Problems

MA310

Spring 1999

1 The Average Traveler

There are two towns, conveniently called "A" and "B" joined by a road. A businesswoman must drive from A to B, conduct some business (what else?) and return to A. She wants her total round trip speed to average 60 miles per hour, but she encountered heavy traffic going to B and averaged only 30 miles per hour on the first half of the trip. What should her average speed be on the return trip to meet her goal?

2 The Quadratic Formula

A *quadratic equation* is an equation of the form

$$ax^2 + bx + c = 0.$$

The solutions to a quadratic equation can be found using the *quadratic formula*:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

Use the quadratic formula to solve the following equations:

1. $2x^2 + 5x - 3 = 0.$
2. $9x^2 - 24x + 16 = 0.$
3. $5x^2 - 2x = 0.$
4. $4x^2 - 49 = 0.$
5. $x^2 + 1 = 0.$
6. $2x^2 - 6x + 17 = 0.$

Write down as many thoughts and questions that come to mind as you can.

3 The Foolish Fly

Two trains are traveling toward each other on the same track. Initially they are 120 miles apart. One is traveling 25 miles per hour, and the other is traveling 35 miles per hour. At the instant that they are 120 miles apart, a fly sitting on the front of the first train flies toward the second train at a speed of 40 miles per hour. The moment it reaches the second train, it reverses direction and flies back toward the first train. Again it reverses direction, and continues to fly back and forth between the two trains until the trains meet. How far does the fly fly?

4 The Early Arrival

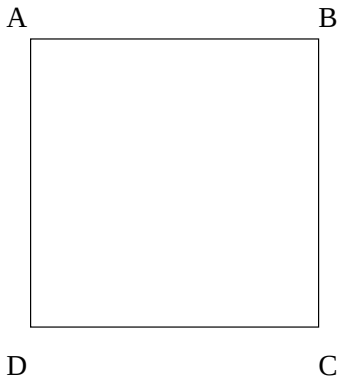
A commuter is in the habit of arriving at his suburban station each evening exactly at five o'clock. His wife always meets the train and drives him home. One day he takes an earlier train, arriving at the station at four. The weather is pleasant, so instead of telephoning home he starts walking along the route always taken by his wife. They meet somewhere on the way. He gets into the car and they drive home, arriving at their house ten minutes earlier than usual. Assuming that the wife always drives at a constant speed, and that on this occasion she left just in time to meet the five o'clock train, can you determine how long the husband walked before he was picked up?

5 Passing Boats

Two ferryboats start at the same instant from opposite sides of a river, traveling across the water on routes at right angles to the shores. Each travels at a constant speed, but one is faster than the other. They pass at a point 720 yards from the nearest shore. Both boats remain in their slips for 10 minutes before starting back. On the return trips they meet 400 yards from the other shore. How wide is the river?

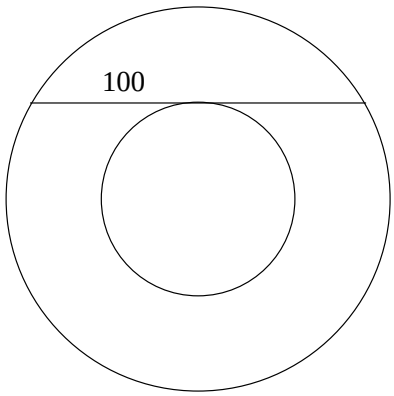
6 Crawling Insects

Four bugs—A, B, C and D—occupy the corners of a square 10 inches on a side. A and C are male and are located at opposite corners. B and D are female, and are located at the two remaining corners. Simultaneously A crawls directly toward B, B toward C, C toward D and D toward A. If all four bugs crawl at the same constant rate, they will describe four congruent logarithmic spirals which meet at the center of the square. How far does each bug travel before they meet? The problem can be solved without calculus.



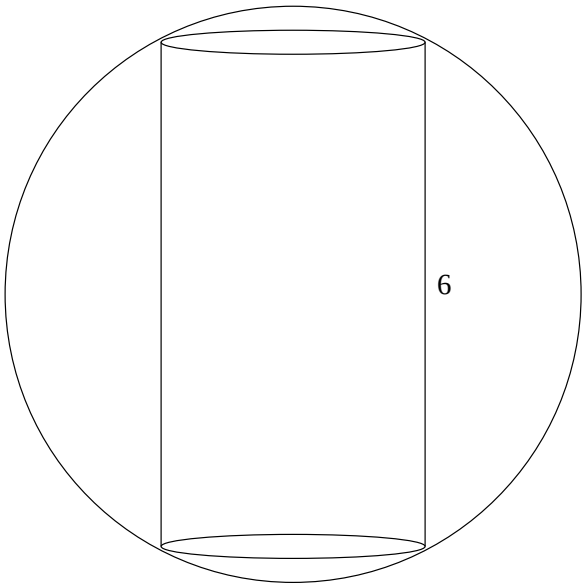
7 Circular Track

A race track is in the shape of the region between two concentric circles, one larger than the other. Suppose that a chord of the larger circle that is tangent to the smaller circle has length 100 meters. What is the area of the track?



