Tools

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1 The Quadratic Formula

A *quadratic equation* is an equation of the form

$$ax^2 + bx + c = 0$$

where $a \neq 0$. The solutions to a quadratic equation can be found using the *quadratic formula*:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

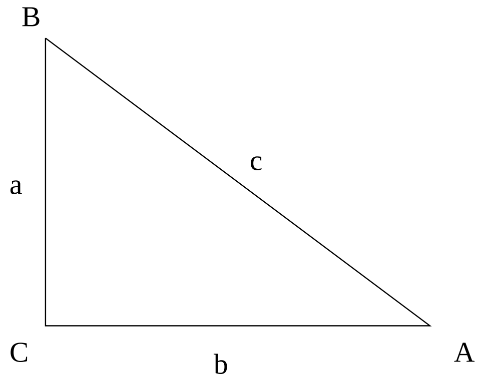
2 Rate, Time and Distance

If an object travels at a constant rate r for a time t over a distance d, then $rt = d$.

If an object travels for a time t over a distance d, but not necessarily at a constant rate, then the average rate is d/t.

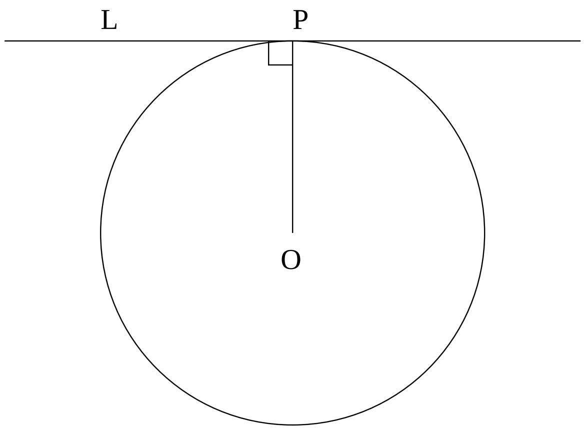
3 The Pythagorean Theorem

If ΔABC is a right triangle with side lengths a and b and hypotenuse length c, then $a^2 + b^2 = c^2$.



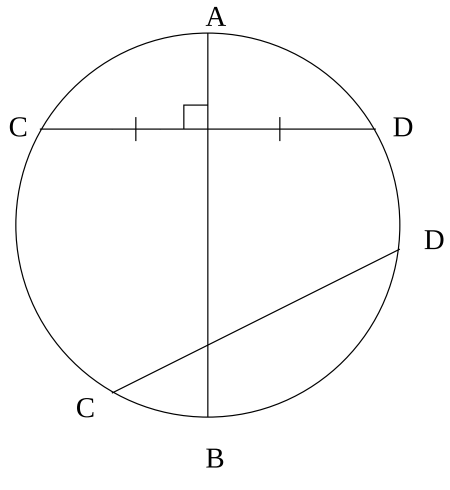
4 Tangents and Radii of Circles

Suppose C is a circle with center O and that P is any point on the circle. Then the line L tangent to C at P is perpendicular to the radius $\overline{OP}$.



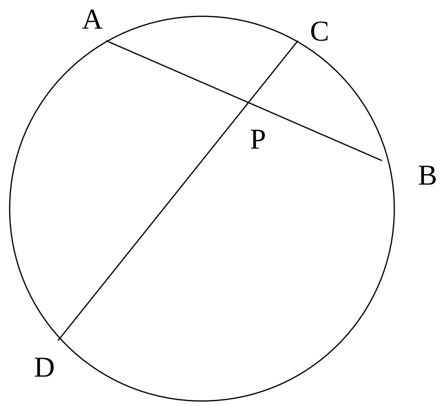
5 Chords and Diameters of Circles

Suppose X is a circle, $\overline{AB}$ is any diameter of the circle, and $\overline{CD}$ is any chord of the circle. Then $\overline{AB}$ bisects $\overline{CD}$ if and only if $\overline{AB}$ is perpendicular to $\overline{CD}$. "If and only if" means *both* of the following statements are true: If $\overline{AB}$ bisects $\overline{CD}$ then $\overline{AB}$ is perpendicular to $\overline{CD}$. If $\overline{AB}$ is perpendicular to $\overline{CD}$ then $\overline{AB}$ bisects $\overline{CD}$.



6 Crossing Chords of Circles

If X is a circle with chords $\overline{AB}$ and $\overline{CD}$ that cross at a point P, then $AP \cdot PB = CP \cdot PD$.



7 Similar Figures, Ratios, and Pi

Assume X_1 and X_2 are two plane figures that are similar. Assume that p_1 is a segment or arc of X_1 that corresponds to a segment or arc p_2 of X_2, and that q_1 is a segment or arc of X_1 that corresponds to a segment or arc q_2 of X_2. Then

$$\frac{\text{length } p_1}{\text{length } q_1} = \frac{\text{length } p_2}{\text{length } q_2}.$$

In particular, the ratio of the circumference C to the diameter d is exactly the same for every circle. We can then *define* π to equal this ratio. That is to say, we *define* $\pi = C/d$. As a consequence, $C = \pi d = 2\pi r$.

8 Area and Circumference of Circles

From the above we know that the circumference C of a circle of radius r is $C = 2\pi r$.

The area of a circle of radius r is $A = \pi r^2$.

This can be proved by calculus or "seen" by cutting up a circle into many, very small, equally-sized, pie-shaped pieces which can then be rearranged into a figure which is approximately a parallelogram of height r and length $C/2$. Its area is therefore $rC/2$ which equals $r2\pi r/2 = \pi r^2$.

