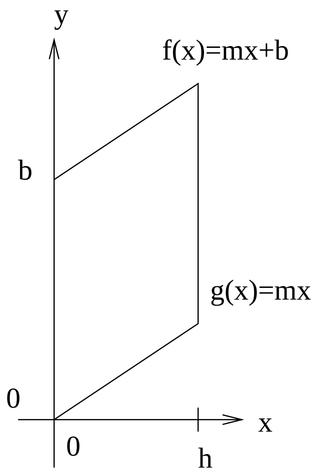
# Area and Volume Problems

1. Use calculus to derive the formula for the area of a parallelogram of base $b$ and height $h$.

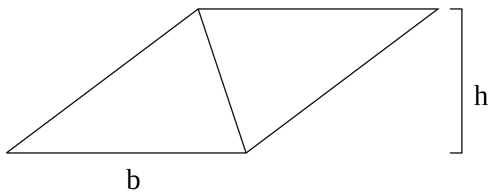
   
   

   The area of the parallelogram is given by the integral of the difference of the function $f(x)$ describing the upper side minus the function $g(x)$ describing the lower side:

   $$\int_0^h (f(x) - g(x))dx = \int_0^h ((mx + b) - (mx))dx = \int_0^h b\,dx = bh.$$

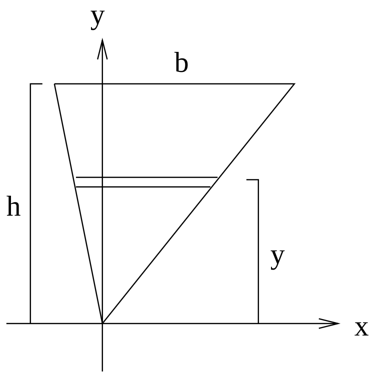
2. Derive the formula for the area of a triangle of base $b$ and height $h$.

   Glue two such triangles together to create a parallelogram with base $b$ and height $h$.

   The area of the triangle is half the area of the parallelogram; namely, $\frac{1}{2}bh$.

   Here is another way: The area of the triangle is the integral of the areas of the horizontal strips.

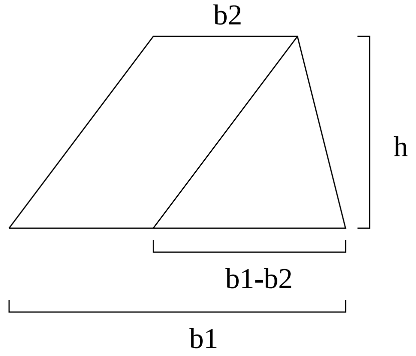



Each strip has length $\frac{y}{h}b$ and thickness $dy$. So the area is

$$\int_0^h \frac{y}{h} b\, dy = \frac{b}{h}\frac{1}{2}y^2\Big]_0^h = \frac{1}{2}bh.$$

3. Derive the formula for the area of a trapezoid with bases $b_1, b_2$ and height $h$.

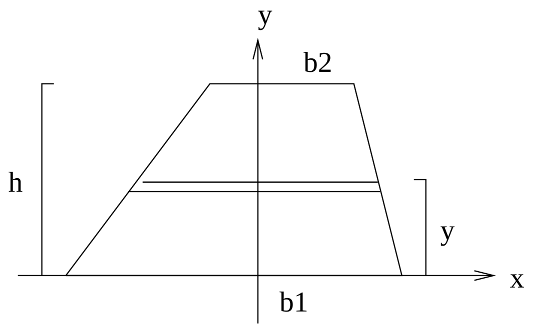
   Suppose $b_1 \geq b_2$. Cut the trapezoid into a parallelogram with base $b_2$ and height $h$, and a triangle with base $b_1 - b_2$ and height $h$.




Then the area is

$$b_2 h + \frac{1}{2}(b_1 - b_2)h = \frac{1}{2}(b_1 + b_2)h.$$

Here is another way: The area of the trapezoid is the integral of the areas of the horizontal strips.

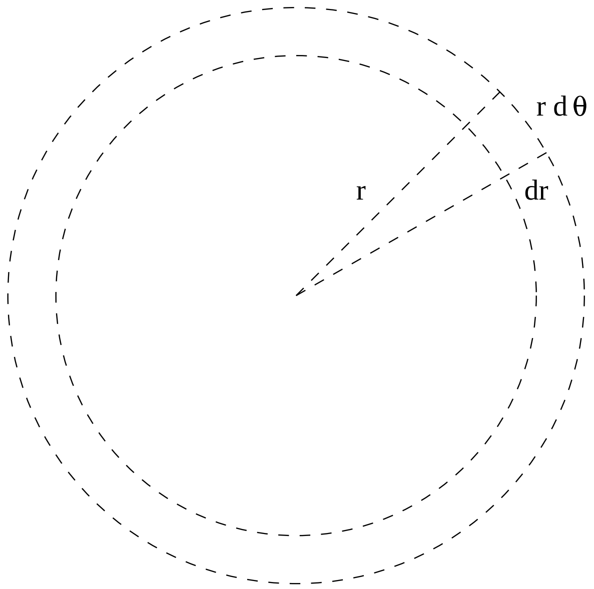



Each strip has length $b_1 + \frac{y}{h}(b_2 - b_1)$ and thickness $dy$. So the area is

$$\int_0^h (b_1 + \frac{y}{h}(b_2 - b_1))dy = \left[b_1 y + \frac{y^2}{2h}(b_2 - b_1)\right]_0^h = b_1 h + \frac{1}{2}h(b_2 - b_1) = \frac{1}{2}(b_1 + b_2)h.$$

