MA 310 — Homework #4

Solutions

1. Prove that for all positive integers n,

$$\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\cdots+\frac{1}{n(n+1)}=\frac{n}{n+1}.$$

Solution. We will prove this by induction on $n \geq 1$.

Base Case. Assume $n = 1$. Then the left hand side equals $\frac{1}{1\cdot 2} = \frac{1}{2}$ and the right hand side equals $\frac{1}{1+1} = \frac{1}{2}$, so the formula is true in this case.

Inductive Step. Assume the formula is true for $n = k \geq 1$ (this is the inductive hypothesis). We need to prove the formula is then also true for $n = k + 1$. Using the inductive hypothesis for the first equality below, we have:

$$\begin{aligned}
\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\cdots+\frac{1}{k(k+1)}+\frac{1}{(k+1)(k+2)} &= \frac{k}{k+1}+\frac{1}{(k+1)(k+2)} \\
&= \frac{k(k+2)+1}{(k+1)(k+2)} \\
&= \frac{k^2+2k+1}{(k+1)(k+2)} \\
&= \frac{(k+1)(k+1)}{(k+1)(k+2)} \\
&= \frac{k+1}{k+2},
\end{aligned}$$

and the right hand side matches the right hand side of the formula when $n = k + 1$.

Therefore the formula is true for all $n \geq 1$ by induction.

2. Solve Example 1.1.4 from the textbook.

Solution. Since no one shakes hands with his/her spouse, the number of times any person can shake hands ranges from 0 to 20. Since each of the 21 people questioned gave different answers, we know the answers given must have been precisely $0, 1, 2, \ldots, 20$, and let's refer to the people by these numbers, using H for the host asking the questions.

Person 20 must have shaken hands with everybody except his/her spouse, so none of those he shook hands with is person 0, and his/her spouse must therefore be person 0. Now remove persons 0 and 20 from the group, and disregard their handshakes. The number of handshakes of the remaining 20 people $(H, 1, 2, 3, \ldots, 19)$ is each reduced by one, leaving 20 people with handshake numbers $H-1, 0, 1, 2, \ldots, 18$. By the same reasoning as before, the new person 18 (who used to be 19) must be married to the new person 0 (who used to be 1). Remove this couple and repeat, and by the same reasoning we eventually get that, based on the original handshake numbers, the couples are 0-20, 1-19, 2-18, 3-17, 4-16, 5-15, 6-14, 7-13, 8-12, and 9-11, leaving H and 10, who must be the final couple. Thus the host's spouse shook hands 10 times. Note: this argument can be reformulated as a proof by induction on the number of couples.

3. Solve Problem 2.2.13 from the textbook.

   Solution. We will prove that n lines divide the plane into $\frac{n^2+n+2}{2}$ regions, $n \geq 0$. The proof will be by induction on $n \geq 0$.

   Base Case. Assume $n = 0$. Then there is only one region, and indeed $\frac{0^2+0+2}{2} = 1$. So the formula is true in this case.

   Inductive Step. Assume the formula is true for $n = k \geq 0$. We need to prove the formula is then also true for $n = k+1$. So consider $k+1$ lines in the plane. Temporarily remove line $k+1$, leaving k lines in the plane, and by the inductive hypothesis, $\frac{k^2+k+2}{2}$ regions. Now when line $k+1$ is restored, it will intersect each of the other k lines in k distinct points. These k points divide line $k+1$ into $k+1$ pieces, each of which corresponds to a region that is split in two by the placement of line $k+1$. So now the number of regions is

$$\begin{aligned}\frac{k^2+k+2}{2} + (k+1) &= \frac{(k^2+k+2)+2(k+1)}{2} \\ &= \frac{k^2+3k+4}{2} \\ &= \frac{(k+1)^2+(k+1)+2}{2},\end{aligned}$$

   which is the desired formula for $n = k+1$.

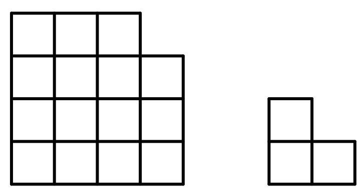
   Therefore the formula is true for all $n \geq 0$ by induction.

4. Let a, b, c be integers satisfying $a^2 + b^2 = c^2$. Prove that abc must be even.

Solution. First note that the square of any odd integer is odd, and the square of any even integer is even. We prove the result by contradiction.

We are given that $a^2 + b^2 = c^2$. Assume abc is odd. Then each of a, b, c must be odd. Thus each of a^2, b^2, c^2 must be odd. From this it follows that $a^2 + b^2$ is even but c^2 is odd, so it is impossible that $a^2 + b^2 = c^2$. This contradiction means that our initial assumption that abc is odd is not valid. Therefore abc is even.

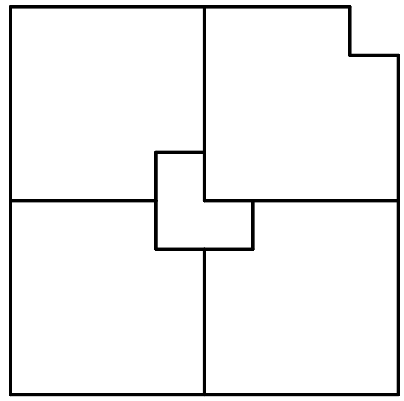
5. For n a positive integer consider an array of $2^n \times 2^n$ squares, with the upper right-hand square removed. Prove that this array can be tiled by "L's" consisting of three squares. In the figure below we show the array $n = 2$, and one "L".



Solution. We will prove this by induction on $n \geq 1$.

Base Case. Assume $n = 1$. Then the $2^1 \times 2^1$ array with one square removed is identical to a single L, and hence is tiled by one L. So the statement is true for $n = 1$.

Inductive Step. Assume the statement is true for $n = k \geq 1$. We need to prove that the statement is then also true for $n = k + 1$. So consider an array of size $2^{k+1} \times 2^{k+1}$ with the upper right-hand square removed. (See the figure below.)



Break the $2^{k+1} \times 2^{k+1}$ array into four $2^k \times 2^k$ arrays. From the upper left-hand array remove the lower right-hand corner, and then tile it by induction. From the lower left-hand array remove the upper right-hand corner, and tile it by induction. From the lower right-hand array remove the upper left-hand corner, and tile it by induction. From the upper right-hand array remove the upper right-hand corner, and tile it by induction. The three missing corners from the first three arrays leave a hole in the middle in the shape of a single L, so place an L here to complete the tiling. Hence we have tiled the array for the case $n = k + 1$.

Therefore the statement is true for all $n \geq 1$ by mathematical induction.