}} = \frac{35}{40}.$$

# The Star Player

The stats for a sports team's season are given in the table. This team has a star player who got injured in the middle of the season. The table records wins and losses and whether the star player was in or not in the game:

| | Wins | Losses | Total |
|---|---|---|---|
| Star in | 12 | 4 | 16 |
| Star out | 15 | 5 | 20 |
| Total | 27 | 9 | 36 |

# The Star Player

The stats for a sports team's season are given in the table. This team has a star player who got injured in the middle of the season. The table records wins and losses and whether the star player was in or not in the game:

| | Wins | Losses | Total |
|---|---|---|---|
| Star in | 12 | 4 | 16 |
| Star out | 15 | 5 | 20 |
| Total | 27 | 9 | 36 |

Suppose a game is chosen at random.

- Let $E$ be the event "the star player was in the game."
- Let $F$ be the event "the team won the game."

# The Star Player ($E$)

| | Wins | Losses | Total |
|---|---|---|---|
| Star in | 12 | 4 | 16 |
| Star out | 15 | 5 | 20 |
| Total | 27 | 9 | 36 |

What is the probability of picking a game that the star player played in?

# The Star Player ($E$)

| | Wins | Losses | Total |
|---|---|---|---|
| Star in | 12 | 4 | 16 |
| Star out | 15 | 5 | 20 |
| Total | 27 | 9 | 36 |

The probability that the star player played is

$$P(E) = \frac{16 \text{ home games}}{36 \text{ games total}} = \frac{16}{36}.$$

# The Star Player ($F$)

| | Wins | Losses | Total |
|---|---|---|---|
| Star in | 12 | 4 | 16 |
| Star out | 15 | 5 | 20 |
| Total | 27 | 9 | 36 |

What is the probability of picking a winning game?

# The Star Player ($F$)

| | Wins | Losses | Total |
|---|---|---|---|
| Star in | 12 | 4 | 16 |
| Star out | 15 | 5 | 20 |
| Total | 27 | 9 | 36 |

The probability that the team won is

$$P(F) = \frac{27 \text{ winning games}}{36 \text{ games total}} = \frac{27}{36}.$$

# The Star Player $(B \bigcap F)$

| | Wins | Losses | Total |
|---|---|---|---|
| Star in | 12 | 4 | 16 |
| Star out | 15 | 5 | 20 |
| Total | 27 | 9 | 36 |

What is the probability of picking a game that has the star player in it and was a winning game?

# The Star Player ($E \bigcap F$)

| | Wins | Losses | Total |
|---|---|---|---|
| Star in | 12 | 4 | 16 |
| Star out | 15 | 5 | 20 |
| Total | 27 | 9 | 36 |

The probability that the game had the star player **and** was a winning game is

$$P(E \bigcap F) = \frac{\text{12 winning games with star}}{\text{36 games total}} = \frac{12}{36}.$$

# The Star Player (?|?)

| | Wins | Losses | Total |
|---|---|---|---|
| Star in | 12 | 4 | 16 |
| Star out | 15 | 5 | 20 |
| Total | 27 | 9 | 36 |

What is the probability of picking a winning game given that the chosen game has the star player in it?

# The Star Player $(F|E)$

| | Wins | Losses | Total |
|---|---|---|---|
| Star in | 12 | 4 | 16 |
| Star out | 15 | 5 | 20 |
| Total | 27 | 9 | 36 |

The probability that the team won **given** that the star player was in is

$$P(F|E) = \frac{12 \text{ winning games with star}}{16 \text{ games with star}} = \frac{12}{16}.$$

# The Star Player (?|?)

| | Wins | Losses | Total |
|---|---|---|---|
| Star in | 12 | 4 | 16 |
| Star out | 15 | 5 | 20 |
| Total | 27 | 9 | 36 |

What is the probability of picking a game that has the star player in it given that the chosen game was a winning game?

# The Star Player $(E|F)$

| | Wins | Losses | Total |
|---|---|---|---|
| Star in | 12 | 4 | 16 |
| Star out | 15 | 5 | 20 |
| Total | 27 | 9 | 36 |

The probability that the star player was in **given** that the team won is

$$P(E|F) = \frac{\text{12 winning games with star}}{\text{27 winning games}} = \frac{12}{27}.$$

# The Star Player: Independence

- Let $E$ be the event "the star player was in the game."
- Let $F$ be the event "the team won the game."

Our computations so far:

- $P(E) = \dfrac{16}{36}$
- $P(F) = \dfrac{27}{36}$
- $P(F|E) = \dfrac{12}{16}$
- $P(E|F) = \dfrac{12}{27}$

Based on these calculations, are the events $E$ and $F$ independent?

# The Star Player: Independence

Let $E$ be the event "the star player was in the game."
Let $F$ be the event "the team won the game."
The events are independent, because

$$P(F) = \frac{27}{36}, \textit{ or } 0.75$$

is equal to

$$P(F|E) = \frac{12}{16}, \textit{ or } 0.75.$$

It looks like the team's likelihood of winning over the season did not get affected by the star player's sidelining.

# The Star Player ($E \cup F$)

| | Wins | Losses | Total |
|---|---|---|---|
| Star in | 12 | 4 | 16 |
| Star out | 15 | 5 | 20 |
| Total | 27 | 9 | 36 |

What is the probability of picking a game that has the star player in it or was a winning game?

