Intro to Contemporary Math

Independence of Irrelevant Alternatives Criteria

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Agenda

- Independence of Irrelevant Alternatives Criteria
- Testing Voting Methods
- The Perfect Voting Method

Announcements

- The fourth homework assignment (HW3) is due next Monday
- Exam 1 is next Wednesday

Today's Criterion

Independence of Irrelevant Alternatives Criterion (IIA): After a winner is declared, if a losing candidate is removed (due to quitting or disqualification) and the election is done again without this candidate, the original winner should win this new election.

Testing for the IIA Criterion (IIA)

To test if a voting method violates (fails) IIA, we must:

1) Find an election using the voting method
2) Determine the winner using the voting method
3) Remove a **losing candidate**
4) Determine the winner of the modified election

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3) Remove a **losing candidate**
4) Determine the winner of the modified election

Outcomes:

- **Invalid:** You remove the original winner.
- **Inconclusive:** Original winner still wins.
  - Perhaps removing a different candidate may work.
- **Violation:** Original winner loses the modified election.

Testing for the IIA Criterion (IIA)

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- **Inconclusive:** Original winner still wins.
  - Perhaps removing a different candidate may work.
- **Violation:** Original winner loses the modified election.

Testing for the IIA Criterion (IIA)

To test if a voting method satisfies (passes) IIA, we must:
1) Study the rules of the voting method
2) Determine if a losing candidate getting removed can affect the performance of other candidates.
If removing losing candidates will never **change the outcome of an election**, the voting method **satisfies the IIA criterion.**

?(9.1) Plurality and IIA

2	3	4
Azure	Cobalt	Blue
Blue	Azure	Azure
Cobalt	Blue	Cobalt

Press the first letter of the color that wins with Plurality.

Plurality and IIA

2	3	4
Azure	Cobalt	Blue
Blue	Azure	Azure
Cobalt	Blue	Cobalt

Blue wins with the plurality method.

The IIA criterion says that if we **remove a losing candidate** and run the plurality method again, **Blue**, the original winner, **should still win**.

Plurality and IIA

Let us see what happens if Cobalt quits:

2	3	4
Azure		Blue
Blue	Azure	Azure
	Blue	

?(9.2) Plurality and IIA

Move up, fill in blanks:

2	3	4
Azure	Azure	Blue
Blue	Blue	Azure

Press the first letter of the color that wins with Plurality now.

?(9.3) Plurality and IIA

2	3	4
Azure	Azure	Blue
Blue	Blue	Azure

Azure now wins. **Blue loses**. This shows:

1) Plurality can violate IIA
2) Plurality always satisfies IIA
3) Nothing (Inconclusive)
4) Nothing (Invalid test)

Plurality and IIA

- This example shows that the plurality method **violates** the IIA criterion, because Blue won the original election, and lost after Cobalt's removal.

Plurality and IIA

What happens if we remove Blue?

2	3	4
Azure	Cobalt	Blue
Blue	Azure	Azure
Cobalt	Blue	Cobalt

?(9.4) Plurality and IIA

2	3	4
Azure	Cobalt	Azure
Cobalt	Azure	Cobalt

Press the first letter of the color that wins with Plurality now.

?(9.5) Plurality and IIA

2	3	4
Azure	Cobalt	Azure
Cobalt	Azure	Cobalt

Azure now wins. This shows:

1) Plurality can violate IIA
2) Plurality always satisfies IIA
3) Nothing (Inconclusive)
4) Nothing (Invalid test)

Plurality and IIA

Removing Blue will make Blue not win the modified election, but this example **does not show a violation** of the IIA criterion, because the original winner was removed, not one of the losing candidates. This is **an invalid example for IIA because we removed the winner**.

Plurality and IIA

What happens if we remove Azure?

2	3	4
Azure	Cobalt	Blue
Blue	Azure	Azure
Cobalt	Blue	Cobalt

?(9.6) Plurality and IIA

2	3	4
Blue	Cobalt	Blue
Cobalt	Blue	Cobalt

Press the first letter of the color that wins with Plurality now.

