MA 341 — Homework #4

Solutions

1. Course Notes 2.10.1.

   Solution. The points on $\overleftrightarrow{NA}$ are given parametrically by

   $$N + t(A - N)$$
   $$(0,0,1) + t(x-0, y-0, z-1)$$
   $$(tx, ty, 1 + t(z-1))$$

   To determine where this ray intersects the plane given by the equation $z = 0$, we need to find t such that $1 + t(z-1) = 0$. This is easy; $t = 1/(1-z)$. Substituting into the parametric equation gives $p = tx = x/(1-z)$ and $q = ty = y/(1-z)$.

2. Course Notes 2.10.2.

   Solution.

   $$\begin{aligned} x^2 + y^2 + z^2 &= \frac{(2p)^2 + (2q^2) + (p^2 + q^2 - 1)^2}{(p^2 + q^2 + 1)^2} \\ &= \frac{4p^2 + 4q^2 + p^4 + q^4 + 1 + 2p^2q^2 - 2p^2 - 2q^2}{(p^2 + q^2 + 1)^2} \\ &= \frac{p^4 + q^4 + 1 + 2p^2 + 2q^2 + 2p^2q^2}{(p^2 + q^2 + 1)^2} \\ &= \frac{(p^2 + q^2 + 1)^2}{(p^2 + q^2 + 1)^2} \\ &= 1. \end{aligned}$$

3. Course Notes 2.10.3.

   Solution. We check $\phi \circ \phi^{-1}$ by substituting the formulas for x, y, z into those for p, q:

   $$\begin{aligned} 1 - z &= 1 - \frac{p^2 + q^2 - 1}{p^2 + q^2 + 1} \\ &= \frac{p^2 + q^2 + 1 - (p^2 + q^2 - 1)}{p^2 + q^2 + 1} \\ &= \frac{2}{p^2 + q^2 + 1}. \end{aligned}$$

So

$$\begin{aligned} p &= \frac{x}{1-z} \\ &= \frac{(2p)/(p^2+q^2+1)}{(2)/(p^2+q^2+1)} \\ &= p, \text{ as required.} \\ q &= \frac{y}{1-z} \\ &= \frac{(2q)/(p^2+q^2+1)}{(2)/(p^2+q^2+1)} \\ &= q, \text{ as required.} \end{aligned}$$

So $\phi \circ \phi^{-1}$ is the identity map.

We check $\phi^{-1}\phi$ by substituting the formulas for p, q into those for x, y, z:

$$\begin{aligned} p^2+q^2+1 &= \frac{x^2+y^2+(1-z)^2}{(1-z)^2} \\ &= \frac{x^2+y^2+1-2z+z^2}{(1-z)^2} \\ &= \frac{2-2z}{(1-z)^2} \\ &= \frac{2}{1-z}. \end{aligned}$$

So

$$\begin{aligned} x &= \frac{2p}{p^2+q^2+1} \\ &= \frac{(2x)/(1-z)}{(2)/(1-z)} \\ &= x, \text{ as required.} \end{aligned}$$

$$
\begin{aligned}
y &= \frac{2q}{p^2+q^2+1} \\
&= \frac{(2y)/(1-z)}{(2)/(1-z)} \\
&= y, \text{ as required.} \\
z &= \frac{p^2+q^2-1}{p^2+q^2+1} \\
&= \frac{p^2+q^2+1-2}{p^2+q^2+1} \\
&= \frac{2/(1-z)-2}{2/(1-z)} \\
&= \frac{2-2(1-z)}{2} \\
&= z, \text{ as required.}
\end{aligned}
$$

4. Assume we know that the Pythagorean Theorem holds in $\mathbf{E}^2$. Use this to derive the formula $\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$ for the distance between the points $A=(x_1,y_1)$ and $B=(x_2,y_2)$. Hint: Consider a third point $C=(x_1,y_2)$.

   Solution. If A and B do not lie on a horizontal or vertical line, consider the right triangle ΔABC with right angle at C. Note that $AC=|y_2-y_1|$, $BC=|x_2-x_1|$, and that $AB^2=BC^2+AC^2$ by the Pythagorean Theorem. In the case that A and B line on a common horizontal line, then $y_2=y_1$ and so $C=A$. Then in this case $AB^2=BC^2+AC^2$ still holds, because this is equivalent to $AB^2=AB^2+0^2$. A similar argument shows that $AB^2=BC^2+AC^2$ in the case that A and B line on a common vertical line. Thus in any case we conclude that

$$AB=\sqrt{BC^2+AC^2}=\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}.$$

5. Assume we know that two lines L_1 and L_2 with respective direction vectors (u_1,v_1) and (u_2,v_2) are perpendicular if and only if (u_2,v_2) is a nonzero multiple of $(v_1,-u_1)$. Consider any right triangle ΔABC with right angle at A. Then there is a direction vector (u,v) and numbers s and t such that $B=A+s(u,v)$ and $C=A+t(v,-u)$.

Use this, together with the distance formula, to prove that the Pythagorean Theorem holds for ΔABC.

Solution. Assume that $A = (x, y)$. Then $B = (x+su, y+sv)$ and $C = (x+tv, y-tu)$, and we can use the distance formula to compute

$$AB = \sqrt{(su)^2 + (sv)^2},$$

$$AC = \sqrt{(tv)^2 + (-tu)^2},$$

and

$$\begin{aligned} BC &= \sqrt{(tv - su)^2 + (-tu - sv)^2} \\ &= \sqrt{(tv)^2 - 2stuv + (su)^2 + (tu)^2 + 2stuv + (sv)^2} \\ &= \sqrt{(tv)^2 + (su)^2 + (tu)^2 + (sv)^2} \\ &= \sqrt{AB^2 + AC^2}. \end{aligned}$$