8 The Bead Problem

A solid ball is made into a bead by drilling a cylindrical hole straight through its center. After the hole has been drilled, the length of the cylindrical hole (which is smaller than the diameter of the original ball) is six inches. What is the volume remaining in the ball?



9 Colored Hats

Three students — Alfred, Beth, and Carla — are blindfolded and told that either a red or a green hat will be placed on each of them. After this is done, the blindfolds are removed; the students are asked to raise a hand if they see a red hat, and to leave the room as soon as they are sure of the color of their own hat. All three hats happen to be red, so all three students raise a hand. Several minutes go by until Carla, who is more astute than the others, leaves the room. How did she deduce the color of her hat?

10 Consecutive Integers

Two students, A and B, are chosen from a math class of highly logical individuals. They are each given one positive integer. Each knows his/her own number, and is trying to determine the other's number. They are informed that their numbers are consecutive. In each of the following scenarios, what can you deduce about the two numbers?

1. First Scenario

   A: I know your number.

   B: I know your number.

2. Second Scenario

   A: I don't know your number.

   B: I know your number.

   A: I know your number.

3. Third Scenario

   A: I don't know your number.

   B: I don't know your number.

   A: I know your number.

   B: I know your number.

4. Fourth Scenario

   A: I don't know your number.

   B: I don't know your number.

   A: I don't know your number.

   B: I don't know your number.

   A: I know your number.

   B: I know your number.

11 Numerous Offspring

"I hear some youngsters playing in the back yard," said Jones, a graduate student in mathematics. "Are they all yours?"

"Heavens, no," exclaimed Professor Smith, the eminent number theorist. "My children are playing with friends from three other families in the neighborhood, although our family happens to be largest. The Browns have a smaller number of children, the Greens have a still smaller number, and the Blacks the smallest of all."

"How many children are there altogether?" asked Jones.

"Let me put it this way," said Smith. "There are fewer than 18 children, and the product of the numbers in the four families happens to be my house number which you saw when you arrived."

Jones took a notebook and pencil from his pocket and started scribbling. A moment later he looked up and said, "I need more information. Is there more than one child in the Black family?"

As soon as Smith replied, Jones smiled and correctly stated the number of children in each family.

Knowing the house number and whether the Blacks had more than one child, Jones found the problem trivial. it is a remarkable fact, however, that the number of children in each family can be determined solely on the basis of the information given above!

12 SETI

Suppose you want to send a message to extraterrestrials that stands a reasonable chance of being understood, even though it is quite unlikely that any recipient is familiar with any human language. One way is to exploit the assumption of the universality of arithmetic. (How reasonable is this assumption? I suggest reading: R. W. Hamming, Mathematics on a distant planet, *American Mathematical Monthly* **105** (1998) 640–650.) The following proposed message to extraterrestrials was devised by Ivan Bell in 1960.

Letters from A through Z (omitting O and X) provide the 24 symbols. (Each symbol is presumably radioed by a combination of beeps, but we need not be concerned with those details.) The punctuation marks are not part of the message but indications of time lapses. Adjacent letters are sent with short pauses between them. A space between letters means a longer pause. Commas, semicolons, and periods represent progressively longer pauses. The longest time lapses come between paragraphs, which are numbered for the reader's convenience; the numbers are not part of the message. To minds in any solar system the message should be crystal clear except for the last paragraph, which is somewhat ambiguous; even if properly deciphered, it could be understood fully only by inhabitants of one of our solar system's planets.

1. A. B. C. D. E. F. G. H. I. J. K. L. M. N. P. Q. R. S. T. U. V. W. Y. Z.

2. A A, B; A A A, C; A A A A, D; A A A A A, E; A A A A A A, F; A A A A A A A, G; A A A A A A A A, H; A A A A A A A A A, I; A A A A A A A A A A, J.

3. A K A L B; A K A K A L C; A K A K A K A L D. A K A L B; B K A L C; C K A L D; D K A L E. B K E L G; G L E K B. F K D L J; J L F K D.

4. C M A L B; D M A L C; I M G L B.

5. C K N L C; H K N L H. D M D L N; E M E L N.

6. J L AN; J K A L AA; J K B L AB; AA K A L AB. J K J L BN; J K J K J L CN. FN K G L FG.

7. B P C L F; E P B L J; F P J L FN.

8. F Q B L C; J Q B L E; FN Q F L J.

9. C R B L I; B R E L CB.

10. J P J L J R B L S L ANN; J P J P J L J R C L T L ANNN. J P S L T; J P T L J R D.

11. A Q J L U; U Q J L A Q S L V.

12. U L WA; U P B L WB; AWD M A L WD L D P U. V L WNA; V P C L WNC. V Q J L WNNA; V Q S L WNNNA. J P EWFGH L EFWGH; S P EWFGH L EFGWH.