?(9.7) Plurality and IIA

2	3	4
Blue	Cobalt	Blue
Cobalt	Blue	Cobalt

Blue wins. This shows:

1) Plurality can violate IIA
2) Plurality always satisfies IIA
3) Nothing (Inconclusive)
4) Nothing (Invalid test)

Plurality and IIA

This test was **inconclusive** - it does not tell us if Plurality can violate IIA or always satisfy it. Blue won after the removal, so it is not a violation, but this alone does not tell us if Plurality always satisfies IIA. In fact, we saw Plurality violate IIA earlier with a different removal.

?(9.8) Borda Count and IIA

5	4
Azure	Blue
Blue	Cobalt
Cobalt	Azure

Press the first letter of the color that wins with Borda count.

Borda Count and IIA

5	4
Azure	Blue
Blue	Cobalt
Cobalt	Azure

- Azure gets $5(3)+4(1)=15+4=19$ points.
- Blue wins with $5(2)+4(3)=10+12=22$ points.
- Cobalt gets only $5(1)+4(2)=13$ points.

Borda Count and IIA

Cobalt is removed:

5	4
Azure	Blue
Blue	
	Azure

?(9.9) Borda Count and IIA

Cobalt is removed:

5	4
Azure	Blue
Blue	Azure

Press the first letter of the color that wins with Borda count.

Borda Count and IIA

5	4
Azure	Blue
Blue	Azure

- Azure gets $5(2)+4(1)=10+4=14$ points.
- Blue gets $5(1)+4(2)=5+8=13$ points.

Borda Count and IIA

Original:

- Azure gets $5(3)+4(1)=15+4=19$ points.
- Blue gets $5(2)+4(3)=10+12=22$ points.

Modified:

- Azure gets $5(2)+4(1)=10+4=14$ points.
- Blue gets $5(1)+4(2)=5+8=13$ points.

Borda Count and IIA

This example shows that the **Borda Count method violates the IIA criterion**, because Blue won the original election, and Azure won after Cobalt's removal.

?(9.10) Plurality with Elimination and IIA

7	8	6
Blue	Cobalt	Azure
Azure	Azure	Blue
Cobalt	Blue	Cobalt

Need 11 first place votes to win. Press the first letter of the color that gets eliminated.

Plurality with Elimination and IIA

Eliminate Azure

7	8	6
Blue	Cobalt	Azure
Azure	Azure	Blue
Cobalt	Blue	Cobalt

Plurality with Elimination and IIA

Eliminate Azure

7	8	6
Blue	Cobalt	
		Blue
Cobalt	Blue	Cobalt

?(9.11) Plurality with Elimination and IIA

7	8	6
Blue	Cobalt	Blue
Cobalt	Blue	Cobalt

Press the first letter of the color that won.

Plurality with Elimination and IIA

7	8	6
Blue	Cobalt	Blue
Cobalt	Blue	Cobalt

Blue won with 13 out of 21 first place votes.

Plurality with Elimination and IIA

Let's remove Cobalt.

7	8	6
Blue	Cobalt	Azure
Azure	Azure	Blue
Cobalt	Blue	Cobalt

Plurality with Elimination and IIA

Let's remove Cobalt.

7	8	6
Blue	Azure	Azure
Azure	Blue	Blue

Plurality with Elimination and IIA

7	8	6
Blue	Azure	Azure
Azure	Blue	Blue

Azure automatically wins with 14 first place votes. Blue loses.

Plurality with Elimination and IIA

This example shows that **PwE violates IIA!** Blue won the original election, and Azure won after Cobalt's removal.

Pairwise Comparison and IIA

Who wins with Pairwise Comparison?