# The Star Player $(E \bigcup F)$

| | Wins | Losses | Total |
|---|---|---|---|
| Star in | 12 | 4 | 16 |
| Star out | 15 | 5 | 20 |
| Total | 27 | 9 | 36 |

To find the probability that the game had the star player, **or** was a winning game, we count:
the 12 games which were winning games with the star,
the 4 other games the star played in, and
the 15 other winning games:

$$P(E \bigcup F) = \frac{12 + 4 + 15}{36 \text{ games total}} = \frac{31}{36}.$$

# Numerator of Conditional Probability

We have been using this formula today to compute conditional probability:

$$P(F|E) = \frac{\text{Number of outcomes in } E \bigcap F}{\text{Number of outcomes in } E}.$$

# Numerator of Conditional Probability

We have been using this formula today to compute conditional probability:

$$P(F|E) = \frac{\text{Number of outcomes in } E \bigcap F}{\text{Number of outcomes in } E}.$$

Sometimes, we might know $P(F|E)$ and the number of outcomes in $E$. We can use them to find the number of outcomes in $E \bigcap F$:

$$\begin{aligned} & (\text{Number of outcomes in E}) \times P(F|E) \\ = \; & \text{Number of outcomes in } E \bigcap F \end{aligned}$$

We have been using this formula today to compute conditional probability:

$$P(F|E) = \frac{\text{Number of outcomes in } E\bigcap F}{\text{Number of outcomes in } E}.$$

Sometimes, we might know $P(F|E)$ and the number of outcomes in $E$. We can use them to find the number of outcomes in $E\bigcap F$:

$$\begin{aligned} & (\text{Number of outcomes in E}) \times P(F|E) \\ = \ & \text{Number of outcomes in } E\bigcap F \end{aligned}$$

Reminder when multiplying a whole number and a fraction:

$$a \times \frac{b}{c} = \frac{a \times b}{c}.$$

# Outcomes in an Intersection

A jar contains beads which are either blue or white, and either round or pointy. If the probability of drawing a blue bead given that it is round is 47/50, and there are 250 round beads in the jar, how many beads are blue and round?

# Outcomes in an Intersection

A jar contains beads which are either blue or white, and either round or pointy. If the probability of drawing a blue bead given that it is round is 47/50, and there are 250 round beads in the jar, how many beads are blue and round?

$$\frac{\text{Number of blue AND round beads}}{\text{Number of round beads}} = \frac{47}{50}$$

# Outcomes in an Intersection

A jar contains beads which are either blue or white, and either round or pointy. If the probability of drawing a blue bead given that it is round is 47/50, and there are 250 round beads in the jar, how many beads are blue and round?

$$\frac{\text{Number of blue AND round beads}}{\text{Number of round beads}} = \frac{47}{50}$$

$$\frac{\text{Number of blue AND round beads}}{\text{250 round beads}} = \frac{47}{50}$$

# Outcomes in an Intersection

A jar contains beads which are either blue or white, and either round or pointy. If the probability of drawing a blue bead given that it is round is 47/50, and there are 250 round beads in the jar, how many beads are blue and round?

$$\frac{\text{Number of blue AND round beads}}{\text{Number of round beads}} = \frac{47}{50}$$

$$\frac{\text{Number of blue AND round beads}}{\text{250 round beads}} = \frac{47}{50}$$

$$\text{Number of blue AND round beads} = 250 \cdot \frac{47}{50}$$

# Outcomes in an Intersection

A jar contains beads which are either blue or white, and either round or pointy. If the probability of drawing a blue bead given that it is round is 47/50, and there are 250 round beads in the jar, how many beads are blue and round?

$$\begin{aligned}
\frac{\text{Number of blue AND round beads}}{\text{Number of round beads}} &= \frac{47}{50} \\
\frac{\text{Number of blue AND round beads}}{\text{250 round beads}} &= \frac{47}{50} \\
\text{Number of blue AND round beads} &= 250 \cdot \frac{47}{50} \\
&= \frac{11,750}{50} = 235.
\end{aligned}$$

$$\frac{\text{Number of blue AND round beads}}{\text{Number of round beads}} = \frac{47}{50}$$

$$\frac{\text{Number of blue AND round beads}}{\text{250 round beads}} = \frac{47}{50}$$

$$\text{Number of blue AND round beads} = 250 \cdot \frac{47}{50}$$

$$= \frac{11,750}{50} = 235.$$

Check using decimals:

$$\frac{\text{Number of blue AND round beads}}{\text{Number of round beads}} = \frac{235}{250} = 0.94;$$

$$\frac{47}{50} = 0.94\checkmark$$

# Outcomes in an Intersection

A jar contains beads which are either blue or white, and either round or pointy. If the probability of drawing a round bead given that it is blue is 1/16, and the jar has 320 blue beads and 480 round beads, how many beads are blue and round?

$$\frac{\text{Number of blue AND round beads}}{\text{Number of blue beads}} = \frac{1}{16}$$

$$\frac{\text{Number of blue AND round beads}}{\text{320 blue beads}} = \frac{1}{16}$$

$$\text{Number of blue AND round beads} = 320 \cdot \frac{1}{16}$$

$$= \frac{320}{16} = 20.$$

Check using decimals:

$$\frac{\text{Number of blue AND round beads}}{\text{Number of blue beads}} = \frac{20}{320} = 0.625;$$

$$\frac{1}{16} = 0.625\checkmark$$

# Fill in the Table

A jar contains beads which are either blue or white, and either round or pointy. There are 200 beads total, with 40 of them being pointy.

| | Round | Pointy | Total |
|---|---|---|---|
| Blue | | | |
| White | | | |
| Total | | | 200 |

Fill in the table. Suppose that:

- $P(\text{Bead is White} \mid \text{Bead is Round}) = 1/8$.
- $P(\text{Bead is Blue} \mid \text{Bead is Pointy}) = 3/20$.