Why is the formula for the circumference equal to the derivative of the formula for the area? Because when r is increased very slightly by an amount Δr, the area A of the circle increases very slightly by an amount ΔA approximately equal to $C\Delta r$ (a "ring" around the circle of length C and width Δr). So $\Delta A/\Delta r \approx C$, and in the limit $dA/dr = C$.

9 Volume and Surface Area of Spheres

The volume of a sphere of radius r is $V = \frac{4}{3}\pi r^3$.

The surface area of a sphere of radius r is $S = 4\pi r^2$.

Why is the formula for the surface area S equal to the derivative of the formula for the volume? Because when r is increased very slightly by an amount Δr, the volume V of the sphere increases very slightly by an amount ΔV approximately equal to $S\Delta r$ (a "shell" around the sphere of area S and width Δr). So $\Delta V/\Delta r \approx S$, and in the limit $dV/dr = S$.

10 Some Trigonometric Formulas

1. The fundamental formula is
$$\sin^2 x + \cos^2 x = 1.$$

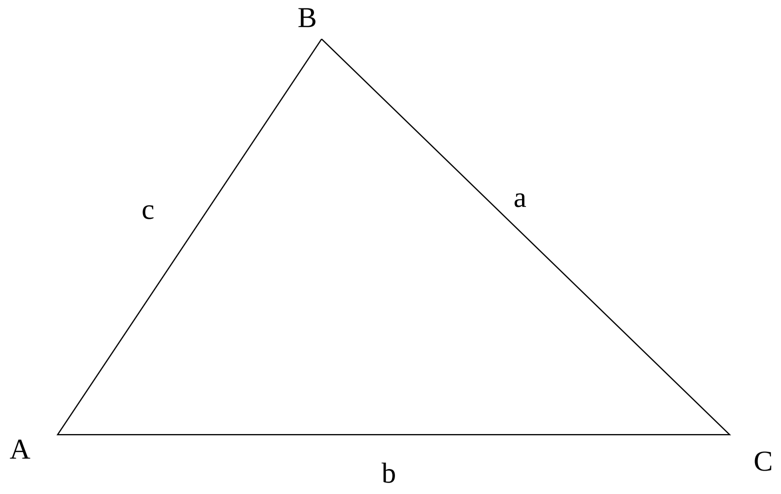
2. Angle Addition Formulas
$$\begin{aligned} \sin(A+B) &= \sin A\cos B + \cos A\sin B, \\ \cos(A+B) &= \cos A\cos B - \sin A\sin B. \end{aligned}$$

3. Setting $B = A$ above (and also using the fundamental formula), we get the Double-Angle Formulas
$$\begin{aligned} \sin 2A &= 2\sin A\cos A, \\ \cos 2A &= \cos^2 A - \sin^2 A \\ &= 2\cos^2 A - 1 \\ &= 1 - 2\sin^2 A. \end{aligned}$$

4. Solving for $\sin A$ and $\cos A$ in the last two formulas gives us the Half-Angle Formulas
$$\cos A = \pm\sqrt{\frac{1+\cos 2A}{2}}; \text{ i.e., } \cos\frac{x}{2} = \pm\sqrt{\frac{1+\cos x}{2}}.$$
$$\sin A = \pm\sqrt{\frac{1-\cos 2A}{2}}; \text{ i.e., } \sin\frac{x}{2} = \pm\sqrt{\frac{1-\cos x}{2}}.$$

11 Laws of Sines and Cosines

Suppose ΔABC is any triangle with side lengths a, b, c.



1. Law of Sines
$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}.$$
2. Law of Cosines
$$c^2 = a^2 + b^2 - 2ab\cos C.$$

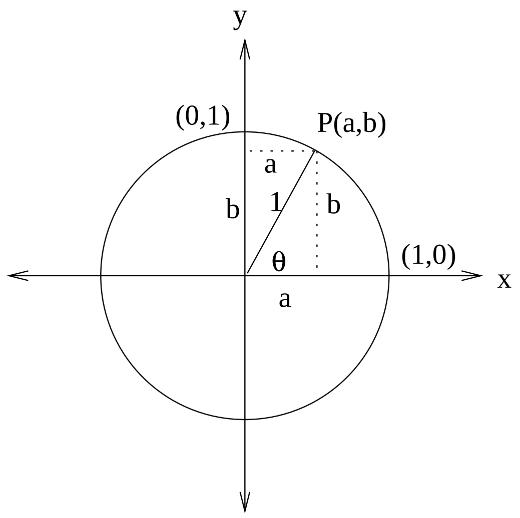
12 Basic Trigonometry

These notes are slight revisions of those written by Sue Foege.

12.1 Definitions of Sine and Cosine

The circle with center at the origin and radius of one unit is often called the *unit circle*. The equation of the unit circle is given by $x^2 + y^2 = 1$.

The trigonometric functions or ratios are often referred to as the *circular* functions. Let θ be an angle of rotation about the origin, measured from the positive x-axis, where a counterclockwise rotation produces a positive angle, as shown here. The point $P(a, b)$ on the unit circle corresponds to θ.



Definition: The *cosine* of θ, denoted $\cos\theta$, is the first, or x coordinate of the corresponding point P on the unit circle. In the figure above, $a = \cos\theta$.

