MA 310 — Homework #6

Due Monday, March 2, in class

1. Solve "Binomial Coefficients" in the file "Problems" by doing the following: For nonnegative integer n consider the expansion of

$$(x+y)^n = c_{n,0}x^n y^0 + c_{n,1}x^{n-1}y^1 + c_{n,2}x^{n-2}y^2 + \cdots + c_{n,n}x^0 y^n.$$

We are going to figure out formulas for these coefficients.

(a) Think carefully about the fact that $(x+y)(x+y)^{n-1} = (x+y)^n$. Then prove (without induction) that

$$c_{n,0} = c_{n,n} = 1, \text{ for all } n \geq 0,$$

and

$$c_{n-1,k-1} + c_{n-1,k} = c_{n,k} \text{ for all } n \geq 1,\ 1 \leq k \leq n-1.$$

Solution. The expansion of $(x+y)^n = (x+y)\cdots(x+y)$ includes the terms x^n and y^n. Thus $c_{n,0} = c_{n,n} = 1$.

Now consider $(x+y)^n = (x+y)(x+y)^{n-1}$. Assume $1 \leq k \leq n-1$. One of the terms on the left-hand side is $c_{n,k}x^{n-k}y^k$. This must come from multiplying (on the right-hand side) x by $c_{n-1,k}x^{n-k-1}y^k$ and from multiplying y by $c_{n-1,k-1}x^{n-k}y^{k-1}$. The result on the right-hand side is $(c_{n-1,k-1} + c_{n-1,k}x^{n-k}y^k$. Thus $c_{n-1,k-1} + c_{n-1,k} = c_{n,k}$.

(b) Now prove by induction on $n \geq 0$ that

$$c_{n,k} = \frac{n!}{k!(n-k)!},\ n \geq 0,\ 0 \leq k \leq n.$$

Solution. Base case. If $n = 0$ then $(x+y)^0 = 1x^0y^0$ so $c_{0,0} = 1$. Verifying the formula, $\frac{0!}{0!0!} = \frac{1}{1} = 1$ also. We can check in general that the formula yields 1 for $c_{n,0}$ and for $c_{n,n}$ for all n, since $\frac{n!}{0!n!} = \frac{n!}{n!0!} = 1$.

For the inductive step, assume the formula is true for $n = r-1$, $r \geq 1$, for all k, $0 \leq k \leq r-1$. Then the formula is also true for $n = r$, and for all $1 \leq k \leq r-1$, since

$$\begin{aligned}
c_{r,k} &= c_{r-1,k-1} + c_{r-1,k} \\
&= \frac{(r-1)!}{(k-1)!(r-k)!} + \frac{(r-1)!}{k!(r-1-k)!} \\
&= \frac{(r-1)!}{(k-1)!(r-k)!}\frac{k}{k} + \frac{(r-1)!}{k!(r-1-k)!}\frac{r-k}{r-k} \\
&= \frac{k(r-1)!}{k!(r-k)!} + \frac{(r-k)(r-1)!}{k!(r-k)!} \\
&= \frac{(k+r-k)(r-1)!}{k!(r-k)!} \\
&= \frac{r!}{k!(r-k)!}.
\end{aligned}$$

This concludes the inductive step.

Therefore the formula is true for all n by induction.

2. Solve "Choosing and Permuting" in the file "Problems."

   Solution. You have n choices for the first book on the shelf, $n-1$ choices for the second, $n-2$ for the third, etc., for a total of k terms. Thus the formula is $n(n-1)(n-2)\cdots(n-k+1)$. This can be rewritten as

$$n(n-1)(n-2)\cdots(n-k+1) = n(n-1)(n-2)\cdots(n-k+1)\frac{(n-k)!}{(n-k)!} = \frac{n!}{(n-k)!}.$$

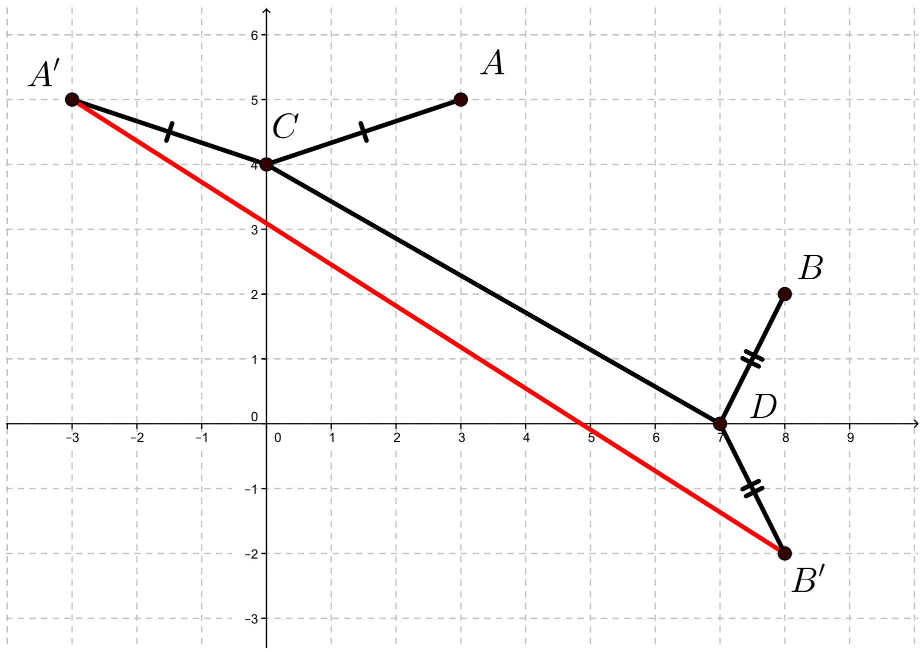
3. Using the solution to the previous problem, solve "Choosing" in the file "Problems."

   Solution. Think about first choosing k books to line up on a shelf, and then removing the books and placing them in the backpack. For each selection of k particular books, these can be permuted in $k!$ ways, and each of these different orderings result in the same collection of books in the backpack. So you must divide the answer to the previous problem by $k!$, giving the answer

$$\frac{n!}{k!(n-k)!}.$$

4. Read Section 3.1 on Symmetry in the text, and especially study Example 3.1.5. Now solve Problem 3.1.13. Include a neat and accurate sketch.

Solution.



Let $A = (3, 5)$ and $B = (8, 2)$. We are looking for a shortest path of the form $ACDB$ in the figure. Consider such a path and reflect A across the y-axis to get $A' = (-3, 5)$ and B across the x-axis to get $B' = (8, -2)$. Then $AC = A'C$ and $BD = B'D$. So the length of the path $ACDB$ also equals the length of the path $A'CDB'$. To make this latter path as short as possible, we need position C and D so that $A'CDB'$ is a straight line segment (indicated by the red line segment in the figure). Thus the shortest path has length equal to the distance $A'B'$, which equals

$$\sqrt{(8+3)^2 + (-2-5)^2} = \sqrt{170}.$$

5. A triangle is inscribed in a given circle. Prove that if the triangle is not equilateral, then there is another triangle with larger area that can be inscribed in the same circle.

Solution.

Let ΔABC be such a triangle. Since the triangle is not equilateral, there must exist two sides, say, $\overline{AB}$ and $\overline{AC}$, that are not equal in length. Then the point A will not lie on the perpendicular bisector of $\overline{BC}$, but moving the point A along the circle to this position A' will result in a triangle with the same base $\overline{BC}$ but strictly greater altitude, hence larger area.