5	5	6	4
D	A	C	B
A	C	B	D
C	B	D	A
B	D	A	C

Pairwise Comparison and IIA

5	5	6	4
D	A	C	B
A	C	B	D
C	B	D	A
B	D	A	C

- A vs B: 10 votes to 10 votes, A and B tie
- A vs C: 14 votes to 6 votes, A wins
- A vs D: 5 votes to 15 votes, D wins

Pairwise Comparison and IIA

5	5	6	4
D	A	C	B
A	C	B	D
C	B	D	A
B	D	A	C

- B vs C: 4 votes to 16 votes, C wins
- B vs D: 15 votes to 5 votes, B wins
- C vs D: 11 votes to 9 votes, C wins

Pairwise Comparison and IIA

- A vs B: 10 to 10, Tie
- A vs C: 14 to 6, A wins
- A vs D: 5 to 15, D wins
- B vs C: 4 to 16, C wins
- B vs D: 15 to 5, B wins
- C vs D: 11 to 9, C wins

- A gets $1\frac{1}{2}$ points
- B gets $1\frac{1}{2}$ points
- C wins with 2 points
- D gets 1 point

Pairwise Comparison and IIA

Remove D:

5	5	6	4
A	A	C	B
C	C	B	A
B	B	A	C

Pairwise Comparison and IIA

Remove D:

5	5	6	4
A	A	C	B
C	C	B	A
B	B	A	C

- A vs B: 10 votes to 10 votes, A and B tie
- A vs C: 14 votes to 6 votes, A wins
- B vs C: 4 votes to 16 votes, C wins

Pairwise Comparison and IIA

- A vs B: 10 votes to 10 votes, A and B tie
- A vs C: 14 votes to 6 votes, A wins
- B vs C: 4 votes to 16 votes, C wins
- Note that these are the same camp sizes from the previous election. The only difference is that the one-on-one comparisons involving D are removed.

Pairwise Comparison and IIA

- A vs B: 10 to 10, Tie
- A vs C: 14 to 6, A wins
- B vs C: 4 to 16, C wins

- A wins with $1\frac{1}{2}$ points
- B gets $\frac{1}{2}$ point
- C gets 1 point

Pairwise Comparison and IIA

- A vs B: 10 to 10, Tie
- A vs C: 14 to 6, A wins
- A vs D: 5 to 15, D wins
- B vs C: 4 to 16, C wins
- B vs D: 15 to 5, B wins
- C vs D: 11 to 9, C wins

- A vs B: 10 to 10, Tie
- A vs C: 14 to 6, A wins
- B vs C: 4 to 16, C wins

Pairwise Comparison and IIA

- **Pairwise Comparison violates IIA!** C won the original election, and A won after D was removed.

IIA Compliance Results

- Plurality: Violation
- Borda count: Violation
- PwE: Violation
- Pairwise comparison: Violation!

IIA and Preference Ballots

All four voting methods we have studied **require voters to rank candidates.** If someone leaves the election, **their loss can upset the rankings**, which can result in different outcomes.

Summary

Method	MA	CO	MO	IIA
Plurality	Satisfy	Violate	Satisfy	Violate
Borda	Violate	Violate	Satisfy	Violate
PwE.	Satisfy	Violate	Violate	Violate
Pairwise	Satisfy	Satisfy	Satisfy	Violate

?(9.12) The Perfect Method?

- Is there a voting method which satisfies:
  - Majority criterion
  - Condorcet criterion
  - Monotonicity criterion
  - IIA criterion

Press 1 for yes, 2 for no

The Perfect Method?

NO

No Perfect Method

Theorem (Arrow's Impossibility Theorem)

*It is **impossible** for a voting method to satisfy all four fairness criteria.*

Summary

- The whole point behind fairness criteria is to not only **state** what a fair election should be like, but allow us to **test** whether a voting method satisfies or violates the criterion.

Fairness Criteria Summary

- The majority criterion and Condorcet criterion involve candidates who are popular in some way:
  - The majority criterion wants a candidate **who has over 50% of the first place votes to win**.
  - The Condorcet criterion wants a candidate **who wins all their pairwise comparisons to win.**

Fairness Criteria Summary

- The monotonicity criterion wants **every first place vote to be helpful, or at least not harmful**.
- The IIA criterion wants to **keep losers who quit (or are otherwise removed) from affecting the election results**.

Next time

- We will review for the exam
- Homework 3 is due next Monday