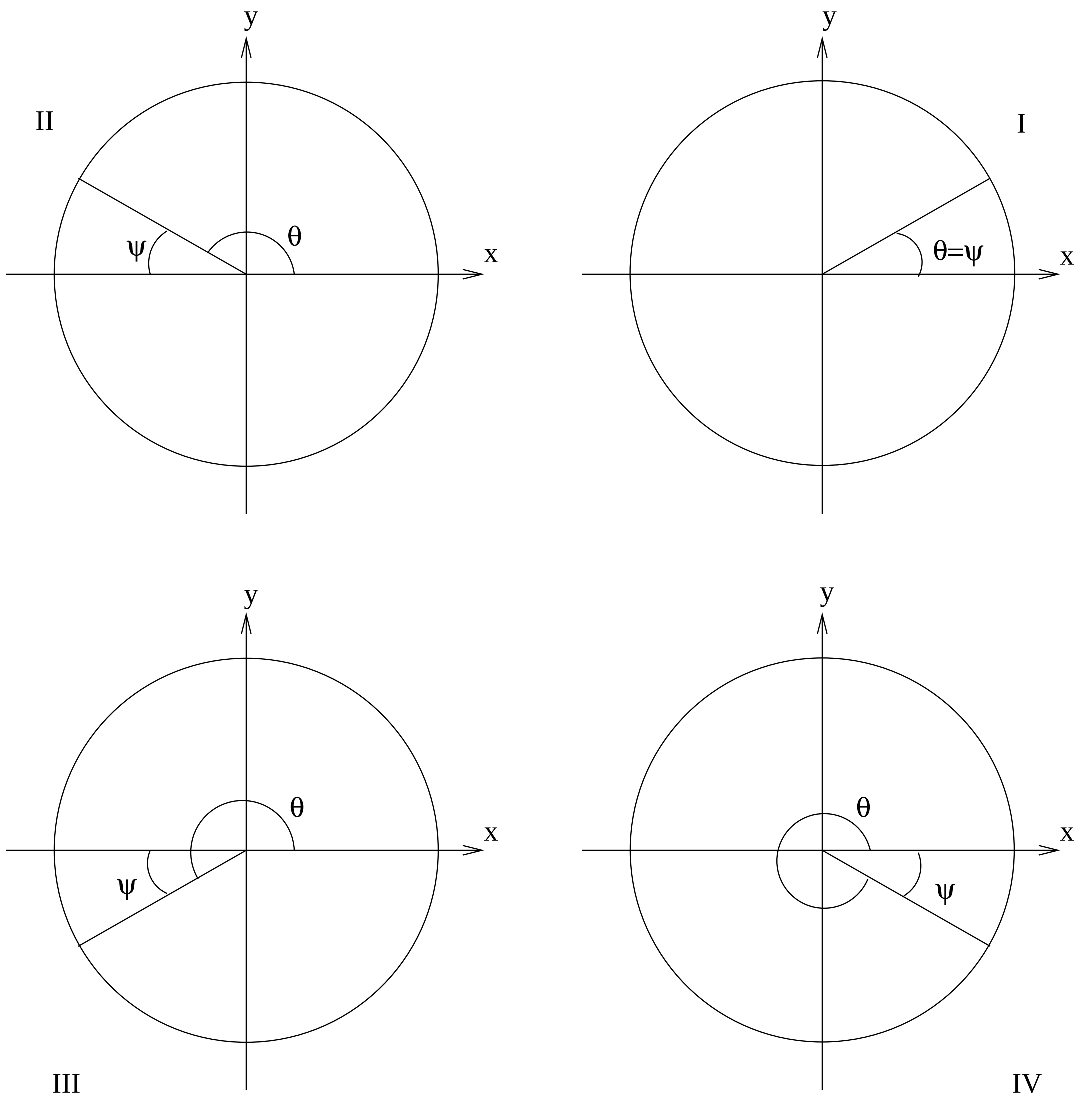
Definition: The *sine* of θ, denoted $\sin\theta$, is the second, or y coordinate of the corresponding point P on the unit circle. In figure above, $b = \sin\theta$.

Since $(\cos\theta, \sin\theta)$ is a point on the unit circle, $\sin^2\theta + \cos^2\theta = 1$.

Note that the largest value of $\sin\theta$ is 1 and is attained, for example, at $\theta = 90°$, $450°$, and $-270°$. The smallest value of $\sin\theta$ is -1 and is attained, for example, at $\theta = 270°$, $630°$, and $-90°$.

Place your finger on the point $(1, 0)$, and trace around the unit circle counterclockwise. While doing this, how does the second coordinate of the point on the unit circle (the sine of the angle) change? It begins at 0 when $\theta = 0°$, rises to 1 when $\theta = 90°$, drops to 0 when $\theta = 180°$, drops to -1 when $\theta = 270°$, and rises to 0 when $\theta = 360°$. How does the first coordinate of the point (the cosine of the angle). It begins at 1 when $\theta = 0°$, drops to 0 when $\theta = 90°$, drops to -1 when $\theta = 180°$, rises to 0 when $\theta = 270°$, and rises to 1 when $\theta = 360°$. Sketch the graphs of sine and cosine.

12.2 Angles in Different Quadrants



The above figure shows four possible positions of the angle θ—one in each of the four quadrants. Another angle, Ψ, is also marked. This is sometimes called the *reference angle* for θ. In each case, $\sin\theta = \pm\sin\Psi$ and $\cos\theta = \pm\cos\Psi$. Also, $\tan\theta = \pm\tan\Psi$, recalling that $\tan\theta = \sin\theta/\cos\theta$. The choice of $\pm$ depends upon the quadrant.

1. Quadrant I

   (a) $\sin\theta = \sin\Psi$.
   (b) $\cos\theta = \cos\Psi$.
   (c) $\tan\theta = \tan\Psi$.