# Fill in the Table

A jar contains beads which are either blue or white, and either round or pointy. There are 200 beads total, with 40 of them being pointy.

| | Round | Pointy | Total |
|---|---|---|---|
| Blue | | | |
| White | | | |
| Total | | 40 | 200 |

Fill in the table. Suppose that:

- $P(\text{Bead is White} \mid \text{Bead is Round}) = 1/8$.
- $P(\text{Bead is Blue} \mid \text{Bead is Pointy}) = 3/20$.

# Fill in the Table (Round Bead Total)

A jar contains beads which are either blue or white, and either round or pointy. There are 200 beads total, with 40 of them being pointy.

| | Round | Pointy | Total |
|---|---|---|---|
| Blue | | | |
| White | | | |
| Total | 160 | 40 | 200 |

- 40 beads are pointy, so there must be

$$200 - 40 = 160$$

round beads.

# Fill in the Table (White Round Beads)

| | Round | Pointy | Total |
|---|---|---|---|
| Blue | | | |
| White | | | |
| Total | 160 | 40 | 200 |

- We have $P(\text{Bead is White} \mid \text{Bead is Round}) = 1/8$. By definition,

$$\frac{\text{Number of white round beads}}{\text{Number of round beads}} = P(\text{White}|\text{Round})$$

$$\frac{\text{Number of white round beads}}{160} = \frac{1}{8}$$

| | Round | Pointy | Total |
|---|---|---|---|
| Blue | | | |
| White | 20 | | |
| Total | 160 | 40 | 200 |

- We have $P(\text{Bead is White} \mid \text{Bead is Round}) = 1/8$. By definition,

$$\begin{aligned}
\frac{\text{Number of white round beads}}{\text{Number of round beads}} &= P(\text{White}|\text{Round}) \\
\frac{\text{Number of white round beads}}{160} &= \frac{1}{8} \\
\text{Number of white round beads} &= \frac{1}{8} \cdot 160 = \frac{160}{8} = 20.
\end{aligned}$$

# Fill in the Table (Blue Pointy Beads)

| | Round | Pointy | Total |
|---|---|---|---|
| Blue | | | |
| White | 20 | | |
| Total | 160 | 40 | 200 |

We know $P(\text{Bead is Blue} \mid \text{Bead is Pointy}) = 3/20$. Use this to find the number of blue pointy beads.

# Fill in the Table (Blue Pointy Beads)

| | Round | Pointy | Total |
|---|---|---|---|
| Blue | | 6 | |
| White | 20 | | |
| Total | 160 | 40 | 200 |

We have $P(\text{Bead is Blue} \mid \text{Bead is Pointy}) = 3/20$. By definition,

$$\begin{aligned}
\frac{\text{Number of blue pointy beads}}{\text{Number of pointy beads}} &= P(\text{Blue}|\text{Pointy}) \\
\frac{\text{Number of blue pointy beads}}{40} &= \frac{3}{20} \\
\text{Number of blue pointy beads} &= \frac{3}{20} \cdot 40 = \frac{3 \cdot 40}{20} = \frac{120}{20} = 6.
\end{aligned}$$

# Fill in the Table (White Pointy Beads)

| | Round | Pointy | Total |
|---|---|---|---|
| Blue | | 6 | |
| White | 20 | | |
| Total | 160 | 40 | 200 |

Now use the totals to fill in missing counts.

- The number of white pointy beads is the number of pointy beads minus the blue pointy beads:

# Fill in the Table (White Pointy Beads)

| | Round | Pointy | Total |
|---|---|---|---|
| Blue | | 6 | |
| White | 20 | 34 | |
| Total | 160 | 40 | 200 |

Now use the totals to fill in missing counts.

- The number of white pointy beads is the number of pointy beads minus the blue pointy beads:

$$40 - 6 = 34.$$

# Fill in the Table (Blue Round Beads)

| | Round | Pointy | Total |
|---|---|---|---|
| Blue | | 6 | |
| White | 20 | 34 | |
| Total | 160 | 40 | 200 |

How many blue round beads are there?

# Fill in the Table (Blue Round Beads)

| | Round | Pointy | Total |
|---|---|---|---|
| Blue | 140 | 6 | |
| White | 20 | 34 | |
| Total | 160 | 40 | 200 |

- The number of blue round beads is the number of round beads minus the blue round beads:

$$160 - 20 = 140.$$

# Fill in the Table (White Bead Total)

| | Round | Pointy | Total |
|---|---|---|---|
| Blue | 140 | 6 | |
| White | 20 | 34 | |
| Total | 160 | 40 | 200 |

Finally, fill in the missing totals.

- The number of white beads is the sum of the number of white round and white point beads:

# Fill in the Table (White Bead Total)

| | Round | Pointy | Total |
|---|---|---|---|
| Blue | 140 | 6 | |
| White | 20 | 34 | 54 |
| Total | 160 | 40 | 200 |

Finally, fill in the missing totals.

- The number of white beads is the sum of the number of white round and white pointy beads:

$$20 + 34 = 54.$$

# Fill in the Table (Blue Bead Total)

| | Round | Pointy | Total |
|---|---|---|---|
| Blue | 140 | 6 | |
| White | 20 | 34 | 54 |
| Total | 160 | 40 | 200 |

How many blue beads are there?

# Fill in the Table (Blue Bead Total)

| | Round | Pointy | Total |
|---|---|---|---|
| Blue | 140 | 6 | 146 |
| White | 20 | 34 | 54 |
| Total | 160 | 40 | 200 |

- The number of blue beads is the sum of the number of blue round and blue pointy beads:

$$140 + 6 = 146.$$

# Continuous Probability Reminders

Use continuous probability when picking random real numbers.