4. Use calculus to derive the area of a circle of radius $R$.

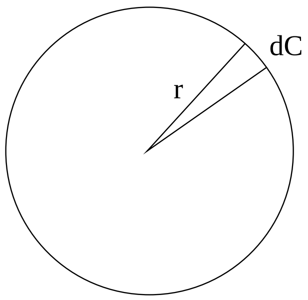
   To calculate the area using polar coordinates, remember that the small unit of area is $r\,dr\,d\theta$.




So the total area is

$$\int_0^{2\pi}\int_0^R r\,dr\,d\theta = \int_0^{2\pi}\left[\frac{1}{2}r^2\right]_0^R d\theta = \int_0^{2\pi}\frac{1}{2}R^2 d\theta = \left[\frac{1}{2}R^2\theta\right]_0^{2\pi} = \pi R^2.$$

Here is another way: Construct a triangle with base being a tiny piece of the circumference, $dC$, joined to the center of the circle.

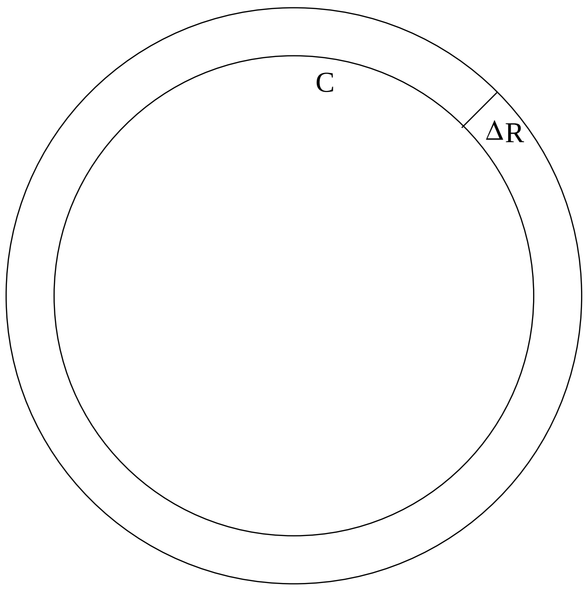



So the height of the triangle is $R$, and its area is $dA = \frac{1}{2}R\,dC$. Integrating this over the circumference of the circle gives the total area

$$A = \frac{1}{2}RC = \frac{1}{2}R2\pi R = \pi R^2.$$

5. Take the derivative of the formula for the area of a circle of radius $R$. What formula do you get? Why does this happen?

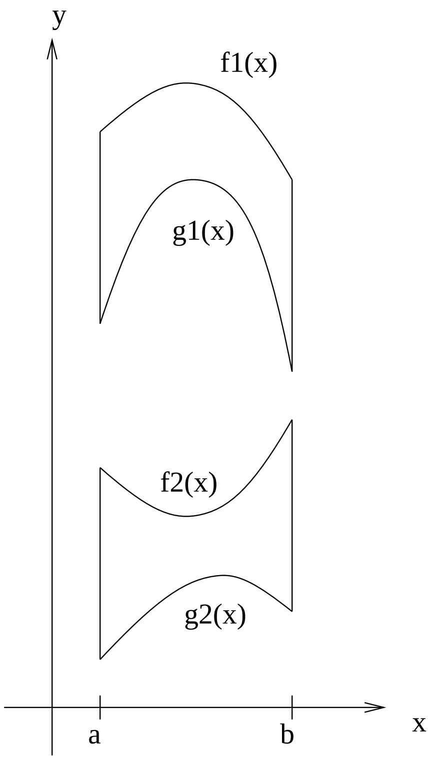
   If $A = \pi R^2$ then $\frac{dA}{dR} = 2\pi R$, which is the formula for the circumference of the circle. This happens because when the radius of the circle is increased by $\Delta R$, then the area increases by $\Delta A \approx C\Delta R$, where $C$ is the circumference.



So $\frac{\Delta A}{\Delta R} \approx C$, and in the limit, $\frac{dA}{dR} = C$.

6. Use calculus to prove Cavalieri's principle: If two 2-dimensional regions have the property that all lines parallel to some fixed line that meet $R_1$ and $R_2$ do so in line segments having equal lengths, whose endpoints are the boundary points of $R$ and $R'$, then $R$ and $R'$ have the same area. (See the illustration on page 79 of the textbook.)

   Suppose $R_1$ has upper boundary $f_1(x)$ and lower boundary $g_1(x)$, and $R_2$ has upper boundary $f_2(x)$ and lower boundary $g_2(x)$.




