MA 341 — Log of Class Activities

Contents

1 Wednesday, January 14

1. Analyzed the Game of 15 and showed it was isomorphic to Tic-Tac-Toe.
2. Discussed the syllabus and the nature of the course. Considered the rough analogy of teaching someone to play a musical instrument or perform heart surgery without ever actually watching them do it.
3. Solved the "Identical" problem.
4. Asked the class to read Chapter 1 and to play and try to analyze the "Game of 100".

2 Friday, January 16

1. Considered the "Strange Purchase" problem.

2. Mentioned the Class Log on the course website.

3. Reviewed the Game of Fifteen proof and analysis. See "Game of Fifteen — Analysis" on website. I also put a link to Tic-Tac-Toe on the website which includes an analysis of the game. Notice that in the analysis we exploit symmetry at times to reduce the work.

4. Discussed the "Game of 100". Some approaches to this problem: get your hands dirty; trial and error; consider a simpler case (e.g., lower goals than 100); work backwards; identify the penultimate step. Defined the notion of "key position" and proved that the key positions for this game are numbers with a remainder of 2 when divided by 7; i.e., congruent to 2 mod 7. Desrirbed the strategy for the problem of the Game of $1,000,000$ in which one may add numbers from 1 to 11—key positions are congruent to 4 mod 12. Described the strategy for the Game of n in which one may add numbers from 1 to ℓ—key positions are congruent to n mod $\ell = 1$. Collected Homework #1.

5. Passed out Polya's Four Phases of Problem Solving (also on the course website) and mentioned a few points relevant to the Game of 100.

6. Asked the class to work on the "Penny Piles" game (see "Problems" on the course website).

3 Wednesday, January 21

1. Considered Problem 1.3.3 in the text, and saw that one solution is the North Pole, but that there are infinitely many other solutions: Just start one mile north of a line of latitude near the South Pole of length $1/k$ miles, $k = 1, 2, 3, \ldots$.

2. Discussed the two Game of 100 problems from the homework and began to find some key positions.

3. Note: A *key* or *good* position is one for which *every* single move from that position goes to a bad position, and a *bad* position is one for which *there exists* a single move from that position to a good position.

4. Worked on the Penny Piles problem. We determined that some of the good positions are $(k, k, 0)$ for $k \geq 0$, and also $(1, 2, 3)$. But there are more....

5. Briefly discussed the Census Taker problem from the text, and the How Many Children problem from the homework. Decided to postpone collection of this particular problem until Monday.

4 Friday, January 23

1. Solved Problem 1.3.2 (lightbulbs).

2. Introduced LaTeX, an excellent typesetting program for mathematics. See the link on the course website for my examples.

3. Discussed the Penny Pile problem. The proposed key positions are 000, 011, 022, 033, 044, 055, 123, 145, 246, 257, 347, and 356. In solving this problem, one useful tactic was *systematic enumeration* and one useful tool to doing this is *lexicographic order.* We saw how to list all possible game positions in lexicographic order. (For each individual position we listed the pile sizes in order from least to greatest, since the order of the three piles in a position does not matter. Thus we exploited a little bit of *symmetry*.) Once we have the list, we can systematically examine positions, starting with the ones with the fewer pennies, to classify them into good or bad.

4. Mentioned (without explanation) that there is an "easy" way to determine whether any position is key or not. Write each pile size in binary, stack these representations, and count the number of 1's in each column. The position is a key position if and only if each of these counts is even. For example, consider position 145.

$$\begin{array}{cc} 1 & 001 \\ 4 & 100 \\ 5 & 101 \\ \hline & 202 \end{array} \longleftarrow \text{All of these numbers are even so this position is a key position.}$$

5. Collected homework #2.

6. Reminder: The "How Many Children?" problem is due Monday.

5 Monday, January 26

1. Tried out the Number Guessing Cards, and described its relationship to representing numbers in binary (base 2).

2. Discussed the problem of How Many Children? Collected homework.

3. Worked on Example 1.1.4 (handshaking couples) — refered to Polya's problem solving guidelines. In particular, there was some insight gained by looking at simpler versions of the problem.

6 Wednesday, January 28

1. Discussed "Thinking about Thinking—Cards".
2. Made suggestions on reading: Read Section 2.1; "skim" through Section 2.2—some of the problems are challenging, but the discussion on how to approach them is helpful; read Section 2.3 carefully.
3. Discussed proofs by mathematical induction. Proved that the following formula is true for all integers $n \geq 1$.

   $$1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}.$$

   Proved for all integers $n \geq 0$ that the number of subsets of $\{1, 2, \ldots, n\}$ is 2^n.
4. Think hard about the current homework assignment in preparation for class on Friday.

