7 Quadrilaterals

This material is summarized from Section 3.7 of Kay.

Definition: If A, B, C, and D are any four points lying in a plane such that no three of them are collinear, and if the points are so situated that no pair of open segments $\overset{\circ\!-\!\circ}{AB}$, $\overset{\circ\!-\!\circ}{BC}$, $\overset{\circ\!-\!\circ}{CD}$, and $\overset{\circ\!-\!\circ}{DA}$ have any points in common, then the set $\diamond ABCD = \overline{AB} \cup \overline{BC} \cup \overline{CD} \cup \overline{DA}$ is a *quadrilateral*, with *vertices* A, B, C, D; *sides* $\overline{AB}$, $\overline{BC}$, $\overline{CD}$, $\overline{DA}$; *diagonals* $\overline{AC}$, $\overline{BD}$, and *angles* $\angle DAB$, $\angle ABC$, $\angle BCD$, $\angle CDA$.

Definition: A quadrilateral is *convex* if its diagonals intersect.

Lemma: The diagonals of a convex quadrilateral intersect at an interior point on each diagonal.

Lemma: If $\diamond ABCD$ is a convex quadrilateral, then D lies in the interior of $\angle ABC$ (and similarly for the other vertices).

Lemma: If A, B, C, and D are consecutive vertices of a convex quadrilateral, then $m\angle BAD = m\angle BAC + m\angle CAD$.

Definition: Two quadrilaterals $\diamond ABCD$ and $\diamond XYZW$ are *congruent* under the correspondence $ABCD \leftrightarrow XYZW$ iff all pairs of corresponding sides and angles under the correspondence are congruent (i.e., CPCF). Such congruence will be denoted by $\diamond ABCD \cong \diamond XYZW$.

Theorem 3.7.1 (SASAS): Suppose that two convex quadrilaterals $\diamond ABCD$ and $\diamond XYZW$ satisfy the SASAS Hypothesis under the correspondence $ABCD \leftrightarrow XYZW$. That is, three consecutive sides and the two angles included by those sides of $\diamond ABCD$ are congruent, respectively, to the corresponding three consecutive sides and two included angles of $\diamond XYZW$. Then $\diamond ABCD \cong \diamond XYZW$. (This is Theorem 1 of Section 3.7 of Kay.)

There are other congruence theorems for, such as $ASASA$, $SASAA$, and $SASSS$.

What about $ASAA$?

Definition: A *rectangle* is a convex quadrilateral having four right angles.

Definition: Let $\overline{AB}$ be any line segment, and erect two perpendiculars at the endpoints A and B. Mark off points C and D on these perpendiculars so that C and D lie on the same side of line $\overleftrightarrow{AB}$, and $BC = AD$. Join C and D. The resulting quadrilateral is a *Saccheri Quadrilateral*. Side $\overline{AB}$ is called the *base*, $\overline{BC}$ and $\overline{AD}$ the *legs*, and side $\overline{CD}$ the *summit*. The angles at C and D are called the *summit angles*.

Lemma: A Saccheri Quadrileral is convex.

Theorem 3.7.2: The summit angles of a Saccheri Quadrilateral are congruent. (This is Theorem 2 of Section 3.7 of Kay.)

Corollary:

1. The diagonals of a Saccheri Quadrilateral are congruent.
2. The line joining the midpoints of the base and the summit of a Saccheri Quadrilateral is the perpendicular bisector of both the base and summit.
3. If each of the summit angles of a Saccheri Quadrilateral is a right angle, the quadrilateral is a rectangle, and the summit is congruent to the base.
4. If the summit angles of a Saccheri Quadrilateral are acute, the summit has greater length than the base.

Theorem 3.7.3: Let $\triangle ABC$ be any triangle. Let M and N be the midpoints of $\overline{AB}$ and $\overline{AC}$, respectively. Let $\overline{BB'}$ and $\overline{CC'}$ be perpendiculars to $\overleftrightarrow{MN}$, with B' and C' lying on $\overleftrightarrow{MN}$. Then $\diamond BCC'B'$ is a Saccheri Quadrilateral with base $\overline{B'C'}$ and summit $\overline{BC}$. Moreover, the angle sum of $\triangle ABC$ equals twice the measure of either summit angle of the quadrilateral, and $MN = \frac{1}{2}B'C'$. (This is Theorem 3 of Section 3.7 of Kay.)

Corollary:

1. The summit angles of a Saccheri Quadrilateral are either acute or right.
2. The summit of a Saccheri Quadrilateral has length greater than or equal to that of the base.
3. The line joining the midpoints of two sides of a triangle has length less than or equal to one-half that of the third side.