   Then

$$\text{area}(R_1) = \int_a^b (f_1(x) - g_1(x))dx$$

   and

$$\text{area}(R_2) = \int_a^b (f_2(x) - g_2(x))dx$$

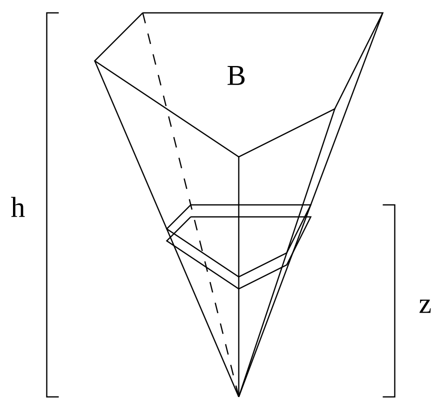
   But by assumption, $f_1(x) - g_1(x) = f_2(x) - g_2(x)$ for all $x$. So the two integrals are the same.

7. Two regions $R_1, R_2$ are *similar* if there is a one-to-one correspondence between the points of $R_1$ and the points of $R_2$, and a constant $k$, such that for every pair of points $A, B$ in $R_1$ and corresponding pair of points $A', B'$ in $R_2$, we have $AB = kA'B'$. If $R_1$ and $R_2$ are similar 2-dimensional regions, what is the relationship between the areas of $R_1$ and $R_2$? (If you are not certain, try this out with simple shapes first.)

   $\text{area}(R_1) = k^2\text{area}(R_2)$. This works for squares, so extends to areas of other figures by calculus.

8. Use calculus to derive the formula for the volume of a pyramid whose base is a polygon of area $B$ and whose height is $h$.

   Calculate volumes by slices.




   The horizontal slice at height $z$ has area $(\frac{z}{h})^2 B$ (by the previous problem) and thickness $dz$. So the volume is

$$\int_0^h (\frac{z}{h})^2 B\, dz = \frac{1}{3h^2} z^3 B \Big]_0^h = \frac{1}{3} Bh.$$

9. Use calculus to derive the formula for the volume of a cylinder of radius $R$ and height $h$.

   We can use cylindrical coordinates. For a small change in $r$, $\theta$, and $z$, the unit of volume is $r\,dr\,d\theta\,dz$. So the total volume is

$$\begin{aligned}
\int_0^h \int_0^{2\pi} \int_0^R r\,dr\,d\theta\,dz &= \int_0^h \int_0^{2\pi} \left[\tfrac{1}{2}r^2\right]_0^R d\theta\,dz \\
&= \int_0^h \int_0^{2\pi} \tfrac{1}{2}R^2 d\theta\,dz \\
&= \int_0^h \left[\tfrac{1}{2}R^2\theta\right]_0^{2\pi} dz \\
&= \int_0^h \pi R^2 dz \\
&= \left[\pi R^2 z\right]_0^h \\
&= \pi R^2 h.
\end{aligned}$$

10. Derive the formula of the surface area of a cylinder of radius $R$ and height $h$.

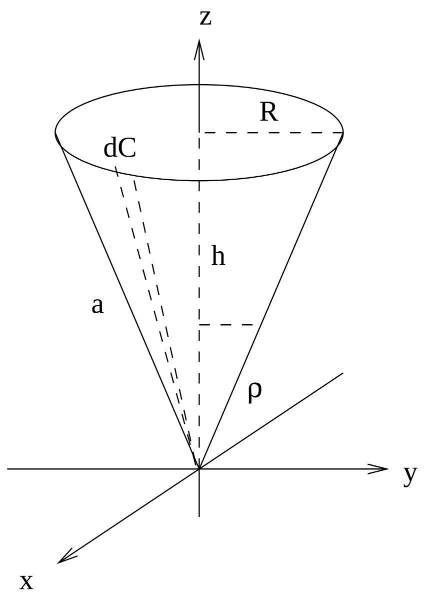
    The top and the bottom of the cylinder have the same area; namely, $\pi R^2$. Cut off the top and bottom and unroll the side of the cylinder into a rectangle. The height of the rectangle is $h$, and its length is the circumference of the cylinder, $2\pi R$. So its area is $2\pi Rh$. So the total area of the cylinder is

$$2\pi R^2 + 2\pi Rh = 2\pi R(R+h).$$

11. Use calculus to derive the formula for the volume of a cone of radius $R$ and height $h$.

    This works exactly the same as the volume of a pyramid, where now $B = \pi R^2$. So the total volume is $\frac{1}{3}Bh = \frac{1}{3}\pi R^2 h$.

12. Derive the formula for the surface area of a cone of radius $R$ and height $h$.



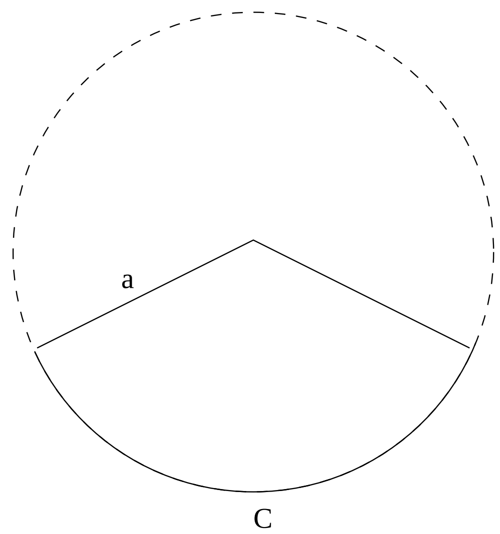
Let $a$ be the "slant height" of the cone, so $a = \sqrt{R^2 + h^2}$. The base of the cone has area $\pi R^2$, so all we need to find is the lateral area. One way is to make a thin triangle with base being $dC$, a small piece of the circumference of the base, and height being $a$. Then the triangle has area $\frac{1}{2}a\,dC$. Integrating this over the circumference gives the area $\frac{1}{2}aC = \pi R\sqrt{R^2 + h^2}$. So the total area is

