Intro to Contemporary Math

Condorcet Criterion

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Agenda and Announcements

- Condorcet Criterion
- Testing Voting Methods
- A new homework assignment will be available on Wednesday. It is due next Monday.

Condorcet Criterion (CO Criterion):

A candidate who **always beats their opponent in their individual comparisons** (Condorcet candidate) **should win.**

- Even if we are not using Pairwise Comparison, we can still do the individual comparisons and look for someone who wins all of theirs
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A voting method:

- Always satisfies the CO criterion if any election with a Condorcet candidate ends in that candidate winning
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- Even if we are not using Pairwise Comparison, we can still do the individual comparisons and look for someone who wins all of theirs
- The CO criterion **does not** demand that every election have a Condorcet candidate.

A voting method:

- Always satisfies the CO criterion if any election with a Condorcet candidate ends in that candidate winning
- Can violate the CO criterion if we can find (or make) an election with a Condorcet candidate who loses

Testing for CO Violation

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Outcomes/Results:

- **Inconclusive:** No Condorcet candidate, or Condorcet candidate wins (with the voting method)
- **Violation:** There is a Condorcet candidate, but the voting method picks someone else as the winner.

Testing for CO Violation Outcomes

Outcomes/Results:

- **Inconclusive:** No Condorcet candidate, or Condorcet candidate wins (with the voting method)
- **Violation:** There is a Condorcet candidate, but the voting method picks someone else as the winner.

If a voting method can violate the CO criterion, it means that a Condorcet candidate **can** lose with the voting method.

Testing for CO Satisfaction

To test if a voting method satisfies (passes) CO, we must:
1) Study the **rules** of the voting method
2) Determine if a Condorcet candidate is guaranteed to win under the rules
If a Condorcet candidate is **guaranteed to win**, the voting method **satisfies the CO criterion.**

Today's Lecture

We will identify a Condorcet candidate **using comparisons**, then see how well **Plurality** and **Pairwise** comparison comply with CO.
We will use this election for both voting methods:

6	2	4	3
A	B	C	C
B	A	A	B
C	C	B	A

?(7.1) A vs. B

6	2	4	3
A	B	C	C
B	A	A	B
C	C	B	A

Press the first letter of the color that wins the A vs. B comparison, or type “Tie” if there is a tie.

A vs. B

6	2	4	3
A	B	C	C
B	A	A	B
C	C	B	A

A's camp has $6+4=10$ voters

B's camp has $2+3=5$ voters

A wins.

?(7.2) A vs. C

6	2	4	3
A	B	C	C
B	A	A	B
C	C	B	A

Press the first letter of the color that wins the A vs. C comparison, or type “Tie” if there is a tie.

A vs. C

6	2	4	3
A	B	C	C
B	A	A	B
C	C	B	A

A's camp has $6+2=8$ voters

C's camp has $4+3=7$ voters

A wins.

?(7.3) B vs. C

6	2	4	3
A	B	C	C
B	A	A	B
C	C	B	A

Press the first letter of the color that wins the B vs. C comparison, or type “Tie” if there is a tie.

B vs. C

6	2	4	3
A	B	C	C
B	A	A	B
C	C	B	A

B's camp has $6+2=8$ voters
C's camp has $4+3=7$ voters
B wins.

Condorcet Candidate

A vs. B	A wins
A vs. C	A wins
B vs. C	B wins

Which color is the Condorcet candidate? Type a letter.

Condorcet Candidate

A vs. B	A wins
A vs. C	A wins
B vs. C	B wins

Azure is the Condorcet Candidate.

- It won A vs. B, and A vs. C, all the comparisons it was in
- It got 2 points in an election with 3 candidates (need $3-1=2$ to be Condorcet candidate)

Now let's see which color(s) Plurality and Pairwise Comparison choose as their winner(s).

?(7.4) Plurality Winner

6	2	4	3
A	B	C	C
B	A	A	B
C	C	B	A

Quick! **Press the first letter of the color that wins** with the Plurality method.