13. GIWIH Y HN; T K C Y T. Z Y CWADAF.

14. D P Z P WNNIB R C Q C.

13 Colorful Triangles and Squares

With an unlimited supply of toothpicks of n different colors, how many different triangles can be formed on a flat surface, using three toothpicks for the three sides of each triangle? (Reflections are considered different, but not rotations.)

How many different squares?

14 Colorful Tetrahedra and Cubes

How many different tetrahedra can be produced by coloring each face a solid color and using n different colors? (Two tetrahedra are the same if they can be turned and placed side by side so that corresponding sides match in color.)

How many cubes with n colors?

15 Slicing Shapes

1. What is the maximum number of pieces into which a pancake can be cut by n straight cuts, each of which crosses each of the others?

2. What is the maximum number of pieces that can be produced by n simultaneous straight cuts of a flat figure shaped like a crescent moon?

3. Into how many parts can the plane be divided by intersecting circles of the same size? Of different sizes? By intersecting ellipses of different sizes?

4. How many pieces of cheesecake can be produced by n simultaneous plane cuts of a cylindrical cake?

5. Into how many regions can space be divided by intersecting spheres?

6. What is the maximum number of pieces that can be produced by n simultaneous plane cuts through a doughnut?

16 Some Summations

1. Guess and prove a formula for the sum of the first n positive integers.
2. Guess and prove a formula for the sum of the first n positive odd integers.
3. Guess and prove a formula for the sum of the squares of the first n positive integers.
4. Guess and prove a formula for the sum of the cubes of the first n positive integers.
5. Guess and prove a formula for the sum of the fourth powers of the first n positive integers.

17 Fibonacci Formula

The Fibonacci numbers are defined by $F_1 = F_2 = 1$ and thereafter $F_n = F_{n-2} + F_{n-1}$, $n = 3, 4, 5, \ldots$. So the first 8 Fibonacci numbers are: 1, 1, 2, 3, 5, 8, 13, 21. Prove the formula:

$$F_n = \frac{1}{\sqrt{5}} \left[\left(\frac{1+\sqrt{5}}{2} \right)^n - \left(\frac{1-\sqrt{5}}{2} \right)^n \right].$$

18 Some Counting Formulas

1. Find and prove a formula for the number of subsets of the set $\{1, 2, 3, \ldots, n\}$.

2. For nonnegative integers n and k, define $\binom{n}{k}$ to be the number of subsets of size k contained in a set of size n. Prove that

$$\binom{n}{k} + \binom{n}{k+1} = \binom{n+1}{k+1}.$$

   Then prove that

$$\binom{n}{k} = \frac{n!}{k!(n-k)!},$$

   remembering that we define $0! = 1$.

3. Prove the binomial formula:

$$(a+b)^n = \sum_{i=0}^{n} \binom{n}{i} a^{n-i} b^i.$$

19 Pairing Off People

1. Find and prove a formula for the number of ways n different men can be completely paired up with n different women.

2. A group of $2n$ individuals are to be paired off into n pairs to play checkers. Find and prove a formula for the number of ways to do this.

20 Towers of Hanoi

The Towers of Hanoi puzzle consists of three posts and n disks of different sizes that are initially stacked on one of the posts, from largest to smallest with the largest disk on the bottom. The goal is to move this stack of disks to a different post by a sequence of moves. Each move consists of moving the disk from top of the pile on any post to the top of the pile on any other post, as long as a larger disk is never placed directly on a smaller one. Try to solve this puzzle with $n = 7$. Can you find a general procedure for n disks? How many moves are required?

21 Polygonal Ears

A convex polygon with n sides can be dissected (triangulated) into triangles by drawing a certain number of non-crossing diagonals.

1. How many diagonals are needed?

2. How many triangles result?

3. A triangle in the triangulation consisting of two consecutive edges of the boundary together with one internal diagonal is called an ear of the triangulation. Prove that if $n \geq 4$ then every triangulation has at least two ears.

22 How Many Triangulations?

Let t_n be the number of ways that a convex polygon with n sides can be triangulated using non-crossing diagonals. Verify that $t_3 = 1$, $t_4 = 2$ and $t_5 = 5$. Calculate t_6 and t_7.

1. Find a way to calculate t_n if you already know $t_3, t_4, \ldots, t_{n-1}$.

2. Can you find a formula for t_n?

23 Triple Duel

Smith, Brown and Jones agree to fight a pistol duel under the following unusual conditions. After drawing lots to determine who fires first, second and third, they take their places at the corners of an equilateral triangle. It is agreed that they will fire single shots in turn and continue in the same cyclic order until two of them are dead. At each turn the man who is firing may aim wherever he pleases. All three duelists know that Smith always hits his target, Brown is 80 percent accurate and Jones is 50 percent accurate. Assuming that all three adopt the best strategy, and that no one is killed by a wild shot not intended for him, who has the best chance to survive? What are the exact survival probabilities of the three men?