2. Quadrant II
   (a) $\sin\theta = \sin\Psi$.
   (b) $\cos\theta = -\cos\Psi$.
   (c) $\tan\theta = -\tan\Psi$.

3. Quadrant III
   (a) $\sin\theta = -\sin\Psi$.
   (b) $\cos\theta = \tan\Psi$.
   (c) $\tan\theta = -\cos\Psi$.

4. Quadrant IV
   (a) $\sin\theta = -\sin\Psi$.
   (b) $\cos\theta = \cos\Psi$.
   (c) $\tan\theta = -\tan\Psi$.

One mnemonic device to remember the signs of the trig functions sine, cosine and tangent (sine/cosine) in the four quadrants is **A S**mart **T**rig **C**lass:

A **All** three trig functions are positive in Quadrant I.

S Only **Sine** is positive in Quadrant II; the others are negative.

T Only **Tangent** is positive in Quadrant III; the others are negative.

C Only **Cosine** is positive in Quadrant IV; the others are negative.

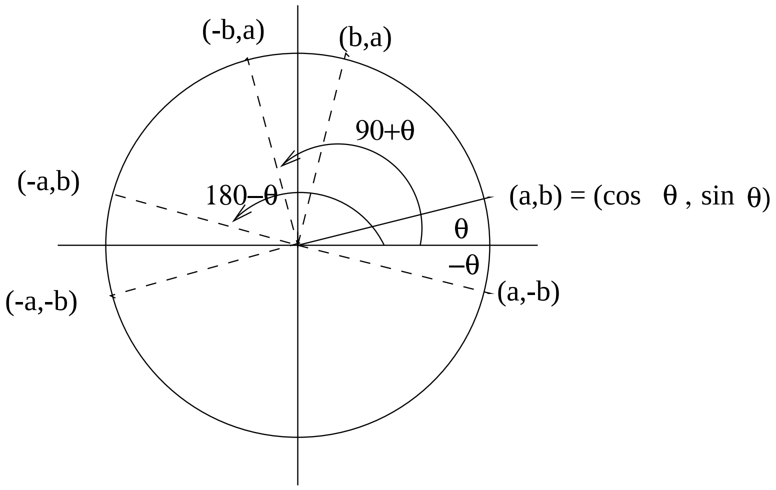
12.3 Some Familiar Angles

Here is a mnemonic to remember the sine and cosine of some common angles:

θ	$0°$	$30°$	$45°$	$60°$	$90°$
$\sin\theta$	$\sqrt{0}/2$	$\sqrt{1}/2$	$\sqrt{2}/2$	$\sqrt{3}/2$	$\sqrt{4}/2$
$\cos\theta$	$\sqrt{4}/2$	$\sqrt{3}/2$	$\sqrt{2}/2$	$\sqrt{1}/2$	$\sqrt{0}/2$

Convince yourself that these values are correct by considering $45° - 45° - 90°$ triangles and $30° - 60° - 90°$ triangles.

12.4 Some Basic Identities



Using the coordinates a and b for a given point on the unit circle, we can find the coordinates of some related points in the above figure. Since $a = \cos\theta$ and $b = \sin\theta$, we easily get the sine and cosine of other angles, such as $-\theta$, $90° + \theta$, $180° + \theta$, $90° - \theta$, and $180° - \theta$. You should be able to verify the following identities.

1. $\cos(-\theta) = a = \cos\theta$.
2. $\sin(-\theta) = -b = -\sin\theta$.
3. $\cos(90° + \theta) = -b = -\sin\theta$.

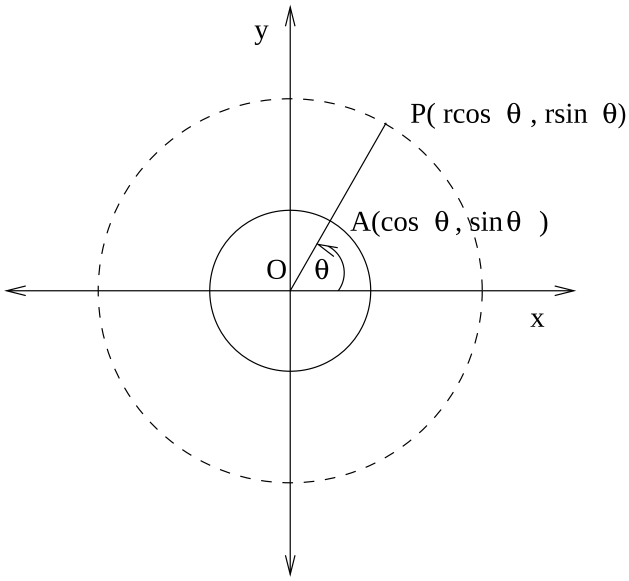
4. $\sin(90° + \theta) = a = \cos\theta$.
5. $\cos(180° + \theta) = -a = -\cos\theta$.
6. $\sin(180° + \theta) = -b = -\sin\theta$.
7. $\cos(90° - \theta) = b = \sin\theta$.
8. $\sin(90° - \theta) = a = \cos\theta$.
9. $\cos(180° - \theta) = -a = -\cos\theta$.
10. $\sin(180° - \theta) = b = \sin\theta$.

12.5 Polar Coordinates of Points in the Euclidean Plane $\mathbf{E}^2$.

We assign coordinates to points in $\mathbf{E}^2$ so that we can locate them. Let P be a point in $\mathbf{E}^2$, and $\overline{OP}$ be the line segment between P and the origin. Let r be the distance between P and the origin. (So r is the length of $\overline{OP}$.) Let θ be the angle, measured in the counterclockwise direction, that $\overline{OP}$ makes with the x-axis. (See the figure below.)

The point P lies on the circle of radius r, centered at the origin, given by the equation $x^2 + y^2 = r^2$. We can verify that $x = r\cos\theta$ and $y = r\sin\theta$ satisfy this equation. $x^2 + y^2 = r^2\cos^2\theta + r^2\sin^2\theta = r^2(cos^2\theta + \sin^2\theta) = r^2$.

(r, θ) are called the *polar coordinates* for the point P.



