MA 310 — Homework #9

Solutions

1. Solve "Logical Implications in Algebraic Reasoning," parts 6–12, using careful logical reasoning.

   (a) Solve $|x+1| + |x-1| = 2$.

   Solution.

   $|x+1| + |x-1| = 2$

   $\Updownarrow$

   $[x+1 \geq 0 \text{ and } x-1 \geq 0 \text{ and } (x+1)+(x-1)=2]$ or

   $[x+1 \geq 0 \text{ and } x-1 \leq 0 \text{ and } (x+1)-(x-1)=2]$ or

   $[x+1 \leq 0 \text{ and } x-1 \geq 0 \text{ and } -(x+1)+(x-1)=2]$ or

   $[x+1 \leq 0 \text{ and } x-1 \leq 0 \text{ and } -(x+1)-(x-1)=2]$

   $\Updownarrow$

   $[x \geq -1 \text{ and } x \geq 1 \text{ and } 2x=2]$ or

   $[x \geq -1 \text{ and } x \leq 1 \text{ and } 2=2]$ or

   $[x \leq -1 \text{ and } x \geq 1 \text{ and } -2=2]$ or

   $[x \leq -1 \text{ and } x \leq 1 \text{ and } -2x=2]$

   $\Updownarrow$

   $[x=1]$ or $[-1 \leq x \leq 1]$ or $[x \in \emptyset]$ or $[x=-1]$

   $\Updownarrow$

   $-1 \leq x \leq 1$

   (b) Solve $\frac{1}{x^2-1} = \frac{1}{3x+3}$.

   Solution.

   $\frac{1}{x^2-1} = \frac{1}{3x+3}$

   $\Updownarrow$

   $[x \neq \pm 1]$ and $[\frac{1}{x^2-1} = \frac{1}{3x+3}]$

   $\Updownarrow$

   $[x \neq \pm 1]$ and $[x^2-1 = 3x+3]$

   $\Updownarrow$

   $[x \neq \pm 1]$ and $[x^2-3x-4=0]$

   $\Updownarrow$

   $[x \neq \pm 1]$ and $[(x-4)(x+1)=0]$

$\Updownarrow$
$[x \neq \pm 1]$ and $[(x-4)=0$ or $(x+1)=0]$
$\Updownarrow$
$[x \neq \pm 1]$ and $[x=4$ or $x=-1]$
$\Updownarrow$
$[x \neq \pm 1]$ and $[x \in \{-1, 4\}]$
$\Updownarrow$
$x=4$

(c) Solve $\frac{x^2}{x-1} = \frac{2-x}{x-1}$.

Solution.
$\frac{x^2}{x-1} = \frac{2-x}{x-1}$
$\Updownarrow$
$[x \neq 1]$ and $[\frac{x^2}{x-1} = \frac{2-x}{x-1}]$
$\Updownarrow$
$[x \neq 1]$ and $[x^2 = 2-x]$
$\Updownarrow$
$[x \neq 1]$ and $[x^2 + x - 2 = 0]$
$\Updownarrow$
$[x \neq 1]$ and $[(x+2)(x-1) = 0]$
$\Updownarrow$
$[x \neq 1]$ and $[(x+2)(x-1) = 0]$
$\Updownarrow$
$[x \neq 1]$ and $[x+2=0$ or $x-1=0]$
$\Updownarrow$
$[x \neq 1]$ and $[x=-2$ or $x=1]$
$\Updownarrow$
$[x \neq 1]$ and $[x \in \{-2, 1\}]$
$\Updownarrow$
$x=-2$

(d) Solve $\frac{1}{\sqrt{x^2-1}} \geq \frac{1}{\sqrt{3x+3}}$.