24 100

This is a two-player game. The first player selects a number between 1 and 6, inclusive. To this number the second player adds a number from 1 through 6. The players continue to alternately increase the current sum, each by adding a number from 1 through 6. The first player to achieve the exact sum 100 wins. What is a good strategy? Who will win?

25 Penny Piles

This is another two-player game. The initial position is three piles of pennies (or some other object). The first pile has 3 pennies, the second pile has 5 pennies, and the third has 7 pennies. Players alternately take turns, which consists of removing any positive number of pennies from any one pile. The winner is the player who removes the very last penny. What is a good strategy? Who will win?

26 One Dollar

Yet another money game! Initially there is no money on the table. Two players alternately add one penny, one nickel, one dime, or one quarter to the money on the table. It is not permissible to allow the total to exceed one dollar, and the first player to reach exactly one dollar wins.

27 Number Guessing Cards

Here are two versions of the Number Guessing Cards. One person chooses a number that appears on at least one card, then indicates on which cards the chosen number appears. The other person then must as quickly as possible determine which number was chosen.

Small Set

CARD 1	CARD 2	CARD 3
1	2	4
3	3	5
5	6	6
7	7	7

Large Set

CARD 1	CARD 2	CARD 3	CARD 4	CARD 5
1	2	4	8	16
3	3	5	9	17
5	6	6	10	18
7	7	7	11	19
9	10	12	12	20
11	11	13	13	21
13	14	14	14	22
15	15	15	15	23
17	18	20	24	24
19	19	21	25	25
21	22	22	26	26
23	23	23	27	27
25	26	28	28	28
27	27	29	29	29
29	30	30	30	30
31	31	31	31	31

28 Weird Multiplication

Why do the following methods work?

1. Suppose you want to calculate 18×14. Write these two numbers at the top of two columns. In the first column, start with 18, divide it by two, and repeat, ignoring any remainders, until you reach the number 1. In the second column, start with 14, double it, and repeat, until you have a column of equal length. Draw a line through any number in the first column that is even, and extend the line to cross out the neighboring number in the second column. Add up the remaining numbers in the second column. This will be the product, and this procedure works for any pair of numbers. Two examples:

18	14
9	28
4	56
2	112
1	224
	252

13	25
6	50
3	100
1	200
	325

2. Number the fingers on each hand 5, 6, 7, 8, and 9, starting with your pinkies and ending with your thumbs. Hold your hands out in front of you with the palms facing you, thumbs up, and pinkies down.

   To multiply two numbers on the above list together, touch the corresponding fingers together. For example, to calculate 8×7, touch the index finger of your left hand to the middle finger of your right hand.

   At and above the index finger on your left hand there are 2 fingers. At and above the middle finger on your right hand there are 3 fingers. Multiply these two numbers together: $2 \times 3 = 6$. The remaining fingers on both hands combined each count 10, and there are 5 of them, for a product of 50. Add 6 and 50 to get the answer: $8 \times 7 = 56$.

   Confused? Let's try another one: 6×8. Touch the ring finger of your left hand to the index finger of your right hand. Multiplying the number of fingers at and above the touching fingers: $4 \times 2 = 8$. Counting fingers below the touching fingers as tens: 40. Add 8 and 40 to get the answer: 48.

29 Deltahedra

A deltahedron is a convex polyhedron, each of whose faces is an equilateral triangle. Each pair of adjacent triangles must "bend outward" (not be coplanar or bend inward).

1. Try to construct all of the deltahedra and draw a good sketch of each one.

2. How many of the deltahedra have an odd number of faces? An even number of faces? Find a proof for what you observe.

3. Assume that each triangle has a side length of 1 unit. Determine the volumes of the following deltahedra:

   (a) Tetrahedron (4 triangles).

   (b) Octahedron (8 triangles).

   (c) The deltahedron with 10 triangles.

   (d) The deltahedron with 14 triangles.

4. You can fit 4 tetrahedra around an octahedron to make a larger tetrahedron.

   (a) Make a sketch to illustrate this.

   (b) What is the relationship of the volume of the large tetrahedron to the volume of one of the small tetrahedra?

   (c) Use this information to determine a relationship between the volume of the octahedron and one of the small tetrahedra.

30 Handshaking

A group of people at a party shake hands with each other. Not everyone shakes hands with everyone else. Prove that the number of people who shake hands an odd number of times is even.

31 The Game of Hex

The following is quoted from Martin Gardner, *The Scientific American Book of Mathematical Puzzles and Diversions,* chapter 8.

Hex is played on a diamond-shaped board made up of hexagons (see Figure 1).