Two sets of polar coordinates (r, θ) and (r', θ') describe precisely the same point when one of the following cases occurs.

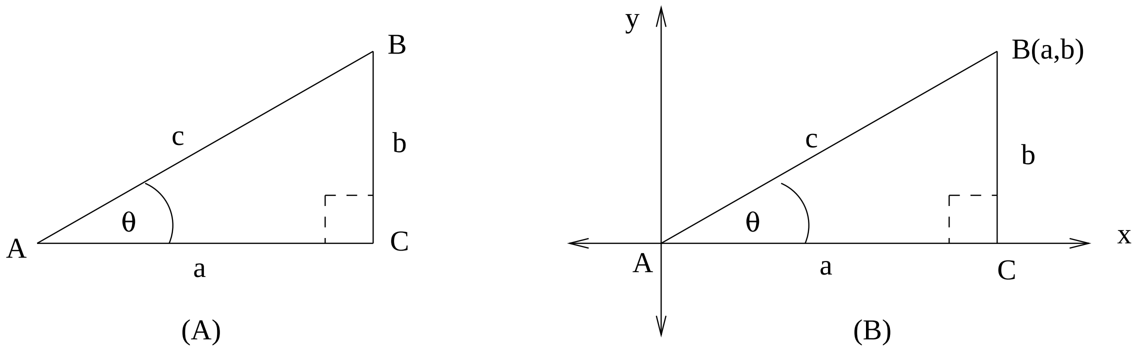
1. $r = r' = 0$, and θ and θ' are arbitrary.

2. $r \neq 0$, $r = r'$, and $\theta' = \theta + k360°$, where k is an integer.

3. $r \neq 0$, $r' = -r$, and $\theta' = \theta + 180° + k360°$, where k is an integer.

12.6 Alternate Definitions of Sine and Cosine

If θ is an acute angle, then we can draw a right triangle, as in (A) of the figure below. The side opposite the angle θ is labeled b, the side adjacent to θ is labeled a and the hypotenuse is labeled c. In (B), we have placed the vertex A at the origin, and the adjacent side on the x-axis. The vertex B is at the point (a, b).



(A) (B)

Alternate definitions for $\sin\theta$ and $\cos\theta$ are:

$$\sin\theta = \frac{\textit{opposite side}}{\textit{hypotenuse}} = \frac{\textit{opp}}{\textit{hyp}} = \frac{\mathbf{O}}{\mathbf{H}} = \frac{b}{c}.$$

$$\cos\theta = \frac{\textit{adjacent side}}{\textit{hypotenuse}} = \frac{\textit{adj}}{\textit{hyp}} = \frac{\mathbf{O}}{\mathbf{H}} = \frac{a}{c}.$$

Note that when $c = 1$, the point $B(a, b)$ lies on the unit circle, and $\sin\theta = \frac{b}{c} = \frac{b}{1} = b$ is the second coordinate of the point $B(a, b)$, while $\cos\theta = \frac{a}{c} = \frac{a}{1} = a$ is the first coordinate of this point. Thus these alternate definitions are consistent with the old definitions of sine and cosine.

12.7 Definitions of Tangent, Secant, and Cosecant

Let θ denote an angle.

Definition: The *tangent* of θ, denoted $\tan\theta$ is

$$\tan\theta = \frac{\sin\theta}{\cos\theta}.$$

Definition: The *secant* of θ, denoted $\sec\theta$ is

$$\sec\theta = \frac{1}{\cos\theta}.$$

Definition: The *cosecant* of θ, denoted $\csc\theta$ is

$$\csc\theta = \frac{1}{\sin\theta}.$$

$$\cot\theta = \frac{\cos\theta}{\sin\theta}.$$

We summarize information about the domain and range of the six trig functions:

1. $\sin\theta$: Domain equals $\mathbf{R}$. Range equals $[-1,1]$.
2. $\cos\theta$: Domain equals $\mathbf{R}$. Range equals $[-1,1]$.
3. $\tan\theta$: Domain equals $\mathbf{R}\setminus\{90^\circ + n180^\circ : n \in \mathbf{Z}\}$. Range equals $\mathbf{R}$.
4. $\cot\theta$: Domain equals $\mathbf{R}\setminus\{n180^\circ : n \in \mathbf{Z}\}$. Range equals $\mathbf{R}$.
5. $\sec\theta$: Domain equals $\mathbf{R}\setminus\{90^\circ + n180^\circ : n \in \mathbf{Z}\}$. Range equals $\mathbf{R}\setminus(-1,1)$.
6. $\csc\theta$: Domain equals $\mathbf{R}\setminus\{n180^\circ : n \in \mathbf{Z}\}$. Range equals $\mathbf{R}\setminus(-1,1)$.

Using the alternate definition of sine and cosine as ratios of sides of right triangles, we can give alternate definitions of tangent, secant, cosecant, and cotangent.

1. $\tan\theta = \frac{\mathbf{O}}{\mathbf{A}}$.
2. $\cot\theta = \frac{\mathbf{A}}{\mathbf{O}}$.
3. $\sec\theta = \frac{\mathbf{H}}{\mathbf{A}}$.

4. $\csc\theta = \frac{\mathbf{H}}{\mathbf{O}}$.

Oscar **H**ad **A** **H**eap **O**f **A**pples is a mnemonic to remember that $\sin = \frac{O}{H}$, $\cos = \frac{A}{H}$, and $\tan = \frac{O}{A}$.