$$\pi R^2 + \pi R\sqrt{R^2 + h^2} = \pi R(R + \sqrt{R^2 + h^2}).$$

Another way to get the lateral surface area is to use spherical coordinates. Every point on the cone has the same value of $\phi$. For a small change in $\theta$ and $\rho$, the unit of surface area is $\frac{\rho}{a}R\,d\theta\,d\rho$. So the total lateral area is

$$\begin{aligned}
\int_0^{2\pi}\int_0^a \tfrac{\rho}{a}R\,d\rho\,d\theta &= \int_0^{2\pi}\left[\tfrac{1}{2a}\rho^2 R\right]_0^a d\theta \\
&= \int_0^{2\pi}\tfrac{1}{2}aR\,d\theta \\
&= \left[\tfrac{1}{2}aR\theta\right]_0^{2\pi} \\
&= \pi aR \\
&= \pi R\sqrt{R^2 + h^2}.
\end{aligned}$$

Here is yet another way to get the lateral surface area. Cut the side of the cone and unfold it to get a sector of a circle.




The radius of the circle is $a$, the slant height of the original cone. The arc length of the sector is $C$, the circumference of the base of the original cone. This is the fraction $\frac{C}{2\pi a}$

of the total circumference of the circle. So the area of the sector is the same fraction of the area of the circle; namely $\frac{C}{2\pi a}\pi a^2 = \frac{Ca}{2} = \pi R\sqrt{R^2+h^2}$.

13. Use calculus to derive the formula for the volume of a sphere of radius $R$.

    Using spherical coordinates, for a small change in $\theta$, $\phi$, and $\rho$ the unit volume is $\rho^2 \sin\phi\, d\rho\, d\phi\, d\theta$. So the total volume is

$$\begin{aligned}
\int_0^{2\pi}\int_0^{\pi}\int_0^{R} \rho^2 \sin\phi\, d\rho\, d\phi\, d\theta &= \int_0^{2\pi}\int_0^{\pi} \left[\tfrac{1}{3}\rho^3 \sin\phi\right]_0^R d\rho\, d\phi\, d\theta \\
&= \int_0^{2\pi}\int_0^{\pi} \tfrac{1}{3}R^3 \sin\phi\, d\phi\, d\theta \\
&= \int_0^{2\pi} \left[-\tfrac{1}{3}R^3 \cos\phi\right]_0^{\pi} d\theta \\
&= \int_0^{2\pi} \tfrac{2}{3}R^3\, d\theta \\
&= \left[\tfrac{2}{3}R^3\theta\right]_0^{2\pi} \\
&= \tfrac{4}{3}\pi R^3.
\end{aligned}$$

14. Use calculus to derive the formula for the surface area of a sphere of radius $R$.

    Using spherical coordinates, all the points on the sphere have constant $\rho = R$. For a small change in $\theta$ and $\phi$, the unit area is $R\, d\phi$ times $R\sin\phi\, d\theta$. So the total surface area is

$$\int_0^{2\pi}\int_0^{\pi} R^2 \sin\phi\, d\phi\, d\theta = \int_0^{2\pi} \left[-R^2\cos\phi\right]_0^{\pi} d\theta = \int_0^{2\pi} 2R^2 d\theta = \left[2R^2\theta\right]_0^{2\pi} = 4\pi R^2.$$

15. Take the derivative of the formula for the volume of a sphere of radius $R$. What formula do you get? Why does this happen?

    If $V = \frac{4}{3}\pi R^3$ then $\frac{dV}{dR} = 4\pi R^2$, which is the formula for the surface area of the sphere. This happens because when the radius of the sphere is increased by $\Delta R$, then the volume increases by $\Delta V \approx S\Delta R$, where $S$ is the surface area. So $\frac{\Delta V}{\Delta R} \approx S$, and in the limit, $\frac{dV}{dR} = S$.

16. Use calculus to prove Cavalieri's principle: If two 3-dimensional regions have the property that all planes parallel to some fixed plane that meet $R_1$ and $R_2$ do so in plane sections having equal areas, whose boundaries lie in the boundaries of $R$ and $R'$, then $R$ and $R'$ have the same volume. (See the illustration on page 79 of the textbook.)

    Suppose $R_1$ and $R_2$ are cut by horizontal planes, the lowest plane being at height $z = a$, and the highest plane being at height $z = b$. Let $A_1(z)$ denote the area of the cross-section of $R_1$ determined by the plane at height $z$, and $A_2(z)$ denote the area of the cross-section of $R_2$ determined by the plane at height $z$. Then

$$\text{volume}(R_1) = \int_a^b A_1(z)dz$$

and

$$\text{volume}(R_2) = \int_a^b A_2(z)dx.$$

But by assumption, $A_1(z) = A_2(z)$ for all $z$. So the two integrals are the same.