- Sample spaces and events are intervals.
- The length of an interval is the right endpoint minus the left endpoint.
-
-

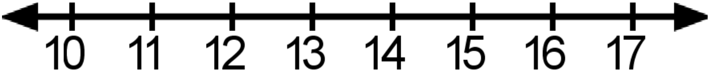
# Continuous Probability Reminders

Use continuous probability when picking random real numbers.

- Sample spaces and events are intervals.
- The length of an interval is the right endpoint minus the left endpoint.
- The probability of an interval event $E$ is the length of $E$ divided by the length of the sample space.
- The intersection of two intervals is the interval formed by their overlap.

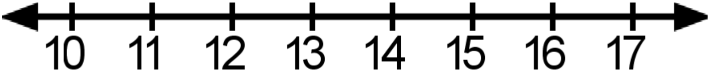
# Continuous Probability Review

Consider the sample space $\Omega = [0,7]$ and event (interval) $F = [3,6]$:



# Continuous Probability Review

Consider the sample space $\Omega = [0, 7]$ and event (interval) $F = [3, 6]$:




- The sample space has length $7 - 0 = 7$.
- 
- 

# Continuous Probability Review

Consider the sample space $\Omega = [0, 7]$ and event (interval) $F = [3, 6]$:


- The sample space has length $7 - 0 = 7$.
- Event $F$ has length $6 - 3 = 3$.
- 

# Continuous Probability Review

Consider the sample space $\Omega = [0, 7]$ and event (interval) $F = [3, 6]$:


- The sample space has length $7 - 0 = 7$.
- Event $F$ has length $6 - 3 = 3$.
- Hence the probability of $F$ is

$$\frac{\text{Length of } F}{\text{Total length}}$$

# Continuous Probability Review

Consider the sample space $\Omega = [0, 7]$ and event (interval) $F = [3, 6]$:


- The sample space has length $7 - 0 = 7$.
- Event $F$ has length $6 - 3 = 3$.
- Hence the probability of $F$ is

$$\frac{\text{Length of } F}{\text{Total length}} = \frac{3}{7}.$$

Notice that $F$ takes up 3/7ths of the total length of the sample space.

# Conditional Probability for Intervals

Let $E$ and $F$ be events in a sample space $\Omega$. Then the probability of event $F$ given that $E$ occurred is

$$P(F|E) = \frac{\text{Length of } E \bigcap F}{\text{Length of } E}.$$

# Conditional Probability for Intervals (Details)

Let $E$ and $F$ be events in a sample space $\Omega$. We have seen that

$$P(F|E) = \frac{P(E \bigcap F)}{P(E)}.$$

In terms of lengths, we have

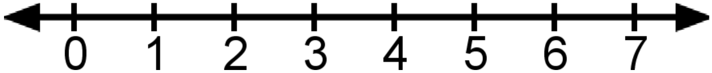
$$P(F|E) = \frac{\frac{\text{Length of } E \bigcap F}{\text{Total length}}}{\frac{\text{Length of } E}{\text{Total length}}},$$

and this simplifies to the fraction

$$P(F|E) = \frac{\text{Length of } E \bigcap F}{\text{Length of } E}.$$

# Conditional Probability for Intervals 1

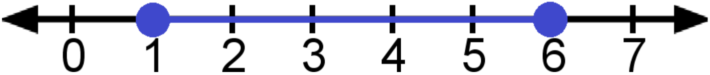
Let $\Omega = [0,7]$, $E = [1,6]$, and $F = [2,5]$.




- 

# Conditional Probability for Intervals 1

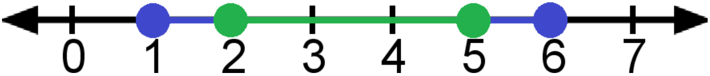
Let $\Omega = [0, 7]$, $E = [1, 6]$, and $F = [2, 5]$.




- 

# Conditional Probability for Intervals 1

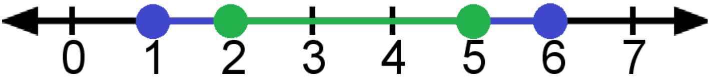
Let $\Omega = [0, 7]$, $E = [1, 6]$, and $F = [2, 5]$.




- 

# Conditional Probability for Intervals 1

Let $\Omega = [0,7]$, $E = [1,6]$, and $F = [2,5]$.




- We have

$$P(F) = \frac{\text{Length of } F}{\text{Total length of } \Omega}$$

# Conditional Probability for Intervals 1

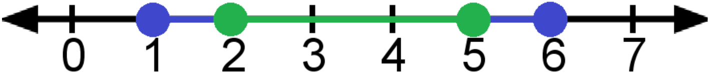
Let $\Omega = [0,7]$, $E = [1,6]$, and $F = [2,5]$.


- We have

$$P(F) = \frac{\text{Length of } F}{\text{Total length of } \Omega} = \frac{5-2}{7-0} = \frac{3}{7}.$$

# Conditional Probability for Intervals 1

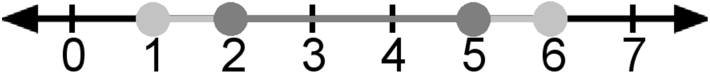
Let $\Omega = [0,7]$, $E = [1,6]$, and $F = [2,5]$.
Let us compute $P(F|E)$.




- Find $E \bigcap F$ and its length:

# Conditional Probability for Intervals 1

Let $\Omega = [0,7]$, $E = [1,6]$, and $F = [2,5]$.
Let us compute $P(F|E)$.

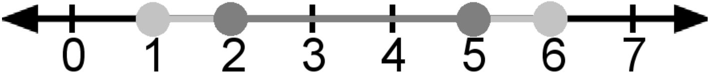



- Find $E \bigcap F$ and its length:

$$E \text{ and } F \text{ overlap on } [2,5],$$

# Conditional Probability for Intervals 1

Let $\Omega = [0, 7]$, $E = [1, 6]$, and $F = [2, 5]$.
Let us compute $P(F|E)$.