Starting with the known trigonometric identity $\sin^2\theta + \cos^2\theta = 1$ we can derive two more identities by dividing both sides of this equation by $\cos^2\theta$ and by $\sin^2\theta$.

$$1 + \tan^2\theta = \sec^2\theta.$$

$$1 + \cot^2\theta = \csc^2\theta.$$

13 Complex Numbers

A complex number is a number of the form $a + bi$ where a and b are real numbers and i^2 is assumed to be -1. Two complex numbers $a + bi$ and $c + di$ are equal if and only if both $a = c$ and $b = d$. To add two complex numbers, $(a + bi) + (c + di) = (a + c) + (b + d)i$. To multiply two complex numbers, $(a + bi)(c + di) = ac + adi + bci + bdi^2 = (ac - bd) + (ad + bc)i$. To divide two complex numbers, use the conjugate of the denominator,

$$\frac{a+bi}{c+di} = \frac{(a+bi)(c-di)}{(c+di)(c-di)} = \frac{(ac+bd)+(bc-ad)i}{c^2+d^2} = \frac{ac+bd}{c^2+d^2} + \frac{bc-ad}{c^2+d^2}i.$$

14 Conjugates

Conjugates of expressions involving square roots are sometimes useful for simplifying or changing rational expressions. They are often used to remove square roots from the denominator of certain rational expressions; see also their use above in dividing complex numbers. For example,

$$\frac{1+\sqrt{2}}{3+\sqrt{5}} = \frac{(1+\sqrt{2})(3-\sqrt{5})}{(3+\sqrt{5})(3-\sqrt{5})} = \frac{1+3\sqrt{2}-\sqrt{5}-\sqrt{10}}{9-5} = \frac{1}{4}(1+3\sqrt{2}-\sqrt{5}-\sqrt{10}).$$

15 Difference of Two Squares

Using conjugates above works because of the property that $(A + B)(A - B) = A^2 - B^2$. You probably first encountered this when factoring polynomials.

16 FOIL

FOIL is a mnemonic to remember how to multiply to binomial expressions: $(A+B)(C+D) = AC + AD + BC + BD$. FOIL stands for "First-Outside-Inside-Last."

17 Extreme Cases

The Circular Track problem gives an example of trying to find an answer by looking at an extreme case. Since the sizes of the two circles were not given, the conclusion is that the answer does not depend upon them, but only on the length of the indicated chord of the large circle that is tangent to the small circle. So we can look at the extreme case when the radius of the smaller circle is 0, in which case the chord of length 100 is a diameter of the larger circle, and the area of the track is simply the area of the larger circle, namely, $\pi(50)^2 = 2500\pi$. This does not *prove* that you get the same answer for different sizes of circles, but at least it gives you evidence of what the answer must be if it only depends upon the chord.

18 Factorials, Permutations and Combinations

$n!$ is defined to be $n(n-1)(n-2)\cdots(3)(2)(1)$.

The number of ways of choosing a subset of size k from a subset of size n of different elements is

$$C_k^n = \binom{n}{k} = \frac{n!}{k!(n-k)!}.$$

The number of ways of choosing and then permuting a subset of size k from a subset of size n of different elements is

$$P_k^n = \frac{n!}{(n-k)!}.$$

19 Pascal's Triangle

$$
\begin{array}{ccccccccccc}
 & & & & & 1 & & & & & \\
 & & & & 1 & & 1 & & & & \\
 & & & 1 & & 2 & & 1 & & & \\
 & & 1 & & 3 & & 3 & & 1 & & \\
 & 1 & & 4 & & 6 & & 4 & & 1 & \\
1 & & 5 & & 10 & & 10 & & 5 & & 1 \\
 & & & & & \vdots & & & & &
\end{array}
$$

Each row begins and ends with 1. Each of the other numbers in each row is the sum of the two numbers above it in the previous row. If we label the rows $0, 1, 2, \ldots$, and the entries in each row $0, 1, 2, \ldots$, then entry k of row n equals the binomial coefficient $\binom{n}{k}$. Also, row n gives the coefficients in the expansion of $(a+b)^n$:

$$
\begin{aligned}
&(a+b)^0 = 1, \\
&(a+b)^1 = a + b, \\
&(a+b)^2 = a^2 + 2ab + b^2, \\
&(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3, \\
&(a+b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4, \\
&(a+b)^5 = a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + b^5, \\
&\vdots
\end{aligned}
$$

The sum of the entries in row n is 2^n. If we divide entry k in row n by 2^n, then we find the probability of getting exactly k heads when flipping a fair coin n times.

20 A Summation Formula

Here is a formula for the sum of the first n natural numbers:

$$1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}.$$

One example of this is $1 + 2 + 3 + \cdots + 100 = 5050$. Gauss as a small child is said to have calculated this by noticing that $1 + 100 = 101$, $2 + 99 = 101$, $3 + 98 = 101$, etc., so the entire sum is $50(101) = 5050$. We can use this idea to prove the formula. Let S be the sum. Then

$$
\begin{aligned}
S &= 1 + 2 + 3 + \cdots + (n-2) + (n-1) + n, \\
S &= n + (n-1) + (n-2) + \cdots + 3 + 2 + 1.
\end{aligned}
$$

Adding these together yields

$$2S = (n+1) + (n+1) + (n+1) + \cdots + (n+1) = n(n+1),$$

So $S = n(n+1)/2$.