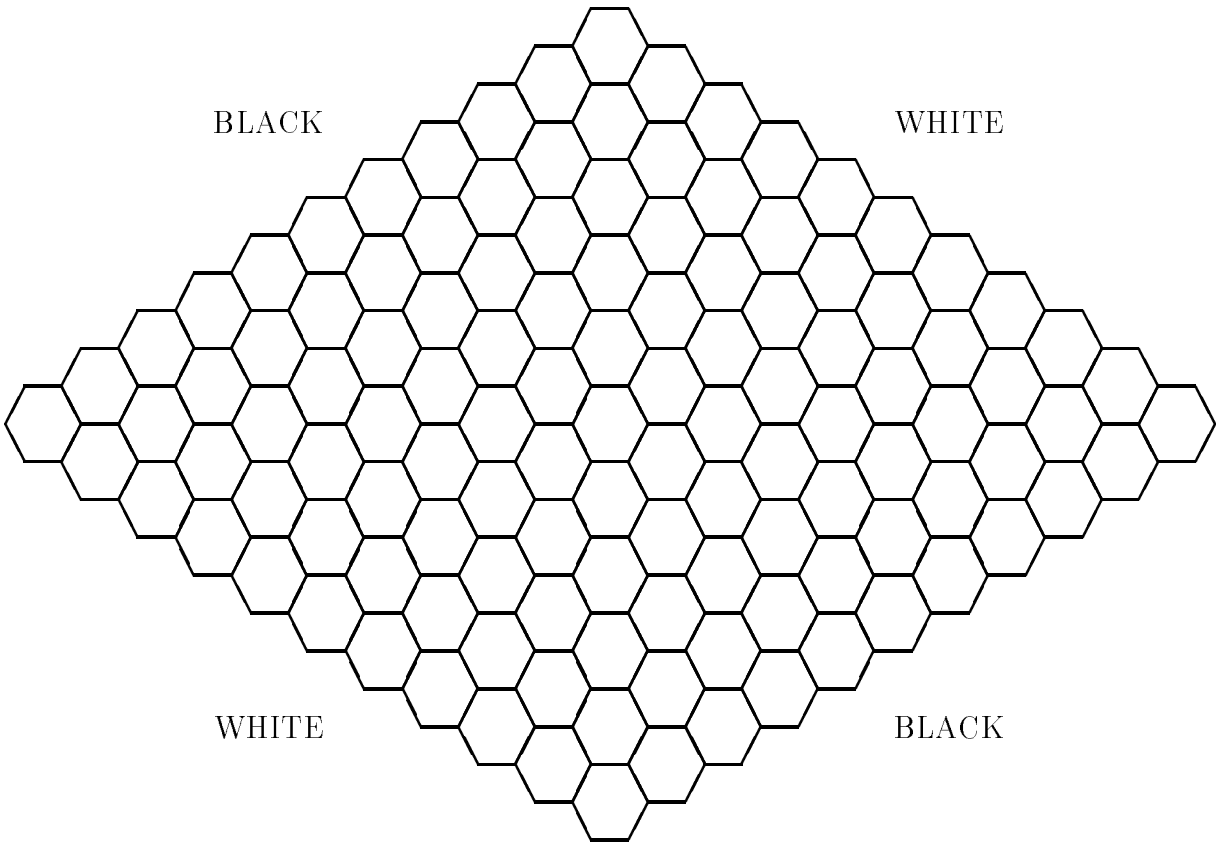


Figure 1: The Game of Hex

The number of hexagons may vary, but the board usually has 11 on each edge. Two opposite sides of the diamond are labeled "black"; the other two sides are "white." The hexagons at the corners of the diamond belong to either side. One player has a supply of black pieces; the other, a supply of white pieces. The

players alternately place one of their pieces on any one of the hexagons, provided the cell is not already occupied by another piece. The objective of "black" is to complete an unbroken chain of black pieces between the two sides labeled "black." "White" tries to complete a similar chain of white pieces between the sides labeled "white."

See Figure 2 for an example of a path (this example also comes from Gardner's article). There will almost certainly be pieces not on the winning path, and there is no obligation to construct the path in any particular order. All that is necessary to win is to have a path joining your two sides of the board somewhere among all the pieces you have played.

