Game of Fifteen — Analysis

Assertion: The Game of Fifteen is Tic-Tac-Toe in Disguise

Proof of the Assertion.

If we are only interested in presenting a proof of the assertion, we might proceed as follows.

> Place the nine cards in the square arrangement corresponding to the “Lo Shu” magic square:
>
> | | | |
> |---|---|---|
> | 8 | 1 | 6 |
> | 3 | 5 | 7 |
> | 4 | 9 | 2 |
>
> Observe that every row and column and each of the two diagonals have sum 15. Observe also that apart from these, there are no other triples of numbers in the Lo Shu that sum to 15. Thus, if we are playing Tic-Tac-Toe on this board, the winning Tic-Tac-Toe lines correspond precisely to the triples of numbers that sum to 15, and hence the winning sets of the Game of Fifteen. Thus the Game of Fifteen and the Game of Tic-Tac-Toe are equivalent; i.e., isomorphic. □

Analysis Leading to the Proof of the Assertion.

However, we are also interested in understanding, and recording, how we might arrive at this proof in the first place. Describing this is essential to understanding how one might come to such a proof, but often omitted when mathematics is published. I know there are cases in which mathematicians who have published a particular theorem can no longer recall how they came up with the proof in the first place. We might summarize our class discussion as follows.

> After playing the game a few times we noticed that it shared some common characteristics with Tic-Tac-Toe: winning sets consisted of three elements; some moves consisted in blocking potential sets of the opponent; sometimes a player won by having two potential sets, only one of which could be blocked; and some games ended in a tie.

We then tried to label the cells in a Tic-Tac-Toe board with the numbers from 1 to 9 to create an equivalence between these two games. We listed of all winning combinations in the Game of Fifteen.

1	5	9
1	6	8
2	4	9
2	5	8
2	6	7
3	4	8
3	5	7
4	5	6

We noticed that 5 appears in four winning combinations, just as the center square of Tic-Tac-Toe lies in four winning lines. This suggests labeling the center square of the Tic-Tac-Toe board with 5.

	5	

Then we observed that the numbers 1, 3, 7, and 9 each appear in only two winning combinations, while 2, 4, 6, and 8 each appear in three winning combinations. This suggests that these odd numbers should appear in the centers of the outer rows and columns of the Tic-Tac-Toe board, since these cells are each only involved in two winning Tic-Tac-Toe lines. Without loss of generality (by symmetry of the square), let's place 1 in the upper center cell.

	1	
	5	

If these games are to be equivalent, this forces us to place 9 in the lower center cell.

	1	
	5	
	9	

Again, without loss of generality, we can place 3 in the left center cell, which then forces us to place 7 in the right center cell.

	1	
3	5	7
	9	

Where should we place 2? Observing that the winning combinations containing 2 involve 7 and 9, we see we must place it in the right lower cell, which forces us to place 8 in the left upper cell.

8	1	
3	5	7
	9	2

The final two placements are then also forced, resulting in the labeling

8	1	6
3	5	7
4	9	2

We can now carefully check that every winning line of Tic-Tac-Toe corresponds to a winning set in the Game of Fifteen, and vice versa, so we have verified that these two games are equivalent; i.e., isomorophic. □

We see that this analysis can be quite a bit longer than the final proof itself. But it is a valuable record of mathematical practices, which we are committed to developing in our students, and may end up being important to refer to when tackling other new problems.