6 Angles

6.1 The Angle Axioms and Basic Theorems

The following is a summary of Section 2.7 of Kay.

Axiom A-1 — Existence of Angle Measure: To every angle $\angle A$ there corresponds a unique, real number $\theta = m\angle A$, $0 < \theta < 180$, called its *measure*.

Axiom A-2 — Angle Addition Postulate: If D lies in the interior of $\angle ABC$, then $m\angle ABC = m\angle ABD + m\angle DBC$, and conversely.

Axiom A-3 — Protractor Postulate: The set of rays having a common origin O and lying on one side of line $\ell = \overleftrightarrow{OA}$, including ray $\overrightarrow{OA}$, may be assigned to the real numbers θ for which $0 \leq \theta < 180$, called *coordinates*, in such a manner that

1. Each ray is assigned a unique coordinate θ.
2. Each coordinate θ is assigned to a unique ray.
3. The coordinate of $\overrightarrow{OA}$ is 0.
4. If rays $\overrightarrow{OP}$ and $\overrightarrow{OQ}$ have coordinates θ and ϕ, respectively, then $m\angle POQ = |\theta - \phi|$.

Definition: Suppose that $\overrightarrow{OA}$, $\overrightarrow{OB}$, and $\overrightarrow{OC}$ are concurrent rays, all having the same endpoint O. Then if these rays are distinct (no two are the same ray), and if $m\angle AOB + m\angle BOC = m\angle AOC$, then $\overrightarrow{OB}$ is said to lie *between* $\overrightarrow{OA}$ and $\overrightarrow{OC}$, and we write $\overrightarrow{OA} - \overrightarrow{OB} - \overrightarrow{OC}$.

Notation: Write $A[a]$ if a is the coordinate of a point A under the Ruler Postulate. Write $\overrightarrow{OA}\ [a]$ if a is the coordinate of a ray $\overrightarrow{OA}$ under the Protractor Postulate.

Theorem 10: If $A[a]$, $B[b]$, and $C[c]$ are three collinear points (and $\overrightarrow{OA}\ [a]$, $\overrightarrow{OB}\ [b]$, $\overrightarrow{OC}\ [c]$ three concurrent rays) with their coordinates, then $A - B - C$ ($\overrightarrow{OA} - \overrightarrow{OB} - \overrightarrow{OC}$) if and only if $a < b < c$ or $c < b < a$. (This is Theorem 1 of Kay.)

Corollary: Suppose that four distinct collinear points are given with their coordinates: $A[a]$, $B[b]$, $C[c]$, $D[d]$. If $A - B - C$ and $A - C - D$, then $A - B - C - D$, and similarly for rays $\overrightarrow{OA}\ [a]$, $\overrightarrow{OB}\ [b]$, $\overrightarrow{OC}\ [c]$, $\overrightarrow{OD}\ [d]$.

Lemma: A segment, ray, or line is a convex set.

Lemma: If A and B are two distinct points, and $C \in \overrightarrow{AB}$, with $A \neq C$, then $\overrightarrow{AB} \subseteq \overrightarrow{AC}$.

Theorem 11: If $C \in \overrightarrow{AB}$ and $A \neq C$, then $\overrightarrow{AB} = \overrightarrow{AC}$. (This is Theorem 2 of Kay.)

Theorem 12 (Segment Construction Theorem): If $\overline{AB}$ and $\overline{XY}$ are any two segments and $AB \neq XY$, then there is a unique point C on ray $\overrightarrow{AB}$ such that $AC = XY$, with $A - C - B$ if $XY < AB$, or $A - B - C$ if $XY > AB$. (This is Theorem 3 of Kay.)

Definition: A point M on a segment $\overline{AB}$ is called a *midpoint* if it has the property that $AM = MB$. Such a midpoint is also said to *bisect* the segment, and any line, segment, or ray passing through that midpoint is also said to *bisect* the segment.

Theorem 13 (Midpoint Construction Theorem): The midpoint of any segment exists and is unique. (This is Theorem 4 of Kay.)

Theorem 14 (Segment Doubling Theorem): There exists a unique point C on ray $\overrightarrow{AB}$ such that B is the midpoint of $\overline{AC}$. (This is Theorem 5 of Kay.)

Definition: A ray $\overrightarrow{OM}$ such that $\overrightarrow{OA} - \overrightarrow{OM} - \overrightarrow{OB}$ is said to be an *angle bisector* of $\angle AOB$ if $m\angle AOM = m\angle OMB$. Any line or ray containing an angle bisector is said to *bisect* the angle.

Theorem 15 (Angle Construction Theorem): If $\angle ABC$ and $\angle XYZ$ are any two nondegenerate angles and $m\angle ABC \neq m\angle XYZ$, then there exists a unique ray $\overrightarrow{BD}$ on the C-side of $\overleftrightarrow{AB}$ such that $m\angle XYZ = m\angle ABD$, and either $\overrightarrow{BA} - \overrightarrow{BD} - \overrightarrow{BC}$ if $m\angle XYZ < m\angle ABC$, or $\overrightarrow{BA} - \overrightarrow{BC} - \overrightarrow{BD}$ if $m\angle XYZ > m\angle ABC$. (This is Theorem $3'$ of Kay.)

Theorem 16 (Angle Bisection Theorem: Every angle has a unique bisector. (This is Theorem $4'$ of Kay.)

Theorem 17 (Angle Doubling Theorem): Given any angle $\angle ABC$ having measure < 90, there exists a ray $\overrightarrow{BD}$ such that $\overrightarrow{BC}$ is the bisector of $\angle ABD$. (This is Theorem $5'$ of Kay.)

6.2 More Theorems on Angles

This is a summary of Section 2.8 of Kay.

Theorem 18 (Crossbar Theorem): If D is in the interior of $\angle BAC$, then ray $\overrightarrow{AD}$ meets segment $\overline{BC}$ at some interior point E. (This is Theorem 1 of Kay.)

Definition: If $A - B - C$ then $\overrightarrow{BA}$ and $\overrightarrow{BC}$ are called *opposite rays*.

Lemma: For every ray $\overrightarrow{PQ}$ there exists a unique opposite ray.

Definition: If the sides of one angle are opposite rays to the respective sides of another angle, the angles are said to form a *vertical pair*.

Definition: Two angles are said to form a *linear pair* iff they have one side in common and the other two sides are opposite rays. We call any two angles whose angle measures sum to 180 a *supplementary pair*, or more simply, *supplementary*, and two angles whose angle measures sum to 90 a *complementary pair*, or *complementary*.

Theorem 19: Two angles which are supplementary (or complementary) to the same angle have equal angle measures. (This is Theorem 2 of Kay.)

Axiom A-4: A linear pair of angles is a supplementary pair.

Theorem 20 (Vertical Pair Theorem): Vertical angles have equal measures. (This is Theorem 3 of Kay.)

Definition: If line ℓ intersects another line m at some point A and forms a supplementary pair of angles at A having equal measures, then ℓ is said to be *perpendicular* to m, and we write $\ell \perp m$.

Definition: An angle having measure 90 is called a *right angle*. Angles having measure less than 90 are *acute angles*, and those with measure greater than 90, *obtuse angles*.

Theorem 21: One line is perpendicular to another line iff the two lines form four right angles at their point of intersection. (This is Theorem 4 of Kay.)

Corollary: Line ℓ is perpendicular to line m iff ℓ and m contains the sides of a right angle.

Theorem 22 (Existence and Uniqueness of Perpendiculars): Suppose that in some plane line m is given an an arbitrary point A on m is located. Then there exists a unique line ℓ that is perpendicular to m at A. (This is Theorem 5 of Kay.)