7 Friday, January 30

1. Presented the Animal Colors problem.
2. Worked on Problem 2.2.13 from book. A closed formula is desirable.
3. Showed a quick way to compute $S = 1 + 2 + 3 + \cdots + 100$.
4. Worked on Problem 5 of Homework #4, “Tiling with L’s”.

8 Monday, February 2

1. Presented the Newspaper Sheet problem.

2. Collected homework.

3. Illustrated Proofs by Contradiction. Gave two proofs that $\sqrt{2}$ is irrational, one being Example 2.3.2 from the text, and the other using the fact that every positive integer greater than 1 has a unique factorization into primes (ignoring the order of the primes.) Discussed why the first proof breaks down when you try to use it to prove that $\sqrt{9}$ is irrational, but can be adapted to prove that $\sqrt{6}$ is irrational.

4. Explained and gave an example of proofs by strong induction. Illustrated this by proving that every positive integer greater than 1 can be factored into primes. This also illustrated starting with a different base case (2).

5. Started working on the Towers of Hanoi problem.

6. I will email out a new homework assignment.

9 Wednesday, February 4

1. Presented the Six Triangles problem.

2. Worked on the Towers of Hanoi homework problem. Let $f(n)$ be the minimum number of moves to solve the n disk problem, $n \geq 1$. The class observed that

$$f(1) = 1 \text{ and } f(n) = 2f(n-1) + 1, \ n > 1.$$

   Most of the class seemed to have an argument for why this must hold, based on analysis of the puzzle. This is an example of a *recursive* formula, since $f(n)$ depends upon earlier values of f. Note that you need to specify a starting value along with the formula in order to completely determine the function. The class also observed that

$$f(n) = 2^n - 1$$

   appeared to hold. This is an example of a *closed* formula, since $f(n)$ is specified in terms of n alone.

   Generally speaking, if you know the truth of the recursive formula, you can prove the closed formula by induction. On the other hand, if you know the truth of a closed formula, you can proved the recursive formula by substitution.

3. Worked briefly on the other problems in Homework #5.

10 Friday, February 6

1. Professor Braun led this class.
2. Worked on Homework #5.
3. Briefly presented information about the power of teaching and learning in a classroom with conscious attention to “Growth Mind-Set”. See the two articles linked to the course website.

11 Monday, February 9

1. Presented problem East and West.

2. Reviewed solutions to Homework #5.

3. Proved 2.3.21: There are infinitely many primes. We proved this by contradiction.

4. Considered problem "Permutations". We argued several ways that the answer is $n!$, including imagining a decision tree, and considering the recursive formula $f(n) = nf(n-1)$ with $f(1) = 1$. Argued that for consistency it helps to define $0! = 1$ also.

12 Wednesday, February 11

Answered questions regarding Friday's exam.

13 Friday, February 13

Exam #1.

14 Monday, February 16

No class due to UK closure.

15 Wednesday, February 18

Worked on “Binomial Coefficients”.

16 Friday, February 20

No class due to UK closure.

17 Monday, February 23

Worked on “Permutations,” “Choosing and Permuting,” and “Choosing”.

18 Wednesday, February 25

1. Exam #2 will be on Wednesday, March 11.

2. Started with "What's my Rule?".

3. Continued working on "Permutations," "Choosing and Permuting," and "Choosing". In particular, we spent a lot of time relating solutions to Choosing and Permuting, and Choosing, when $n = 10$ and $k = 3$, ultimately seeing why you divide by 3! to move from one solution to the other. This kind of deep understanding is even more important than just knowing the final formulas. Along the way we also saw that the answer is also $(8+7+6+\cdots+1)+(7+6+5+\cdots+1)+(7+5+4+\cdots+1)+\cdots+(2+1)+(1)$ which after regrouping becomes $1\cdot 8+2\cdot 7+3\cdot 6+\cdots+8\cdot 1$. What is the easiest way to see that this computation gives the same solution in this case? In general?

4. Looked at the problem "Binomial Coefficients, Revisited," in the special case of thinking about $(x+y)\cdot(x+y)\cdot(x+y)\cdots(x+y) = (x+y)^{10}$, to understand why the coefficient of x^3y^7 is equal to the number of ways of choosing 3 objects out of 10.

19 Friday, February 27

1. Discussed the monk problem, Example 2.1.2.
2. Discussed Example 3.1.5.
3. Discussed Example 3.1.17 (though in class we integrated from 0 to π).
4. Worked on the last two homework problems, which relate to symmetry.