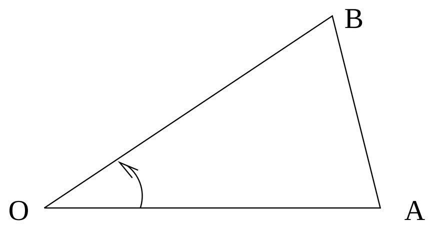
17. If $R_1$ and $R_2$ are similar 3-dimensional regions, what is the relationship between the volumes of $R_1$ and $R_2$? Between the surface areas of $R_1$ and $R_2$?

    Suppose there is a one-to-one correspondence between the points of $R_1$ and the points of $R_2$, and a constant $k$, such that for every pair of points $A, B$ in $R_1$ and corresponding pair of points $A', B'$ in $R_2$, we have $AB = kA'B'$. Then volume$(R_1) = k^3$volume$(R_2)$ and surface area$(R_1) = k^2$surface area$(R_2)$.

18. Recall that if that if $OAB$ is a triangle in $\mathbf{E}^2$ with coordinates $O = (0,0)$, $A = (x_1, y_1)$, $B = (x_2, y_2)$, then its area is

$$\frac{1}{2}\left\| \begin{matrix} x_1 & y_1 \\ x_2 & y_2 \end{matrix} \right\|$$

Actually, it turns out that if the angle $\angle AOB$ is counterclockwise, measured from ray $\overrightarrow{OA}$ to $\overrightarrow{OB}$,




then we can drop the absolute value above, and the area is in fact just equal to the determinant

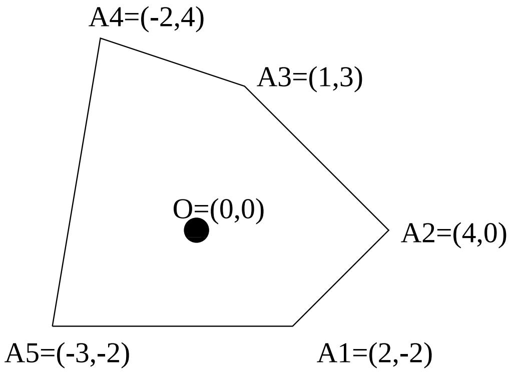
$$\frac{1}{2}\left| \begin{matrix} x_1 & y_1 \\ x_2 & y_2 \end{matrix} \right|$$

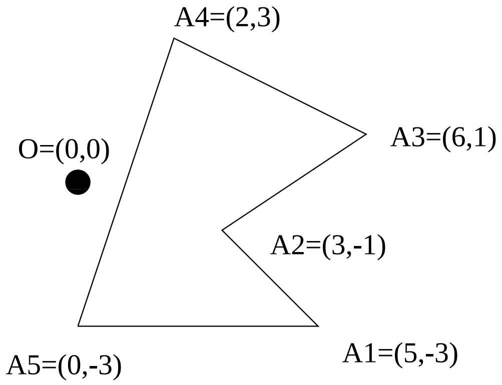
Now suppose that you have a convex polygon $A_1, A_2, \ldots, A_n$ in the plane $\mathbf{E}^2$ (listing its vertices in counterclockwise order) that contains the origin in its interior. Suppose that the vertex $A_i$ has coordinates $(x_i, y_i)$, $i = 1, \ldots, n$. Prove that the area of the polygon is

$$\frac{1}{2}\left(\begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix} + \begin{vmatrix} x_2 & y_2 \\ x_3 & y_3 \end{vmatrix} + \cdots + \begin{vmatrix} x_{n-1} & y_{n-1} \\ x_n & y_n \end{vmatrix} + \begin{vmatrix} x_n & y_n \\ x_1 & y_1 \end{vmatrix}\right)$$

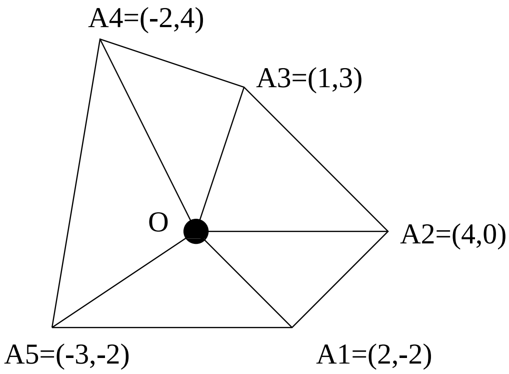
Actually, it turns out that this formula works even if the polygon is not convex, and even it does not contain the origin in its interior, as long as it does not cross itself.

Use this formula to calculate the areas of the following polygons:




We can see why the formula works by using the first polygon and constructing triangles from every side of the polygon to the origin $O$.




The triangle with vertices $(0, 0)$, $(x_i, y_i)$, and $(x_{i+1}, y_{i+1})$ has area

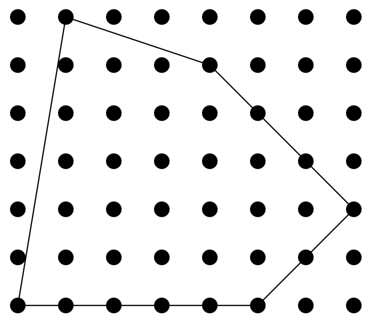
$$\frac{1}{2}\begin{vmatrix} x_i & y_i \\ x_{i+1} & y_{i+1} \end{vmatrix}$$

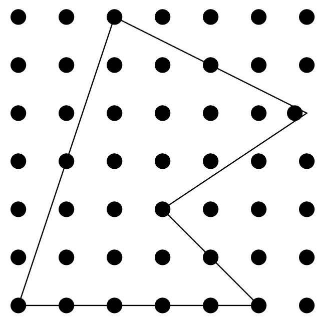
Then the total area of the polygon equals the sum of the areas of the triangles.