Plurality Winner

6	2	4	3
A	B	C	C
B	A	A	B
C	C	B	A

Cobalt has the most 1st place votes (7), so it wins with Plurality. Watch for first place votes from multiple columns (groups of voters)!

?(7.5) CO Testing: Plurality

- Plurality winner: Cobalt
- Condorcet candidate: Azure

Does this election show:
1) Plurality can violate CO
2) Plurality always satisfies CO
3) an inconclusive result

CO Testing: Plurality

Violation:
Azure is the Condorcet candidate, but the Plurality method **picked a different candidate**, Cobalt, so it **did not pick the Condorcet candidate**.

?(7.6) Pairwise Comparison Winner

From earlier:

A vs. B	A wins
A vs. C	A wins
B vs. C	B wins

Press the first letter of the color that wins with Pairwise Comparison.

Pairwise Comparison Winner

A vs. B	A wins
A vs. C	A wins
B vs. C	B wins

A has 2 points (2 wins), B has 1 point, C has no points.
A wins with the **most points**.

?(7.7) CO Testing: Pairwise Comparison

- Pairwise winner: Azure
- Condorcet candidate: Azure

Does this election show:
1) Pairwise comparison can violate CO
2) Pairwise comparison always satisfies CO
3) an inconclusive result

CO Testing: Pairwise Comparison

Inconclusive!

Just because the Condorcet candidate won this election with

Pairwise comparison does not tell us if they always win!

Pairwise comparison obeyed CO this time, but will it disobey in another election? Let's **look at the rules.**

?(7.8) CO and Pairwise Comparison

Pairwise comparison rules: Whoever gets the most points from winning or tying comparisons wins the election

- If an election has a Condorcet candidate, will they always win with Pairwise Comparison?
  Type **YES**, or **NO**.

CO and Pairwise Comparison

Yes.
To win with Pairwise Comparison, a candidate must win the most individual comparisons. A Condorcet candidate **wins all** of its comparisons, **so it definitely won the most**.
Thus, The Pairwise Comparison method **always satisfies** the Condorcet criterion.

The MA and CO Criteria

We saw on Friday that a majority candidate is also a Condorcet candidate. Thus,

- If a voting method violates MA, then it also violates CO
- If a voting method satisfies CO, then it also satisfies MA

Note: It is possible to have a Condorcet candidate which is not a majority candidate.

@Home: Borda Count and PwE

Please read this example for testing if Borda Count and Plurality with Elimination can violate the Condorcet Criterion at home.
We will use this election for both voting methods:

2	7	7	9	4
Azure	Denim	Blue	Cobalt	Azure
Blue	Azure	Azure	Denim	Blue
Cobalt	Blue	Denim	Azure	Denim
Denim	Cobalt	Cobalt	Blue	Cobalt

A vs. B

2	7	7	9	4
Azure	Denim	Blue	Cobalt	Azure
Blue	Azure	Azure	Denim	Blue
Cobalt	Blue	Denim	Azure	Denim
Denim	Cobalt	Cobalt	Blue	Cobalt

Which color wins A vs. B?

A vs. B

2	7	7	9	4
Azure	Denim	Blue	Cobalt	Azure
Blue	Azure	Azure	Denim	Blue
Cobalt	Blue	Denim	Azure	Denim
Denim	Cobalt	Cobalt	Blue	Cobalt

A's camp has 2+7+9+4=22 voters

B's camp has 7 voters

A wins.

A vs. C

2	7	7	9	4
Azure	Denim	Blue	Cobalt	Azure
Blue	Azure	Azure	Denim	Blue
Cobalt	Blue	Denim	Azure	Denim
Denim	Cobalt	Cobalt	Blue	Cobalt

Which color wins A vs. C?

A vs. C

2	7	7	9	4
Azure	Denim	Blue	Cobalt	Azure
Blue	Azure	Azure	Denim	Blue
Cobalt	Blue	Denim	Azure	Denim
Denim	Cobalt	Cobalt	Blue	Cobalt

A's camp has 2+7+7+4=20 voters
C's camp has 9 voters
A wins.