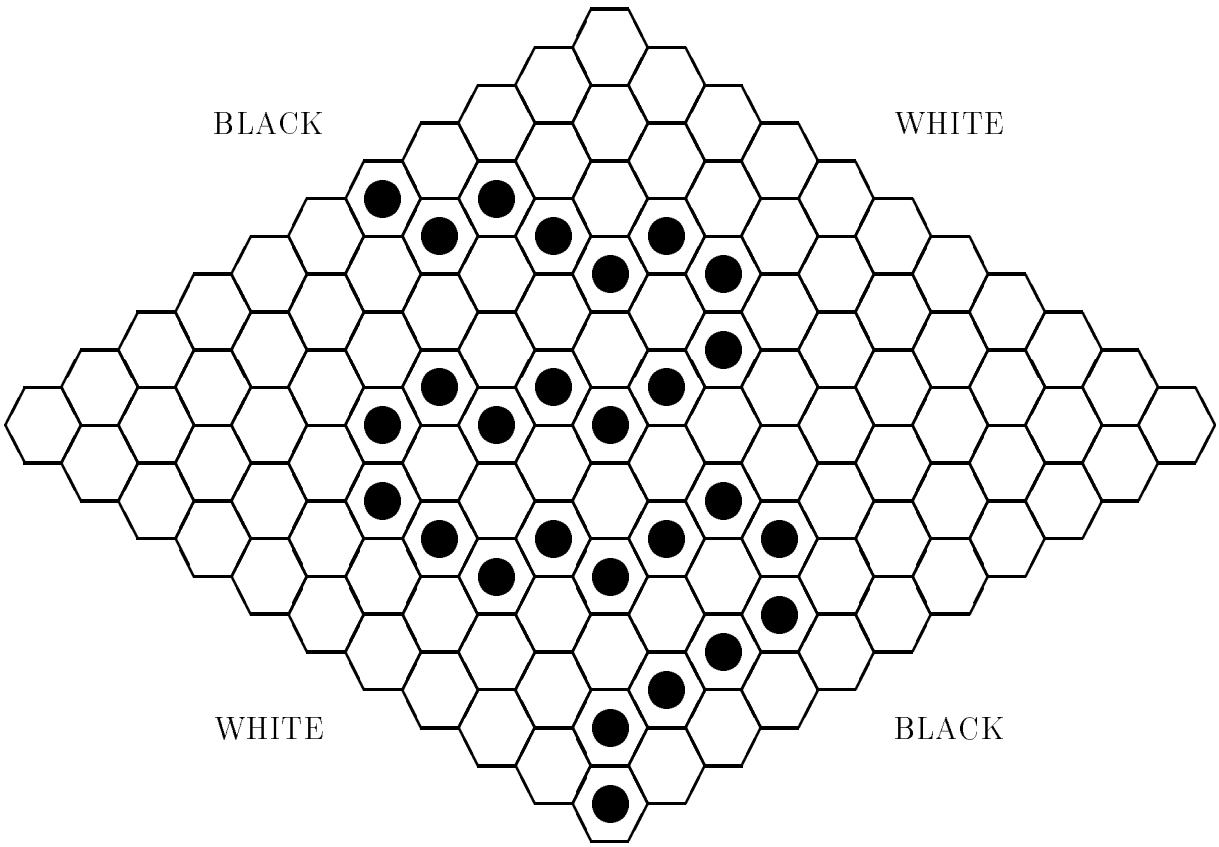


Figure 2: An Example of a Path in Hex

Prove the following:

1. There can be no tie, and hence either the first player or the second player has a winning strategy.

2. The first player has a winning strategy. Suggestion: Assume that the second player has a winning strategy. Then the first player can make any first move, and then pretend to be the second player, following the second player's strategy.

You may be interested to know that although the first player has a winning strategy, no one knows what it is!

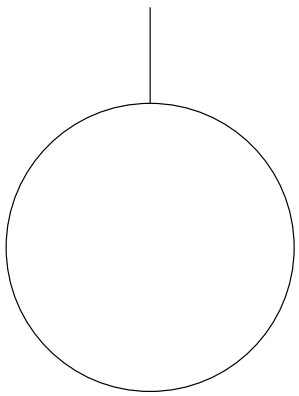
32 How Far Away is the Horizon?

> *It was the first time that Poole had seen a genuine horizon since he had come to Star City, and it was not quite as far away as he had expected.... He used to be good at mental arithmetic—a rare achievement even in his time, and probably much rarer now. The formula to give the horizon distance was a simple one: the square root of twice your height times the radius—the sort of thing you never forgot, even if you wanted to...*
>
> —Arthur C. Clarke, *3001*, Ballantine Books, New York, 1997, page 71

In the above passage, Frank Poole uses a formula to determine the distance to the horizon given his height above the ground.

1. Use algebraic notation to express the formula Poole is using.

2. Beginning the diagram below, use one of the circle theorems to derive your own formula. You will need to add some more elements to the diagram.



3. Compare your formula to Poole's; you will find that they do not match. How are they different?

4. When I was a boy it was possible to see the Atlantic Ocean from the peak of Mt. Washington in New Hampshire. This mountain is 6288 feet high. How far away is the horizon? Express your answer in miles. Assume that the radius of the Earth is 4000 miles. Use both your formula and Poole's formula and comment on the results. Why does Poole's formula work so well, even though it is not correct?

33 Curve Stitching

Consider all lines PQ determined by points $P(-a, a)$ and $Q(b, b)$, where $a + b = 1$. What shape is outlined when these lines are graphed?