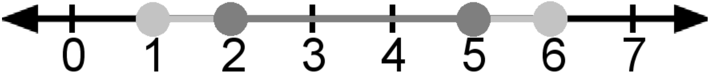

- Find $E \bigcap F$ and its length:

$$E \text{ and } F \text{ overlap on } [2, 5],$$

which has length $5 - 2 = 3$. Hence

# Conditional Probability for Intervals 1

Let $\Omega = [0,7]$, $E = [1,6]$, and $F = [2,5]$.
Let us compute $P(F|E)$.




- Find $E \bigcap F$ and its length:

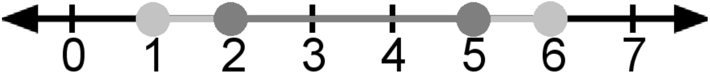
$$E \text{ and } F \text{ overlap on } [2,5],$$

which has length $5 - 2 = 3$. Hence

$$P(F|E) = \frac{\text{Length of } E \bigcap F}{\text{Length of } E}$$

# Conditional Probability for Intervals 1

Let $\Omega = [0,7]$, $E = [1,6]$, and $F = [2,5]$.
Let us compute $P(F|E)$.




- Find $E \bigcap F$ and its length:

$$E \text{ and } F \text{ overlap on } [2,5],$$

which has length $5 - 2 = 3$. Hence

$$P(F|E) = \frac{\text{Length of } E \bigcap F}{\text{Length of } E} = \frac{3}{6-1} = \frac{3}{5}.$$

Notice that $E \bigcap F$ takes up 3/5ths of the total length of $E$.

# Conditional Probability Practice 1

Let $\Omega = [24, 47]$, $E = [29, 43]$, and $F = [34, 38]$. Compute $P(F|E)$.

Hints:

1. Identify the intersection of $[29, 43]$ and $[34, 38]$ as an interval.
2. What is the length of the intersection?
3. What is the length of the given event?
4. Answer the question by dividing the appropriate lengths.

# Conditional Probability Practice 1

Let $\Omega = [24, 47]$, $E = [29, 43]$, and $F = [34, 38]$. Compute $P(F|E)$.


Find $E \bigcap F$ and its length:

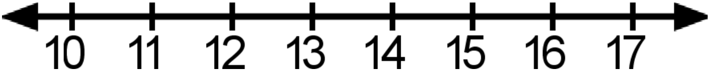
$$E \text{ and } F \text{ overlap on } [34, 38],$$

which has length $38 - 34 = 4$. Hence

$$P(F|E) = \frac{\text{Length of } E \bigcap F}{\text{Length of } E} = \frac{4}{43 - 29} = \frac{4}{14}.$$

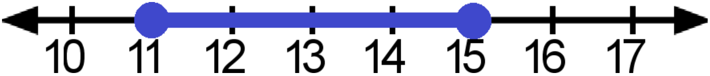
# Conditional Probability for Intervals 2

Now let $\Omega = [0, 7]$, $E = [1, 5]$, and $F = [3, 6]$. Find $P(F|E)$.



# Conditional Probability for Intervals 2

Now let $\Omega = [0, 7]$, $E = [1, 5]$, and $F = [3, 6]$. Find $P(F|E)$.



# Conditional Probability for Intervals 2

Now let $\Omega = [0, 7]$, $E = [1, 5]$, and $F = [3, 6]$. Find $P(F|E)$.

# Conditional Probability for Intervals 2

Now let $\Omega = [0, 7]$, $E = [1, 5]$, and $F = [3, 6]$. Find $P(F|E)$.


- Find $E \bigcap F$ and its length:

# Conditional Probability for Intervals 2

Now let $\Omega = [0,7]$, $E = [1,5]$, and $F = [3,6]$. Find $P(F|E)$.


- Find $E \bigcap F$ and its length:

$$E \bigcap F = [3,5], \text{ so its length is } 5 - 3 = 2.$$

- 

# Conditional Probability for Intervals 2

Now let $\Omega = [0,7]$, $E = [1,5]$, and $F = [3,6]$. Find $P(F|E)$.


- Find $E \bigcap F$ and its length:

$$E \bigcap F = [3,5], \text{ so its length is } 5 - 3 = 2.$$

- Compute $P(F|E)$ using lengths:

$$P(F|E) = \frac{\text{Length of } E \bigcap F}{\text{Length of } E} \qquad .$$

# Conditional Probability for Intervals 2

Now let $\Omega = [0,7]$, $E = [1,5]$, and $F = [3,6]$. Find $P(F|E)$.


- Find $E \bigcap F$ and its length:

$$E \bigcap F = [3,5], \text{ so its length is } 5 - 3 = 2.$$

- Compute $P(F|E)$ using lengths:

$$P(F|E) = \frac{\text{Length of } E \bigcap F}{\text{Length of } E} = \frac{2}{5-1} = \frac{2}{4}.$$

# Conditional Probability Practice 2

Let $\Omega = [47, 78]$, $E = [51, 60]$, and $F = [54, 63]$. Compute $P(F|E)$.

Hints:

1. Identify the intersection of $[51, 60]$ and $[54, 63]$ as an interval.
2. What is the length of the intersection?
3. What is the length of the given event?
4. Answer the question by dividing the appropriate lengths.

# Conditional Probability Practice 2

Let $\Omega = [47, 78]$, $E = [51, 60]$, and $F = [54, 63]$. Compute $P(F|E)$.