A vs. D

2	7	7	9	4
Azure	Denim	Blue	Cobalt	Azure
Blue	Azure	Azure	Denim	Blue
Cobalt	Blue	Denim	Azure	Denim
Denim	Cobalt	Cobalt	Blue	Cobalt

Which color wins A vs. D?

A vs. D

2	7	7	9	4
Azure	Denim	Blue	Cobalt	Azure
Blue	Azure	Azure	Denim	Blue
Cobalt	Blue	Denim	Azure	Denim
Denim	Cobalt	Cobalt	Blue	Cobalt

A's camp has 2+7+4=13 voters

D's camp has 7+9=16 voters

D wins.

B vs. C

2	7	7	9	4
Azure	Denim	Blue	Cobalt	Azure
Blue	Azure	Azure	Denim	Blue
Cobalt	Blue	Denim	Azure	Denim
Denim	Cobalt	Cobalt	Blue	Cobalt

Which color wins B vs. C?

B vs. C

2	7	7	9	4
Azure	Denim	Blue	Cobalt	Azure
Blue	Azure	Azure	Denim	Blue
Cobalt	Blue	Denim	Azure	Denim
Denim	Cobalt	Cobalt	Blue	Cobalt

B's camp has 2+7+7+4=20 voters
C's camp has 9 voters
B wins.

B vs. D

2	7	7	9	4
Azure	Denim	Blue	Cobalt	Azure
Blue	Azure	Azure	Denim	Blue
Cobalt	Blue	Denim	Azure	Denim
Denim	Cobalt	Cobalt	Blue	Cobalt

Which color wins B vs. D?

B vs. D

2	7	7	9	4
Azure	Denim	Blue	Cobalt	Azure
Blue	Azure	Azure	Denim	Blue
Cobalt	Blue	Denim	Azure	Denim
Denim	Cobalt	Cobalt	Blue	Cobalt

B's camp has 2+7+4=13 voters

D's camp has 7+9=16 voters

D wins.

C vs. D

2	7	7	9	4
Azure	Denim	Blue	Cobalt	Azure
Blue	Azure	Azure	Denim	Blue
Cobalt	Blue	Denim	Azure	Denim
Denim	Cobalt	Cobalt	Blue	Cobalt

Which color wins C vs. D?

C vs. D

2	7	7	9	4
Azure	Denim	Blue	Cobalt	Azure
Blue	Azure	Azure	Denim	Blue
Cobalt	Blue	Denim	Azure	Denim
Denim	Cobalt	Cobalt	Blue	Cobalt

C's camp has 2+9=11 voters

D's camp has 7+7+4=18 voters

D wins.

Condorcet Candidate

A vs. B	A wins
A vs. C	A wins
A vs. D	D wins

B vs. C	B wins
B vs. D	D wins
C vs. D	D wins

Which color is the Condorcet candidate?

Condorcet Candidate

A vs. B	A wins
A vs. C	A wins
A vs. D	D wins

B vs. C	B wins
B vs. D	D wins
C vs. D	D wins

Denim:

- It won A vs. D, B vs. D, and C vs. D, all the comparisons it was in
- It got 3 points in an election with 4 candidates (need 4−1=3 to be Condorcet candidate)

Now let's see which color(s) Borda Count and Plurality with Elimination choose as their winner(s).

Borda Count Winner

2	7	7	9	4
Azure	Denim	Blue	Cobalt	Azure
Blue	Azure	Azure	Denim	Blue
Cobalt	Blue	Denim	Azure	Denim
Denim	Cobalt	Cobalt	Blue	Cobalt

Compute each color's Borda point total. **Which color that wins** with the Borda Count method?

Borda Count Winner

2	7	7	9	4	
Azure	Denim	Blue	Cobalt	Azure	4 points
Blue	Azure	Azure	Denim	Blue	3 points
Cobalt	Blue	Denim	Azure	Denim	2 points
Denim	Cobalt	Cobalt	Blue	Cobalt	1 point

- Azure: $2(4)+7(3)+7(3)+9(2)+4(4)=84$
- Blue: $2(3)+7(2)+7(4)+9(1)+4(3)=69$
- Cobalt: $2(2)+7(1)+7(1)+9(4)+4(1)=58$
- Denim: $2(1)+7(4)+7(2)+9(3)+4(2)=79$
- Azure has the most points, so it wins with Borda Count.