34 Various Loci

1. Let ℓ be a horizontal line in the plane and $F(r, s)$ be a point not on that line. Let $e > 0$ be a positive real number. Find an equation to describe all points $P(x, y)$ such that the ratio of PF divided by the distance from P to ℓ equals e. What kind of curves do you get?

2. Let $Q(-c, 0)$ and $R(c, 0)$ be given points on the x-axis. Find an equation to describe all points $P(x, y)$ such that the sum of the distances $PQ + PR$ is a constant. What kind of curves do you get?

3. Let $Q(-c, 0)$ and $R(c, 0)$ be given points on the x-axis. Find an equation to describe all points $P(x, y)$ such that the difference of the distances $PQ - PR$ is a constant. What kind of curves do you get?

4. A wagon sits at the point $(0, L)$ and a child stands at the point $(0, 0)$, holding a rope of length L attached to the wagon. The child begins walking along the positive x-axis, pulling the wagon along with the rope. Find an equation to describe the curve that the wagon follows.

5. A wheel of radius a rolls along the positive x-axis. There is a point P on the circumference of the wheel that initially touches the x-axis at the origin. Describe the curve traced out by $P(x, y)$ as the wheel rolls, giving x and y in terms of the angle t about which the wheel has turned about its center. (Initially $t = 0$. When P comes to rest on the x-axis again, $t = 2\pi$.)

6. A string is wound tightly many times around a circle of radius a centered at the origin. The end of the string is initially at the point $(a, 0)$. Describe the curve traced out by the end $P(x, y)$ of the string as it is unwound, keeping the string taut. Let's assume that it is unwound in a counterclockwise direction. Suggestion: express the location of the point P in polar coordinates. Initially $\theta = 0$ and $r = a$.

35 Some Reflections on Light

Light originates at a given point $P(a, b)$, reflects off a mirror ℓ at some point R, and reaches a given point $Q(c, d)$. It follows the path so that the total distance traveled is minimum. Prove that the angle of incidence equals the angle of reflection. Suggestion: Let ℓ be the x-axis, and choose P and Q to lie above the x-axis.

36 South-East-North

An old riddle runs as follows. An explorer walks one mile due south, turns and walks one mile due east, turns again and walks one mile due north. He finds himself back where he started. He shoots a bear. What color is the bear?

How many spots on the globe are there from which one could walk a mile south, a mile east, a mile north and find himself back at his original position?

37 More Handshaking

My wife and I recently attended a party at which there were four other married couples. Various handshakes took place. No one shook hands with himself (or herself) or with his (or her) spouse, and no one shook hands with the same person more than once. After all the handshakes were over, I asked each other person, including my wife, how many hands he (or she) had shaken. To my surprise each gave a different answer. How many hands did my wife shake?

38 Pigeonhole Problems

Quite simply, the Pigeonhole Principle states that if $|X| > |Y|$, then there are no one-to-one functions from X into Y. So if $m > n$ and there are m pigeons and n pigeonholes in which they roost, there must be at least one pigeonhole with more than one pigeon.

This obvious principle can be used to prove some perhaps less-than-obvious facts.

1. Prove that there are at least two humans on earth with the same number of hairs on their heads.

2. There are 100 people at a party. Assume that if person A knows person B, then person B knows person A. Prove that there are at least two people at the party who know the same number of people.

3. In a bureau drawer there are 60 socks, all identical except for their color: 10 pairs are red, 10 are blue, and 10 are green. The socks are all mixed up in the drawer, and the room the bureau is in is totally dark. What is the smallest number of socks you must remove to be sure you have at least one matching pair?

4. Prove that when a fraction a/b is expressed in decimal form, the resulting number will be either a terminating decimal or one that repeats with a period no longer than b.

5. Prove that if five points are placed anywhere on or in a square of side length 1, at least two points will be no farther apart than $\sqrt{2}/2$.

6. Prove that if five points are placed anywhere on or in an equilateral triangle of side length 1, at least two points will be no farther apart than $1/2$.

7. Try counting the edges around the faces of a polyhedron. You will find that there are always two faces somewhere bounded by the same number of edges. Why?

8. Prove that no matter how a set S of 10 positive integers smaller than 100 is chosen there will always be two completely different selections from S that have the same sum. For example, in the set 3, 9, 14, 21, 26, 35, 42, 59, 63, 76 there are the selections 14, 63, and 35, 42, both of which add to 77; similarly, the selection 3, 9, 14 adds up to 26, a number that is a member of the set.

9. A physician testing a new medication instructs a test patient to take 48 pills over a 30-day period. The patient is at liberty to distribute the pills however he likes over this period as long as he takes at least one pill a day and finishes all 48 pills by the

end of the 30 days. Prove that no matter how the patient decides to arrange things, there will be some stretch of consecutive days in which the total number of pills taken is 11.

10. Suppose some set of 101 numbers $a_1, a_2, \ldots, a_{101}$ is chosen from the numbers $1, 2, \ldots, 200$. Prove that it is impossible to choose such a set without taking two numbers for which one divides the other evenly; that is, with no remainder.