Hints:

1. Identify the intersection of $[51, 60]$ and $[54, 63]$ as an interval.
2. What is the length of the intersection?
3. What is the length of the given event?
4. Answer the question by dividing the appropriate lengths.

# Conditional Probability Practice 2

Let $\Omega = [47, 78]$, $E = [51, 60]$, and $F = [54, 63]$. Compute $P(F|E)$.


Find $E \bigcap F$ and its length:

$$E \text{ and } F \text{ overlap on } [54, 60],$$

which has length $60 - 54 = 6$. Hence

$$P(F|E) = \frac{\text{Length of } E \bigcap F}{\text{Length of } E} = \frac{6}{60 - 51} = \frac{6}{9}.$$

# Multiplying Integers and Fractions

If you have a whole number or integer $a$ and a fraction $b/c$, then

$$a \cdot \frac{b}{c} = \frac{a \cdot b}{c}.$$

Examples:

$$3 \cdot \frac{4}{5}$$

$$-2 \cdot \frac{7}{9}$$

# Multiplying Integers and Fractions

If you have a whole number or integer $a$ and a fraction $b/c$, then

$$a \cdot \frac{b}{c} = \frac{a \cdot b}{c}.$$

Examples:

$$3 \cdot \frac{4}{5} = \frac{3 \cdot 4}{5}$$

$$-2 \cdot \frac{7}{9}$$

# Multiplying Integers and Fractions

If you have a whole number or integer $a$ and a fraction $b/c$, then

$$a \cdot \frac{b}{c} = \frac{a \cdot b}{c}.$$

Examples:

$$\begin{aligned} 3 \cdot \frac{4}{5} &= \frac{3 \cdot 4}{5} \\ &= \frac{12}{5} \end{aligned}$$

$$-2 \cdot \frac{7}{9}$$

# Multiplying Integers and Fractions

If you have a whole number or integer $a$ and a fraction $b/c$, then

$$a \cdot \frac{b}{c} = \frac{a \cdot b}{c}.$$

Examples:

$$\begin{aligned} 3 \cdot \frac{4}{5} &= \frac{3 \cdot 4}{5} \\ &= \frac{12}{5} \end{aligned}$$

$$-2 \cdot \frac{7}{9} = \frac{-2 \cdot 7}{9}$$

# Multiplying Integers and Fractions

If you have a whole number or integer $a$ and a fraction $b/c$, then

$$a \cdot \frac{b}{c} = \frac{a \cdot b}{c}.$$

Examples:

$$3 \cdot \frac{4}{5} = \frac{3 \cdot 4}{5} = \frac{12}{5}$$

$$-2 \cdot \frac{7}{9} = \frac{-2 \cdot 7}{9} = \frac{-14}{9} = -\frac{14}{9}.$$

# Die Throwing Game

Someone presents the following game. They have a fair six-sided die. To play, you make a bet of \$1, and they throw the die. If they roll a 1, you win \$3. Notice your chances of winning a game is

$$P(\text{"Win"}) = \frac{1 \text{ outcome}}{6 \text{ sides total}} = \frac{1}{6}.$$

# Die Throwing Game

Someone presents the following game. They have a fair six-sided die. To play, you make a bet of \$1, and they throw the die. If they roll a 1, you win \$3. Notice your chances of winning a game is

$$P(\text{"Win"}) = \frac{1 \text{ outcome}}{6 \text{ sides total}} = \frac{1}{6}.$$

Suppose you play the game 10 times, and at the end, you have lost \$4. On average, you **lost**

$$\frac{\$4}{10 \text{ games}} = \$0.40$$

per game.

# Die Throwing Game Questions

- Were you unlucky?
- Is it possible to end up with more money than you started with?
- Is this game fair?

# Expected Value

**Expected value** is the average gain (or loss) of an experiment if the procedure is repeated many times.

- We can compute the expected value by multiplying the gain from each outcome by the probability of that outcome, then adding up the products.
-

# Expected Value

**Expected value** is the average gain (or loss) of an experiment if the procedure is repeated many times.

- We can compute the expected value by multiplying the gain from each outcome by the probability of that outcome, then adding up the products.
- Alternatively, group the outcomes into events (usually win and lose) and multiply the gain from each event by the probability of that event, then adding up the products.

# Die Throwing Game

Someone presents the following game. They have a fair six-sided die. To play, you make a bet of \$1, and they throw the die. If they roll a 1, you win \$3. What is the expected value of this game?

We have two events, their payouts, and probabilities:

- 
- 

# Die Throwing Game

Someone presents the following game. They have a fair six-sided die. To play, you make a bet of $1, and they throw the die. If they roll a 1, you win $3. What is the expected value of this game?

We have two events, their payouts, and probabilities:

- "They roll a 1:"
- "They do not roll a 1:"

# Die Throwing Game

Someone presents the following game. They have a fair six-sided die. To play, you make a bet of $1, and they throw the die. If they roll a 1, you win $3. What is the expected value of this game?

We have two events, their payouts, and probabilities:

- "They roll a 1:" They pay you $3, but earlier, you gave them $1, so you end up gaining $2.
- "They do not roll a 1:"

# Die Throwing Game

Someone presents the following game. They have a fair six-sided die. To play, you make a bet of \$1, and they throw the die. If they roll a 1, you win \$3. What is the expected value of this game?

We have two events, their payouts, and probabilities:

- "They roll a 1:" They pay you \$3, but earlier, you gave them \$1, so you end up gaining \$2.

$$P(\text{"Gain \$2"}) = \frac{1 \text{ outcome}}{6 \text{ sides total}} = \frac{1}{6}.$$

- "They do not roll a 1:"

# Die Throwing Game

Someone presents the following game. They have a fair six-sided die. To play, you make a bet of $1, and they throw the die. If they roll a 1, you win $3. What is the expected value of this game?

