Homework #7 Solutions

Let $f(n) = 1^3 + 2^3 + 3^3 + \cdots + n^3$. We can make a table of values and calculate differences.

n	0		1		2		3		4		5		6
$f(n)$	0		1		9		36		100		225		441
		1		8		27		64		125		216	
			7		19		37		61		91		
				12		18		24		30			
					6		6		6				
						0		0					

Since the fifth row of differences seems to be zeros, we try to find a polynomial of degree 4, and use the first five values of $f(n)$.

We try

$$\begin{aligned} f(n) &= c_0(n-1)(n-2)(n-3)(n-4) + c_1 n(n-2)(n-3)(n-4) + c_2 n(n-1)(n-3)(n-4) \\ &\quad + c_3 n(n-1)(n-2)(n-4) + c_4 n(n-1)(n-2)(n-3) \end{aligned}$$

Substituting in $n = 0, 1, 2, 3, 4$ gives us

$$\begin{aligned} 0 &= c_0(-1)(-2)(-3)(-4) \text{ so } c_0 = 0 \\ 1 &= c_1(1)(-1)(-2)(-3) \text{ so } c_1 = -1/6 \\ 9 &= c_2(2)(1)(-1)(-2) \text{ so } c_2 = 9/4 \\ 36 &= c_3(3)(2)(1)(-1) \text{ so } c_3 = -36/6 = -6 \\ 100 &= c_4(4)(3)(2)(1) \text{ so } c_4 = 100/24 = 25/6 \end{aligned}$$

Thus

$$\begin{aligned} f(n) &= -(1/6)n(n-2)(n-3)(n-4) + (9/4)n(n-1)(n-3)(n-4) \\ &\quad -6n(n-1)(n-2)(n-4) + (25/6)(n)(n-1)(n-2)(n-3). \end{aligned}$$

Simplify with algebra to get

$$f(n) = (1/4)n^4 + (1/2)n^3 + (1/4)n^2.$$

Now to prove this by induction on $n \geq 1$.

For $n = 1$ the formula gives $(1/4) + (1/2) + (1/4) = 1$, which is correct.

Now assume that the formula is true for $n = k - 1$, $k \geq 1$, and prove it is true for $n = k$.

$$\begin{aligned} f(k) &= 1^3 + 2^3 + 3^3 + \cdots + (k-1)^3 + k^3 \\ &= ((1/4)(k-1)^4 + (1/2)(k-1)^3 + (1/4)(k-1)^2) + k^3 \\ &= \ldots \text{algebra} \ldots \\ &= (1/4)k^4 + (1/2)k^3 + (1/4)k^2 \end{aligned}$$

This concludes the inductive step, and so the formula is true by induction.