Applying the formula to the two polygons, we get areas $\frac{1}{2}(8+12+10+16+10)=28$ and $\frac{1}{2}(4+9+16-6+15)=19$.

19. Suppose a polygon (not necessarily convex, but not self-crossing) has the property that each of its vertices has integer coordinates. Let $B$ be the number of integer points on its boundary, and $I$ be the number of integer points in its interior. Consider some examples and conjecture a formula for the area of the polygon in terms of $B$ and $I$. This is called *Pick's Formula.*

Use this formula to calculate the areas of the following polygons:





With a bit of experimentation you can guess the formula

$$A=\frac{1}{2}B+I-1.$$

Applying this formula to the two polygons, we get areas $\frac{1}{2}(12)+23-1=28$ and $\frac{1}{2}(12)+14-1=19$.

20. Recall that $\mathbf{S}^2$ is the sphere of radius 1 centered at the origin. Suppose that $A$ and $A'$ are any two points on $\mathbf{S}^2$ that are exactly opposite each other. Connect $A$ and $A'$ with two great half-circles that meet at $A$ (and also at $A'$) at an angle $A<180°$. These two half-circles determine a region of the sphere called a *lune.* Explain why the area of this lune is $\frac{\pi}{90}A$. (See page 463 of the textbook.)

The lune consists of a fraction of the total surface area $S$ of the sphere. This fraction is the same as the fraction of the total 360° angle that $A$ consists of; namely, $\frac{A}{360}$. So the area of the lune is

$$\frac{A}{360}S=\frac{A}{360}4\pi=\frac{A}{90}\pi.$$

21. Complete the discussion on page 464 of the textbook to obtain a formula for the area of a spherical triangle.

Referring to the book for the definitions of regions $K$, $I$, $II$, and $III$, we can first see that $K \cup I$ is the lune determined by angle $A$, $K \cup II$ is the lune determined by angle $C$, and $K \cup III$ is the lune determined by angle $B$. So for areas we have

$$\begin{aligned} K + I &= \tfrac{A}{90}\pi \\ K + II &= \tfrac{C}{90}\pi \\ K + III &= \tfrac{B}{90}\pi \end{aligned}$$

Adding these together gives

$$3K + I + II + III = \frac{\pi}{90}(A + B + C).$$

We can also observe that $K \cup I \cup II \cup III$ occupies exactly half the total area of the sphere—this is most easily seen by examining a physical model. So

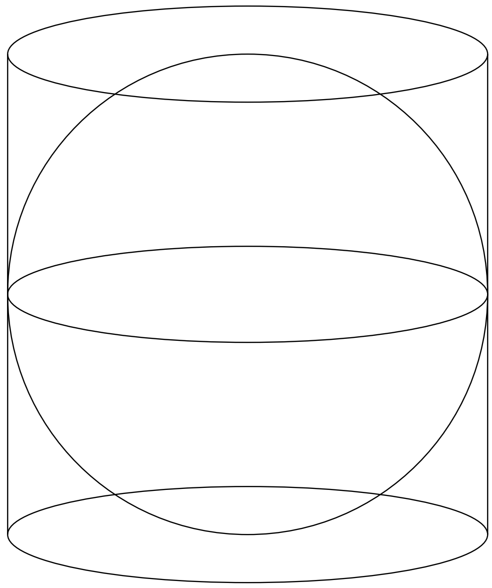
$$K + I + II + III = \frac{1}{2}4\pi = 2\pi.$$

Subtracting this equation from the previous one yields

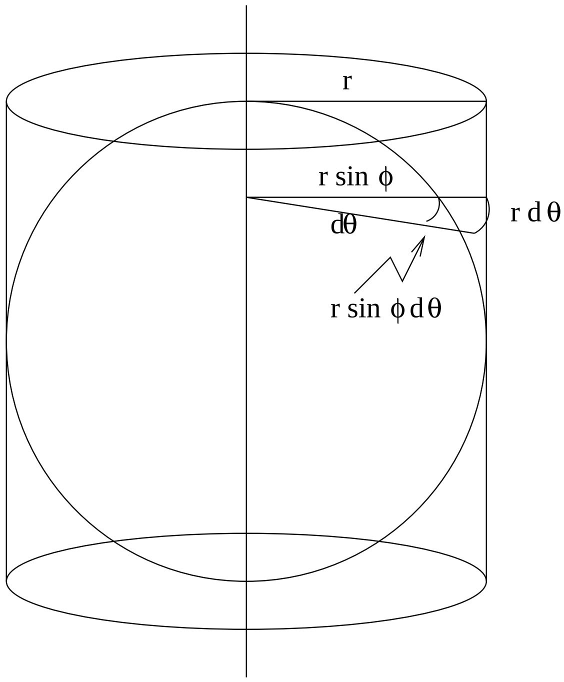
$$\begin{aligned} 2K &= \tfrac{\pi}{90}(A + B + C) - 2\pi \\ K &= \tfrac{\pi}{180}(A + B + C) - \pi \\ &= \tfrac{\pi}{180}(A + B + C - 180). \end{aligned}$$