21 Another Summation Formula

Another formula you may have seen before is

$$1 + r + r^2 + r^3 + \cdots + r^n = \frac{1 - r^{n+1}}{1 - r}.$$

One way to prove this is to let S be the sum. Then

$$\begin{aligned} S &= 1 + r + r^2 + r^3 + \cdots + r^n, \\ rS &= r + r^2 + r^3 + \cdots + r^n + r^{n+1}. \end{aligned}$$

Subtracting them yields

$$\begin{aligned} S - rS &= 1 - r^{n+1}, \\ S(1-r) &= 1 - r^{n+1}, \\ S &= \frac{1 - r^{n+1}}{1 - r}. \end{aligned}$$

You might recognize this formula in another guise. Change $n+1$ to n and multiply both sides by $1 - r$ to get

$$1 - r^n = (1 - r)(1 + r + r^2 + r^3 + \cdots + r^{n-1}).$$

Some special cases of this are when $n = 2, 3, 4$:

$$\begin{aligned} 1 - r^2 &= (1 - r)(1 + r), \\ 1 - r^3 &= (1 - r)(1 + r + r^2), \\ 1 - r^4 &= (1 - r)(1 + r + r^2 + r^3). \end{aligned}$$

You should recognize these as special cases of some formulas you learned when factoring polynomials.

22 Induction

When proving that a statement $P(n)$ holds for all positive integers $n = 1, 2, 3, \ldots$, two steps are necessary:

1. First prove that the statement is true when $n = 1$.
2. Second prove that the truth of the statement for $n = k + 1$, $k \geq 1$, follows from the truth of the statement for $n = k$. (Alternatively, prove that the truth of the statement for $n = k + 1$ follows from the truth of the statement for $n = 1, \ldots, k$.)

Example 1: Prove that $1 + 2 + 3 + \cdots + n = n(n + 1)/2$.

1. Set $n = 1$. Then $1 = 1(1 + 1)/2$, so the statement is true for $n = 1$.
2. Assume that the statement is true for $n = k$ (this is called the induction hypothesis). So we assume that $1 + 2 + 3 + \cdots + k = k(k + 1)/2$. Prove that the statement is true for $n = k+1$. So we want to prove that $1+2+3+\cdots+k+(k+1) = (k+1)(k+2)/2$. We can do this as follows:

$$\begin{aligned} 1 + 2 + 3 + \cdots + k + (k + 1) &= (1 + 2 + 3 + \cdots + k) + (k + 1) \\ &= k(k + 1)/2 + (k + 1) \text{ (using the induction hypothesis)} \\ &= (k + 1)(k/2 + 1) \\ &= (k + 1)(k + 2)/2. \end{aligned}$$

Therefore by mathematical induction, the formula has been proved.

Example 2: Prove that the number of subsets of an n-element set is 2^n. For convenience, assume that the set is $\{1, 2, 3, \ldots, n\}$.

1. Set $n = 1$. Then the number of subsets of the set $\{1\}$ is 2^1, so the statement is true for $n = 1$.
2. Assume that the statement is true for $n = k$. So we assume that the number of subsets of the set $\{1, 2, 3, \ldots, k\}$ is 2^k. Prove that the statement is true for $n = k + 1$. So we want to prove that the number of subsets of the set $\{1, 2, 3, \ldots, k + 1\}$ is 2^{k+1}. We can do this as follows: Every subset of $\{1, 2, 3, \ldots, k + 1\}$ is one of two types: (1) Those that do not contain the element $k+1$, and (2) those that do contain the element $k+1$. The subsets of type (1) are all subsets of $\{1, 2, 3, \ldots, k\}$, so by the induction hypothesis there are 2^k of them. The subsets of type (2) are all subsets of the form $S \cup \{k + 1\}$, where S is a subset of $\{1, 2, 3, \ldots, k\}$. By the induction hypothesis there are 2^k such subsets S, so there are 2^k subsets of type (2). So altogether, the number of subsets of $\{1, 2, 3, \ldots, k + 1\}$ is $2^k + 2^k = 2 \cdot 2^k = 2^{k+1}$.

Therefore by mathematical induction, the statement has been proved.

Look at pages 114–121 in the book *How to Solve It* to see another explanation of mathematical induction and other examples.

23 Finite Differences

The method of finite differences can sometimes be used to guess a formula $f(n)$ (but not to prove it). The way I am about to describe it requires that the values of $f(n)$ be given for integer values of n beginning with $n = 0$ (rather than 1). Let's use the example of guessing the formula for the sum of the cubes of the first n positive integers. We calculate a few values:

$$\begin{gathered}
0 \text{ (the sum of no positive integers at all is just 0)}, \\
1 = 1, \\
1 + 8 = 9, \\
1 + 8 + 27 = 36, \\
1 + 8 + 27 + 64 = 100,
\end{gathered}$$

and then arrange them in a row. Then we build a difference table beneath them by subtraction:

$$\begin{array}{ccccccccccccccc}
0 & & 1 & & 9 & & 36 & & 100 & & 225 & & 441 & & 784 \\
& 1 & & 8 & & 27 & & 64 & & 125 & & 216 & & 343 & \\
& & 7 & & 19 & & 37 & & 61 & & 91 & & 127 & & \\
& & & 12 & & 18 & & 24 & & 30 & & 36 & & & \\
& & & & 6 & & 6 & & 6 & & 6 & & & & \\
& & & & & 0 & & 0 & & 0 & & & & &
\end{array}$$

We continue generating rows by this process until it appears that we get a row of 0's. This does not always happen, but when it does, we can build a polynomial formula for $f(n)$. Using the first entries in each row, multiply the first entry of the first row by $1/0!$, the first entry of the second row by $n/1!$, the first entry of the third row by $n(n-1)/2!$, the first entry of the fourth row by $n(n-1)(n-2)/3!$, the first entry of the fifth row by $n(n-1)(n-2)(n-3)/4!$, etc., and then add the results. (Remember that 0! is defined to be 1.) In our example, we get

$$f(n) = 0\frac{1}{0!} + 1\frac{n}{1!} + 7\frac{n(n-1)}{2!} + 12\frac{n(n-1)(n-2)}{3!} + 6\frac{n(n-1)(n-2)(n-3)}{4!},$$

which simplifies (you try it!) to

$$f(n) = \left[\frac{n(n+1)}{2}\right]^2.$$

At this point we have *not* proved that this formula is correct, but at least we have a good guess for the formula, and we can try to prove it, e.g., by induction.