Solution.
$\frac{1}{\sqrt{x^2-1}} \geq \frac{1}{\sqrt{3x+3}}$

$\Updownarrow$

$[x^2 - 1 > 0]$ and $[3x + 3 > 0]$ and $[\frac{1}{\sqrt{x^2-1}} \geq \frac{1}{\sqrt{3x+3}}]$

$\Updownarrow$

$[(x + 1)(x - 1) > 0]$ and $[x > -1]$ and $[\sqrt{x^2 - 1} \leq \sqrt{3x + 3}]$

$\Updownarrow$

$[(x + 1 > 0$ and $x - 1 > 0)$ or $(x + 1 < 0$ and $x - 1 < 0)$ $]$ and $[x > -1]$ and $[x^2 - 1 \leq 3x + 3]$

$\Updownarrow$

$[(x > -1$ and $x > 1)$ or $(x < -1$ and $x < 1)$ $]$ and $[x > -1]$ and $[x^2 - 3x - 4 \leq 0]$

$\Updownarrow$

$[x > 1$ or $x < -1]$ and $[x > -1]$ and $[(x - 4)(x + 1) \leq 0]$

$\Updownarrow$

$[x > 1$ or $x < -1]$ and $[x > -1]$ and $[(x - 4 \geq 0$ and $x + 1 \leq 0)$ or $(x - 4 \leq 0$ and $x + 1 \geq 0)]$

$\Updownarrow$

$[x > 1$ or $x < -1]$ and $[x > -1]$ and $[(x \geq 4$ and $x \leq -1)$ or $(x \leq 4$ and $x \geq -1)]$

$\Updownarrow$

$[x > 1$ or $x < -1]$ and $[x > -1]$ and $[x \in \emptyset$ or $-1 \leq x \leq 4]$

$\Updownarrow$

$[x > 1]$ and $[\ -1 \leq x \leq 4]$

$\Updownarrow$

$1 < x \leq 4$

(e) Solve $x(2x + 3) = x(x - 5)$.

Solution.

$x(2x + 3) = x(x - 5)$

$\Updownarrow$

$x(2x + 3) - x(x - 5) = 0$

$\Updownarrow$

$x(2x + 3 - x + 5) = 0$

$\Updownarrow$

$x(x + 8) = 0$

$\Updownarrow$

$x = 0$ or $x - 8 = 0$

$\Updownarrow$

$x = 0$ or $x = 8$

$\Updownarrow$

$x \in \{0, 8\}$

(f) Solve $\frac{1}{\frac{1}{x}} = x$.

Solution.

$\frac{1}{\frac{1}{x}} = x$

$\Updownarrow$

$x \neq 0$ and $\frac{1}{\frac{1}{x}} = x$

$\Updownarrow$

$x \neq 0$ and $x = x$

$\Updownarrow$

$x \in \mathbf{R} \setminus \{0\}$

(g) Solve $\sqrt{x^2 - 5x + 5} = \sqrt{x - 3}$.

Solution.

$\sqrt{x^2 - 5x + 5} = \sqrt{x - 3}$

$\Updownarrow$

$x^2 - 5x + 5 \geq 0$ and $x - 3 \geq 0$ and $\sqrt{x^2 - 5x + 5} = \sqrt{x - 3}$

$\Updownarrow$

$x^2 - 5x + 5 \geq 0$ and $x \geq 3$ and $x^2 - 5x + 5 = x - 3$

$\Updownarrow$

$x^2 - 5x + 5 \geq 0$ and $x \geq 3$ and $x^2 - 6x + 8 = 0$

$\Updownarrow$

$x^2 - 5x + 5 \geq 0$ and $x \geq 3$ and $(x - 2)(x - 4) = 0$

$\Updownarrow$

$x^2 - 5x + 5 \geq 0$ and $x \geq 3$ and $[x - 2 = 0$ or $x - 4 = 0]$

$\Updownarrow$

$x^2 - 5x + 5 \geq 0$ and $x \geq 3$ and $[x = 2$ or $x = 4]$

$\Updownarrow$

$x = 4$

2. Solve "Outdoor Barbeque".

Solution.

	Nurse	Secretary	Teacher	Pilot	Hamburger	Chicken	Steak	Hot Dogs
Tom	0	0	0	1	0	0	0	1
John	0	1	0	0	1	0	0	0
Fred	1	0	0	0	0	0	1	0
Bill	0	0	1	0	0	1	0	0
Hamburger	0	1	0	0				
Chicken	0	0	1	0				
Steak	1	0	0	0				
Hot Dogs	0	0	0	1				

3. Solve "Socks".

Solution.

If you take only 3 socks, they may be of different colors, so 3 is not enough. But if you take 4 socks, you must have at least 2 socks of the same color—because if you have a set of socks with no matching colors, you can have at most one of each color; namely, a total of 3.