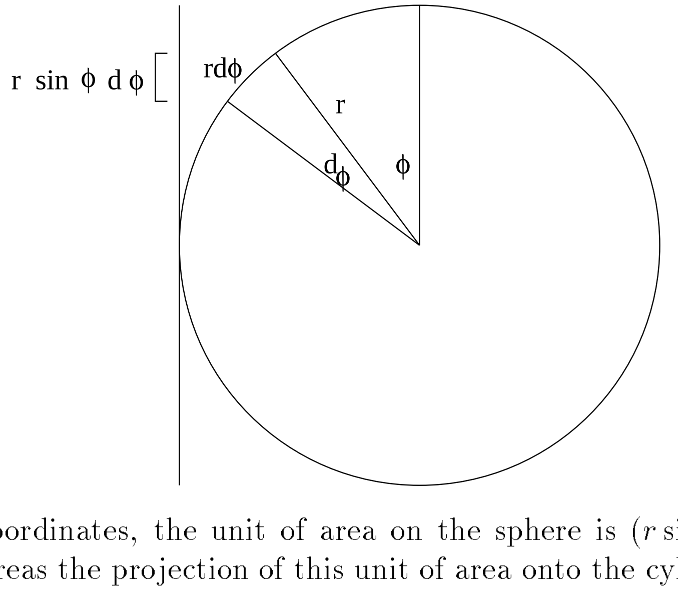
So the total area is proportional to the amount by which the angle sum of the triangle exceeds $180°$.

22. Suppose a sphere is inscribed in a cylinder whose radius is the radius of the sphere, and whose height is the diameter of the sphere.



Let $R$ be any region on the sphere. Project this region out horizontally onto the cylinder to get the region $R'$ on the cylinder. Use calculus to prove that $R$ and $R'$ have the same areas. This is one way to make maps of the earth that accurately represent areas of countries, although the shapes of the countries are distorted.




Using spherical coordinates, the unit of area on the sphere is $(r \sin\phi\, d\theta) \cdot (r\, d\phi) = r^2 \sin\phi\, d\theta\, d\phi$, whereas the projection of this unit of area onto the cylinder gives a unit of area $(r\, d\theta) \cdot (r \sin\phi\, d\phi) = r^2 \sin\phi\, d\theta\, d\phi$. Since these units of area are identical in size, integrating them over regions $R$ and $R'$ will yield the same result.

23. Assume that $OABC$ is a tetrahedron with coordinates $O = (0, 0, 0)$, $A = (x_1, y_1, z_1)$,

$B = (x_2, y_2, z_2)$, $C = (x_3, y_3, z_3)$. Give two examples to show that the volume is

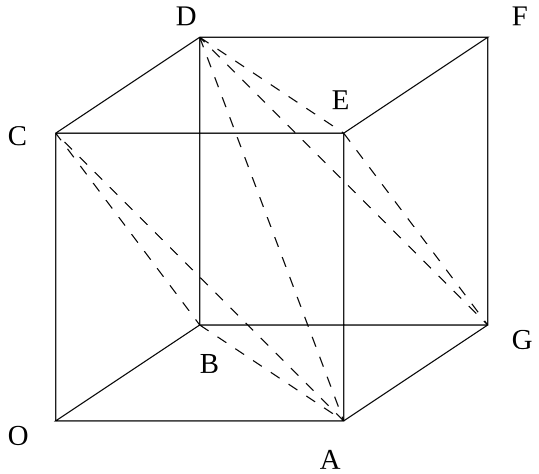
$$\frac{1}{6}\left\|\begin{array}{ccc} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{array}\right\|$$

Explain why the volume of the tetrahedron is $\frac{1}{6}$th of the parallelepiped determined by $A$, $B$, and $C$.

There are many examples to choose from. Here is a simple one: $A = (1, 0, 0)$, $B = (0, 2, 0)$, $C = (0, 0, 3)$. According to the formula, the volume should be

$$\frac{1}{6}\left\|\begin{array}{ccc} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{array}\right\| = 1.$$

The base triangle $\triangle OAB$ has area 1, and the height of the tetrahedron is 3, so its volume is $\frac{1}{3}$base$\times$height$= \frac{1}{3} \cdot 1 \cdot 3 = 1$.




You can cut the parallelepiped into six tetrahedra as shown above: $OABC$, $ABCD$, $ACDE$, $ABDG$, $ADEG$, $DEFG$. Each of these tetrahedra have the same volume, as can be demonstrated by showing that certain pairs of them have bases of common area, and equal heights. For example, $OABC$ has base $\triangle OBC$ and height $OA$, while $ABCD$ has base $\triangle BCD$ and height $OA$. So these two tetrahedra have the same volume.

24. Try to find a formula analogous to that of problem (18) that applies to a convex 3-dimensional polyhedron, all of whose faces are triangles. What could you do if not all faces are triangles?

Suppose that the vertices of the polyhedron have coordinates $P_i = (x_i, y_i, z_i)$, $i = 1, \ldots, n$.