24 Recursion

Informally speaking, a recursive definition defines something (e.g., a function or a procedure) in terms of itself (but not in such a way to create an unending process!). For example, we can define the function $f(n)$ for $n!$ the following way:

$$f(n) = \begin{cases} 1 & \text{if } n = 1, \\ nf(n-1) & \text{if } n > 1. \end{cases}$$

Often definitions or descriptions by recursion lend themselves to proofs by induction.

Here is another example, for the Towers of Hanoi. We define the procedure move(n, a, b, c) for moving a stack of disks numbered $1, \ldots, n$ from peg a to peg b using peg c:

1. If $n > 1$ then move$(n-1, a, c, b)$.
2. Move disk n from a to b.
3. If $n > 1$ then move$(n-1, c, b, a)$.

25 Congruence of Integers Modulo n

If two integers a and b have the same remainder when divided by positive integer n, then we say that a and b are congruent modulo n. This is written $a \cong b \pmod{n}$. For example, $5 \cong 14 \pmod{3}$. It is not hard to see that $a \cong b \pmod{n}$ is equivalent to $a - b$ being a multiple of n.

26 Binary Representations

Any positive integer can be represented in base 2 using the two digits 0 and 1. Going from right to left, the digits represent the powers of 2: 2^0, 2^1, 2^2, 2^3, ...

Some examples:

Base 10	Base 2
1	1
2	10
3	11
4	100
5	101
6	110
7	111
100	1100100

In the last example, we see that $100 = 1 \cdot 2^6 + 1 \cdot 2^5 + 0 \cdot 2^4 + 0 \cdot 2^3 + 1 \cdot 2^2 + 0 \cdot 2^1 + 0 \cdot 2^0$.

27 Some Volumes

Some general principles: Suppose P_1 and P_2 are two similar figures. Assume that the ratio of two corresponding one-dimensional lengths is a/b. Then the ratio of two corresponding two-dimensional areas is a^2/b^2, and the ratio of two corresponding three-dimensional volumes is a^3/b^3.

The volume of a prism with base of area B and height h is Bh (area of base times height).

The volume of a pyramid with base of area A and height h is $\frac{1}{3}Bh$. You can prove this with calculus by placing the pyramid with apex at the origin and with base perpendicular to the z-axis. Then we can calculate the volume by making slices perpendicular to the z-axis. Let $A(z)$ be the area of the slice at position z. Then $A(z)/B = z^2/h^2$, so $A(z) = z^2B/h^2$. We get the volume by integrating from $z = 0$ to $z = h$:

$$V = \int_0^h A(z)dz = \int_0^h \frac{B}{h^2}z^2 dz = \left[\frac{B}{h^2}\frac{z^3}{3}\right]_0^h = \frac{B}{h^2}\frac{h^3}{3} = \frac{1}{3}Bh.$$

28 Drawing Polyhedra With Maple

Here are the commands that you can give to Maple to draw a square-based pyramid. You need to know the coordinates of the vertices, and for each face, you need to know the order in which the vertices appear around the face. The first command loads some graphics macros. The next five lines give the coordinates of the five vertices. The next five lines list which vertices are on each face. The next line defines the polyhedron as the collection of faces. The last line displays it. You don't have to use the symbols $v1, v2, \ldots, f1, f2, \ldots, p$; you can chooses other names for the vertices, faces, and polyhedron.

```
with(plots);
v1:=[1,1,0];
v2:=[-1,1,0];
v3:=[-1,-1,0];
v4:=[1,-1,0];
v5:=[0,0,1];
f1:=[v1,v2,v3,v4];
f2:=[v1,v2,v5];
f3:=[v2,v3,v5];
f4:=[v3,v4,v5];
f5:=[v4,v1,v5];
p:=[f1,f2,f3,f4,f5];
polygonplot3d(p,scaling=constrained);
```

29 Lines and Tangent Lines

The equation of the line passing through point (x_1, y_1) with slope m is given by $(y - y_1) = m(x - x_1)$.

If you have a differentiable function $y = f(x)$ and a particular value of x, say, $x = c$, then the slope of the tangent line when $x = c$ is $f'(c)$. Since the point on the graph at this value of x is $(c, f(c)$, then the equation of the tangent line is $(y - f(c)) = f'(c)(x - c)$.

30 Conic Sections

The general equation of a conic section is

$$Ax^2 + Bx + Cy^2 + Dy + Exy + F = 0$$

In the problems that we have been looking at, $E = 0$. In this case the axes of the conic section are horizontal and/or vertical. The graph can be determined by "completing the squares" of the x and y. The graph may be degenerate (empty, a single point, a line, or a pair of crossing lines). If the graph is not degenerate, then we can tell what curve we will have:

- $A = B$: circle.
- $A \neq B$ but A and B have the same sign: ellipse.
- A and B have opposite signs: hyperbola.
- $A = 0$ or $B = 0$ (but not both): parabola.

31 Maximization and Minimization

If $f(x)$ is a differentiable function, or differentiable except, say, at a finite set of of points, we can try to find where it achieves its minimum or maximum value by looking at $f'(x)$. When $f'(x) > 0$ the function is increasing. When $f'(x) < 0$ the function is decreasing. When $f'(x) = 0$ or is undefined, we have a candidate for a local maximum or minimum. We then have to examine the behavior of the function by looking at its first and/or second derivatives to see if we have found a global minimum or a global maximum or neither.