CO Testing: Borda Count

- Borda winner: Azure
- Condorcet candidate: Denim

Does this election show:
1) Borda count can violate CO
2) Borda count always satisfies CO
3) an inconclusive result

CO Testing: Borda Count

Violation:
Denim is the Condorcet candidate, but the Borda count method picked Azure (and did not pick the Condorcet candidate).

PwE Votes to Win

2	7	7	9	4
Azure	Denim	Blue	Cobalt	Azure
Blue	Azure	Azure	Denim	Blue
Cobalt	Blue	Denim	Azure	Denim
Denim	Cobalt	Cobalt	Blue	Cobalt

What is the minimum number of first place votes needed to win in PwE?

PwE Votes to Win

2	7	7	9	4
Azure	Denim	Blue	Cobalt	Azure
Blue	Azure	Azure	Denim	Blue
Cobalt	Blue	Denim	Azure	Denim
Denim	Cobalt	Cobalt	Blue	Cobalt

A candidate needs over 50% of the first place votes to win.
There are 29 voters.
Add one to get 30.
Divide by 2 to get 15, which rounds up to 15 (no decimal).
Need 15 out of 29 first place votes to win.

PwE Winner Round 1

2	7	7	9	4
Azure	Denim	Blue	Cobalt	Azure
Blue	Azure	Azure	Denim	Blue
Cobalt	Blue	Denim	Azure	Denim
Denim	Cobalt	Cobalt	Blue	Cobalt

Need 15 out of 29 first place votes to win.

What color that gets eliminated first?

PwE Winner Round 1

Azure: only 6 first place votes

2	7	7	9	4
Azure	Denim	Blue	Cobalt	Azure
Blue	Azure	Azure	Denim	Blue
Cobalt	Blue	Denim	Azure	Denim
Denim	Cobalt	Cobalt	Blue	Cobalt

PwE Winner Round 1

2	7	7	9	4
	Denim	Blue	Cobalt	
Blue			Denim	Blue
Cobalt	Blue	Denim		Denim
Denim	Cobalt	Cobalt	Blue	Cobalt

PwE Winner Round 2

2	7	7	9	4
Blue	Denim	Blue	Cobalt	Blue
Cobalt	Blue	Denim	Denim	Denim
Denim	Cobalt	Cobalt	Blue	Cobalt

Which color is eliminated next?

PwE Winner Round 2

Denim: only 7 first place votes

2	7	7	9	4
Blue	Denim	Blue	Cobalt	Blue
Cobalt	Blue	Denim	Denim	Denim
Denim	Cobalt	Cobalt	Blue	Cobalt

PwE Winner Round 2

2	7	7	9	4
Blue		Blue	Cobalt	Blue
Cobalt	Blue			
	Cobalt	Cobalt	Blue	Cobalt

PwE Winner Decision

2	7	7	9	4
Blue	Blue	Blue	Cobalt	Blue
Cobalt	Cobalt	Cobalt	Blue	Cobalt

Which color is the winner?

PwE Winner Decision

2	7	7	9	4
Blue	Blue	Blue	Cobalt	Blue
Cobalt	Cobalt	Cobalt	Blue	Cobalt

Blue has 20 first place votes, over 50%. It wins with PwE.

CO Testing: PwE

- PwE winner: Blue
- Condorcet candidate: Denim

Does this election show:
1) PwE can violate CO
2) PwE always satisfies CO
3) an inconclusive result

CO Testing: PwE

Violation:
Denim is the Condorcet candidate, but PwE picked Blue (and did not pick the Condorcet candidate).

Next time

- We will introduce a fairness criterion about first place votes being good.
- The fourth homework assignment will be available on Wednesday. It is due next Monday.