11. Consider a circle C with a radius of 16 and an annulus, or ring, A, with an outer radius of 3 and an inner radius of 2. Prove that wherever one might sprinkle a set S of 650 points inside C the annulus A can always be placed on the figure so that it covers at least 10 of the points.

12. The next example concerns a marching band whose members are lined up in a rectangular array of m rows and n columns. Viewing the band from the left side, the bandmaster notices that some of the shorter members are hidden in the array. He rectifies this aesthetic flaw by arranging the musicians in each row in nondecreasing order of height from left to right, so that each one is of height greater than or equal to that of the person to his left (from the viewpoint of the bandmaster). When the bandmaster goes around to the front, however, he finds that once again some of the shorter members are concealed. he proceeds to shuffle the musicians within their columns so that they are arranged in nondecreasing order of height from front to back. At this point he hesitates to go back to the left side to see what this adjustment has done to his carefully arranged rows. When he does go, however, he is pleasantly surprised to find that the rows are still arranged in nondecreasing order of height from left to right! Shuffling an array within its columns in this manner does not undo the nondecreasing order in its rows. Can you prove this?

13. Take the numbers from $1, 2, 3, \ldots, n^2 + 1$ and arrange them in a sequence in any order. Prove that when the arrangement is scanned from left to right, it must contain either an increasing subsequence of length (at least) $n + 1$ or a decreasing subsequence of length (at least) $n + 1$. For example, when $n = 3$, the arrangement $6, 5, 9, 3, 7, 1, 2, 8, 4, 10$ includes the decreasing subsequence $6, 5, 3, 1$. (As this example demonstrates, a subsequence need not consist of consecutive elements of the arrangement.)

14. A lattice point is a point in a coordinate plane for which both coordinates are integers. Prove that no matter what five lattice points might be chosen in the plane at least one of the segments that joins two of the chosen points must pass through some lattice point in the plane.

15. Six circles (including their circumferences and interiors) are arranged in the plane so that no one of them contains the center of another. Prove that they cannot have a point in common.

16. Prove that in any row of $mn+1$ distinct real numbers there must be either an increasing subsequence of length (at least) $m+1$ or a decreasing subsequence of length (at least) $n+1$.

39 Intersecting Circles

Three circles of unit radius intersect at a common point O. Prove that the other three intersection points of the pairs of circles lie on a circle that also has a unit radius.

40 Dissections

1. Prove that any triangle can be dissected into a finite number of pieces that can be rearranged to make a rectangle.

2. Prove that any rectangle can be dissected into a finite number of pieces that can be rearranged to make a square.

3. Find a way to dissect an equilateral triangle into exactly four pieces that can be rearranged to make a square.

41 Desert Excursion

An unlimited supply of gasoline is available at one edge of a desert 800 miles wide, but there is no source on the desert itself. A truck can carry enough gasoline to go 500 miles (this will be called one "load"), and it can build up its own refueling stations at any spot along the way. These caches may be any size, and it is assumed that there is no evaporation loss. What is the minimum amount (in loads) of gasoline the truck will require in order to cross the desert? What is the widest desert that can be crossed?

42 The Chessboard and the Dominoes

The props for this problem are a chessboard and 32 dominoes. Each dominoe is of such size that it exactly covers two adjacent squares on the board. The 32 dominoes therefore can cover all 64 of the chessboard squares. But now suppose we cut off two squares at diagonally opposite corners of the board and discard one of the dominoes. Is it possible to place the 31 dominoes on the board so that all the remaining 62 squares are covered? If so, show how it can be done. If not, prove it impossible.

43 Which Way?

A logician vacationing in the South Seas finds himself on an island inhabited by the two proverbial tribes of liars and truth-tellers. Members of one tribe always tell the truth, members of the other always lie. He comes to a fork in a road and has to ask a native bystander which branch he should take to reach a village. He has no way of telling whether the native is a truth-teller or a liar. The logician thinks a moment, then asks one question only. From the reply he knows which road to take. What question does he ask?

44 Mixed Labels

Imagine that you have three boxes, one containing two black marbles, one containing two white marbles, and the third, one black marble and one white marble. The boxes were labeled for their contents—BB, WW and BW—but someone has switched the labels so that every box is now incorrectly labeled. You are allowed to take one marble at a time out of any box, without looking inside, and by this process of sampling you are to determine the contents of all three boxes. What is the smallest number of drawings needed to do this?

45 The Rectangle and the Oil Well

An oil well being drilled in flat prairie country struck pay sand at an underground spot exactly 21,000 feet from one corner or a rectangular plot of farmland, 18,000 feet from the opposite corner, and 6,000 feet from a third corner. How far is the underground spot from the fourth corner?

46 Generalized Pythagoras

Is there a good analogue to the Pythagorean theorem for three-dimensional objects?