20 Monday, March 2

Discussed the first problem in the homework. Students may submit solutions on Wednesday if they wish. In particular, we saw how this problem leads to an understanding of the construction of Pascal's triangle.

21 Wednesday, March 4

1. Today's theme is to guess formulas for functions. Once you have come up with a credible formula, you can then try to prove it by, say, induction.

2. Worked on a problem like "Creating Formulas" part 1. One easy answer is $f(x) = 0$ for all x.

3. Worked on a problem like "Creating Formulas" part 2. We saw that we could use piecewise defined functions, but we also saw we could use $f(x) = (x-a)(x-b)(x-c)$.

4. Worked on a problem like "Creating Formulas" part 4. Again, we can use piecewise defined functions, but another way to do this, with a single polynomial, given values of $f(a) = p$, $f(b) = q$, $f(c) = r$, is to write

   $$f(x) = d_1(x-b)(x-c) + d_2(x-a)(x-c) + d_3(x-a)(x-b),$$

   and then substitute $x = a$ to solve for d_1, etc. (You have seen similar expressions when doing partial fractions.) The ultimate result will be a parabola (given three points) unless they are collinear. We saw how Wolframalpha and GeoGebra could also calculate this parabola.

5. Began working on "Creating Formulas" part 7.

6. Mentioned briefly that if you make a table of values for $f(0), f(1), f(2), \cdots$ you can sometimes detect patterns by looking at differences of these values, and differences of these differences, etc.

22 Friday, March 6

No class due to UK closure.

23 Monday, March 9

1. Reviewed solutions to the homework.

2. Briefly discussed logical statements. Consider the statement "A implies B". The statement "not B implies not A" is the contrapositive, and is logically equivalent to the original statement. The statement "B implies A" is the converse, and is not logically equivalent to the original statement, but IS logically equivalent to "not A implies not B", which is the inverse of the original statement.

24 Wednesday, March 11

Exam #2.

25 Friday, March 13

1. Quick problem: “Sum of Two Primes.”
2. Worked on “8 Card Trick” and “Counterfeit Coins” part 1.

26 Monday, March 23

1. Worked on "How Many Questions?". We saw that efficient questioning involved cutting down the possibilities by half with each question.
2. Worked on homework.

27 Wednesday, March 25

1. Showed that you can approach the "How Many Questions?" problem by asking about the digits of the number as expressed in binary.

2. Worked on homework.

28 Friday, March 27

1. Dr. Ben Braun led this class, starting with working on the Collatz problem. See, for example, `http://en.wikipedia.org/wiki/Collatz_conjecture`.

2. Introduced and discussed the eight Mathematical Practices of the Common Core State Standards for Mathematics; see `http://www.corestandards.org/Math/Practice` and `http://www.corestandards.org/wp-content/uploads/Math_Standards.pdf`.

29 Monday, March 30

1. Considered the problem "50 Words".
2. I encourage everyone to look at the problem "Two-Cube Calendar".
3. I spoke a bit about the importance of the Mathematical Practices.
4. Talked a bit about the problem "Venn Diagrams".
5. Worked on the problem "Watching TV". We saw several solution methods.
6. Solved the problem using a spreadsheet with a list of the 32 possibilities, and then systematically sorting and eliminating disallowed possibilities.

30 Wednesday, April 1

1. The solution to the "Two-Cube Calendar" was revealed.
2. Worked on "Logical Implications in Algebraic Reasoning." See separate handout.

31 Friday, April 3

Worked on "Logical Implications in Algebraic Reasoning."

32 Monday, April 6

Worked on "Logical Implications in Algebraic Reasoning."

33 Wednesday, April 8

Finished up logic problems.

34 Friday, April 10

1. Worked on the Pigeonhole Principle.

2. Exam #3 will be on Wednesday, April 22.

35 Monday, April 13

1. Worked on homework problems.
2. Discussed how to convert repeating decimals to fractions, and vice versa.

36 Wednesday, April 15

1. Solved the problem “Same Sums”.
2. Began working on the problem “Planar Clusters”.

37 Friday, April 17

Continued working on Planar Clusters. Professor Braun facilitated.

38 Monday, April 20

Answered questions relating to upcoming Exam #3.

39 Wednesday, April 22

Exam #3.

40 Friday, April 24

Workerd on Planar Clusters and Regular and Semiregular Tilings.

41 Monday, April 27

Workerd on Planar Clusters and Regular and Semiregular Tilings.

42 Wednesday, April 29

Workerd on Planar Clusters and Regular and Semiregular Tilings.

43 Friday, May 1

1. Answered some questions about Planar Clusters and Regular and Semiregular Tilings.
2. Made some concluding comments on the course.
3. Shared some useful websites and iPad apps.