We have two events, their payouts, and probabilities:

- "They roll a 1:" They pay you $3, but earlier, you gave them $1, so you end up gaining $2.

$$P(\text{"Gain \$2"}) = \frac{\text{1 outcome}}{\text{6 sides total}} = \frac{1}{6}.$$

- "They do not roll a 1:" Earlier, you gave them $1, so you lost $1, or gained $-\$1$.

# Die Throwing Game

Someone presents the following game. They have a fair six-sided die. To play, you make a bet of \$1, and they throw the die. If they roll a 1, you win \$3. What is the expected value of this game?

We have two events, their payouts, and probabilities:

- "They roll a 1:" They pay you \$3, but earlier, you gave them \$1, so you end up gaining \$2.

$$P(\text{"Gain \$2"}) = \frac{1 \text{ outcome}}{6 \text{ sides total}} = \frac{1}{6}.$$

- "They do not roll a 1:" Earlier, you gave them \$1, so you lost \$1, or gained $-\$1$.

$$P(\text{"Gain -\$1"}) = \frac{5 \text{ outcomes}}{6 \text{ sides total}} = \frac{5}{6}.$$

- "They roll a 1:" They pay you \$3, but earlier, you gave them \$1, so you end up gaining \$2.

$$P(\text{"Gain \$2"}) = \frac{\text{1 outcome}}{\text{6 sides total}} = \frac{1}{6}.$$

- "They do not roll a 1:" Earlier, you gave them \$1, so you lost \$1, or gained $-\$1$.

$$P(\text{"Gain -\$1"}) = \frac{\text{5 outcomes}}{\text{6 sides total}} = \frac{5}{6}.$$

Multiply each gain by its probability, and add to get your average winnings:

$$(\$2) \cdot \frac{1}{6} + (-\$1) \cdot \frac{5}{6}$$

- "They roll a 1:" They pay you \$3, but earlier, you gave them \$1, so you end up gaining \$2.

$$P(\text{"Gain \$2"}) = \frac{\text{1 outcome}}{\text{6 sides total}} = \frac{1}{6}.$$

- "They do not roll a 1:" Earlier, you gave them \$1, so you lost \$1, or gained $-\$1$.

$$P(\text{"Gain -\$1"}) = \frac{\text{5 outcomes}}{\text{6 sides total}} = \frac{5}{6}.$$

Multiply each gain by its probability, and add to get your average winnings:

$$(\$2) \cdot \frac{1}{6} + (-\$1) \cdot \frac{5}{6} = \frac{2}{6} - \frac{5}{6}$$

- "They roll a 1:" They pay you \$3, but earlier, you gave them \$1, so you end up gaining \$2.

$$P(\text{"Gain \$2"}) = \frac{\text{1 outcome}}{\text{6 sides total}} = \frac{1}{6}.$$

- "They do not roll a 1:" Earlier, you gave them \$1, so you lost \$1, or gained $-\$1$.

$$P(\text{"Gain -\$1"}) = \frac{\text{5 outcomes}}{\text{6 sides total}} = \frac{5}{6}.$$

Multiply each gain by its probability, and add to get your average winnings:

$$(\$2)\cdot\frac{1}{6} + (-\$1)\cdot\frac{5}{6} = \frac{2}{6} - \frac{5}{6} = \boxed{-\frac{3}{6}} = -\$0.50.$$

You should expect to lose \$0.50 (50 cents) per game.

# Die Throwing Game

Our expected value from the die throwing game is $-\$0.50$.

- This does not mean you will lose exactly \$0.50 on a single game.
  Rather, your **average loss** is likely to be near \$0.50 per game.
-

# Die Throwing Game

Our expected value from the die throwing game is $-\$0.50$.

- This does not mean you will lose exactly \$0.50 on a single game.
  Rather, your **average loss** is likely to be near \$0.50 per game.
- If you played the game, say, 10 times, you should expect to end up with

$$(-\$0.50) \cdot (10) = -\$5.$$

  You will lose most of the time, and win a few times, and most likely end up losing \$5.

# Raffle Prize

You purchase one raffle ticket for \$10. The charity sold 1,000 tickets, and one of them will be drawn. The winner receives a \$4,000 prize. Let us compute the expected value. We have two events: "win" and "lose."

-

# Raffle Prize

You purchase one raffle ticket for \$10. The charity sold 1,000 tickets, and one of them will be drawn. The winner receives a \$4,000 prize. Let us compute the expected value. We have two events: “win” and “lose.”

- Win: You receive \$4,000, but you paid \$10, so you gain \$3,990.

# Raffle Prize

You purchase one raffle ticket for \$10. The charity sold 1,000 tickets, and one of them will be drawn. The winner receives a \$4,000 prize. Let us compute the expected value. We have two events: "win" and "lose."

- Win: You receive \$4,000, but you paid \$10, so you gain \$3,990.
  To win, your ticket must be drawn:

$$P(Win) = \frac{\text{1 ticket that is yours}}{\text{1,000 tickets total}} = \frac{1}{1,000}.$$

# Raffle Prize

You purchase one raffle ticket for \$10. The charity sold 1,000 tickets, and one of them will be drawn. The winner receives a \$4,000 prize. Let us compute the expected value. We have two events: "win" and "lose."

- Win: You receive \$4,000, but you paid \$10, so you gain \$3,990.
  To win, your ticket must be drawn:

$$P(Win) = \frac{\text{1 ticket that is yours}}{\text{1,000 tickets total}} = \frac{1}{1,000}.$$

If you lose, how much money do you gain?

a) 0

b) \$10 (gain \$10)

c) $-$\$10 (lose \$10)

# Raffle Prize

You purchase one raffle ticket for \$10. The charity sold 1,000 tickets, and one of them will be drawn. The winner receives a \$4,000 prize. Let us compute the expected value. We have two events: "win" and "lose."

- Win: You receive \$4,000, but you paid \$10, so you gain \$3,990.

  To win, your ticket must be drawn:

$$P(Win) = \frac{\text{1 ticket that is yours}}{\text{1,000 tickets total}} = \frac{1}{1,000}.$$

- Lose: You receive nothing, and paid \$10, so you gain $-\$10$.

What is the probability of losing?

# Raffle Prize

- Win: You receive \$4,000, but you paid \$10, so you gain \$3,990.
  To win, your ticket must be drawn:
  $$P(Win) = \frac{\text{1 ticket that is yours}}{\text{1,000 tickets total}} = \frac{1}{1,000}.$$
- Lose: You receive nothing, and paid \$10, so you gain $-\$10$.
  To lose, your ticket must **not** be drawn:
  $$P(Lose) = \frac{\text{999 other tickets}}{\text{1,000 tickets total}} = \frac{999}{1,000},$$

What is the expected value?

- Win: gain \$3,990.
  To win, your ticket must be drawn:
  $$P(Win) = \frac{\text{1 ticket that is yours}}{\text{1,000 tickets total}} = \frac{1}{1,000}.$$
- Lose: gain $-\$10$.
  To lose, your ticket must **not** be drawn:
  $$P(Lose) = \frac{\text{999 other tickets}}{1,000 \text{ tickets total}} = \frac{999}{1,000},$$

Multiply each gain by its probability, and add to get the expected value:

$$\begin{aligned}(\$3,990)\frac{1}{1,000} + (-\$10)\frac{999}{1,000} &= \frac{3,990}{1,000} - \frac{9,990}{1,000} \\ &= -\frac{6,000}{1,000} \\ &= \boxed{-\$6}.\end{aligned}$$

# Die Throwing Game (2)

Someone presents the following game. They have a fair six-sided die. To play, you make a bet of \$1, and they throw the die. If they roll a 1, you win \$3. If they roll an even number, you win \$1. What is the expected value of this game? Note: there are three events: win big, win little, and lose.

# Die Throwing Game (2)

We have **3** events:

- Win big: You receive \$3, but you paid \$1, so you gain \$2. To win big, a 1 is rolled:

$$P(\text{Win big}) = \frac{1 \text{ outcome}}{6 \text{ sides}} = \frac{1}{6}.$$

# Die Throwing Game (2)

We have **3** events:

- Win little: You receive \$1, but you paid \$1, so you gain \$0.
  To win little, a 2, 4, or 6 is rolled. This describes 3 outcomes:

$$P(\text{Win little}) = \frac{3 \text{ outcomes}}{6 \text{ sides}} = \frac{3}{6}.$$

# Die Throwing Game (2)

We have **3** events:

- Lose: You receive \$0, and paid \$1, so you gain $-\$1$.
  To lose, a 3 or 5 (2 outcomes) is rolled:

$$P(\text{Lose}) = \frac{2 \text{ outcomes}}{6 \text{ sides}} = \frac{2}{6}.$$

# Die Throwing Game (2)

We have **3** events:

- Win big: gain \$2, probability 1/6
- Win little: gain \$0, probability 3/6
- Lose: gain −\$1, probability 2/6

Multiply each gain by its probability, and add to get the expected value:

$$\begin{aligned}(\$2)\frac{1}{6}+(\$0)\frac{3}{6}+(-\$1)\frac{2}{6} &= \frac{2}{6}+0-\frac{2}{6}\\ &= \boxed{\$0}.\end{aligned}$$

# Fair Games Remarks

- A game is fair if its expected value is **zero**. On average, neither side gains or loses money.
- If the expected value is negative, the player will lose money on average.
- Most lotteries and casino games have negative expected values (they want to make money off of the players).

# Insurance and Expected Value

A driver is shopping for auto insurance. The probability that they will have an accident in the next year is $3/100$. An insurance company charges \$800 for a policy that pays \$25,000 in the event of an accident. What is the expected value for the driver?
Note: you should get an integer (positive or negative whole number).

# Insurance and Expected Value

A driver is shopping for auto insurance. The probability that they will have an accident in the next year is 3/100. An insurance company charges \$800 for a policy that pays \$25,000 in the event of an accident. What is the expected value for the driver?

We have **2** events:

- Accident: Driver receives \$25,000, but paid \$800. Gains

$$25,000 - 800 = 24,200$$

with probability 3/100.

# Insurance and Expected Value

We have **2** events:

- Accident: Driver receives \$25,000, but paid \$800. Gains

$$25,000 - 800 = 24,200$$

  with probability 3/100.

- No accident: Driver receives \$0, paid \$800, so gains $-$\$800. Use complements to find probability:

$$\begin{aligned} P(\text{No accident}) &= 1 - P(\text{Accident}) \\ &= 1 - \frac{3}{100} \\ &= \frac{100}{100} - \frac{3}{100} = \frac{97}{100}. \end{aligned}$$

# Insurance and Expected Value

- Accident: Gains \$24,200 with probability 3/100.
- No accident: Gains $-$\$800 with probability 97/100.

Expected value:

$$\begin{aligned}(24,200)\frac{3}{100}+(-800)\frac{97}{100} &= \frac{72,600}{100}-\frac{77,600}{100}\\ &= \frac{-5,000}{100}=\boxed{\text{-\$50.}}\end{aligned}$$

Typically, insurance policies have a negative expected value for the driver (buyer), but the added security of having a policy may make the cost worth it.