If the structure contains the origin in its interior, then you can construct a tetrahedron from each triangular face to the origin and add up the volumes of the tetrahedra. For any given face with vertices $P_j, P_k, P_\ell$, the volume of the tetrahedron will be

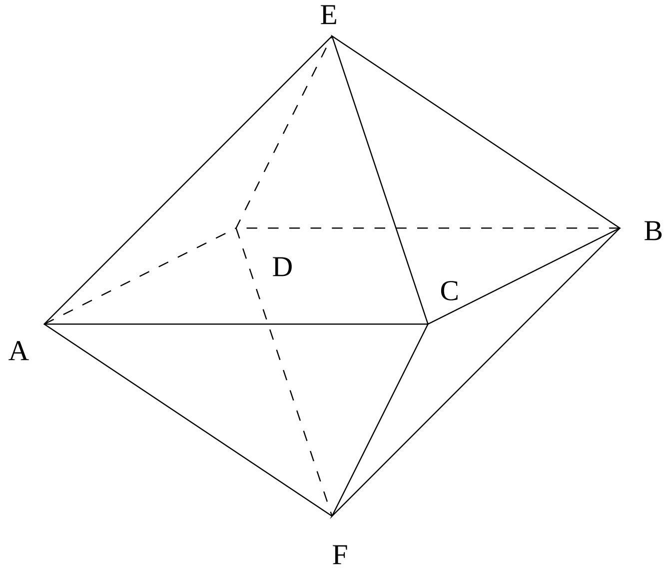
$$\frac{1}{6}\left\| \begin{array}{ccc} x_j & y_j & z_j \\ x_k & y_k & z_k \\ x_\ell & y_\ell & z_\ell \end{array} \right\|.$$

Adding up these volumes, one per face, gives the volume of the entire polyhedron.

It turns out that if the vertices of each face are listed in the determinant in counterclockwise order as viewed from the outside, then the determinant will be automatically positive, and the absolute value can be dropped. Once this is done, it can also be proved that the formula works even if the polyhedron is not convex, and even if does not contain the origin in its interior, so long as it is a closed structure and not self-intersecting.

If not all faces are triangles, the nontriangular faces can be be first subdivided into triangles by drawing some diagonals. Then the formula can be applied.

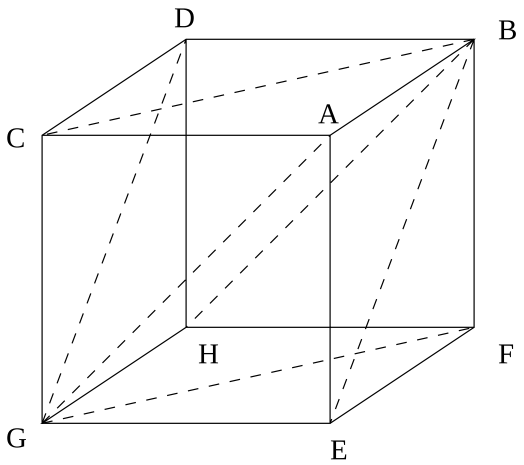
Example: an octahedron. The vertices are $A = (1, 0, 0)$, $B = (-1, 0, 0)$, $C = (0, 1, 0)$, $D = (0, -1, 0)$, $E = (0, 0, 1)$, $F = (0, 0, -1)$. The faces, with vertices listed in counterclockwise order, are $ACE$, $CBE$, $BDE$, $DAE$, $CAF$, $BCF$, $DBF$, $ADF$.



The volume is

$$\begin{aligned}
V &= \tfrac{1}{6}\left(\begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} + \begin{vmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{vmatrix} + \begin{vmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{vmatrix} + \begin{vmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{vmatrix}\right. \\
&+ \left.\begin{vmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{vmatrix} + \begin{vmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{vmatrix} + \begin{vmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & -1 \end{vmatrix} + \begin{vmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{vmatrix}\right) \\
&= \tfrac{1}{6}(1+1+1+1+1+1+1+1) = \tfrac{4}{3}.
\end{aligned}$$

Example: a cube. The vertices are $A = (1,1,1)$, $B = (-1,1,1)$, $C = (1,-1,1)$, $D = (-1,-1,1)$, $E = (1,1,-1)$, $F = (-1,1,-1)$, $G = (1,-1,-1)$, $H = (-1,-1,-1)$. Cut each of the six squares with a diagonal. The triangles, with vertices listed in counterclockwise order, are $GAC$, $AGE$, $EBA$, $BEF$, $DGC$, $GDH$, $BHD$, $HBF$, $BCA$, $CBD$, $FGH$, $GFE$.



The volume is

$$
\begin{aligned}
V = {} & \tfrac{1}{6}\left(\begin{vmatrix} 1 & -1 & -1 \\ 1 & 1 & 1 \\ 1 & -1 & 1 \end{vmatrix} + \begin{vmatrix} 1 & 1 & 1 \\ 1 & -1 & -1 \\ 1 & 1 & -1 \end{vmatrix} + \begin{vmatrix} 1 & 1 & -1 \\ -1 & 1 & 1 \\ 1 & 1 & 1 \end{vmatrix} + \begin{vmatrix} -1 & 1 & 1 \\ 1 & 1 & -1 \\ -1 & 1 & -1 \end{vmatrix}\right. \\
& + \begin{vmatrix} -1 & -1 & 1 \\ 1 & -1 & -1 \\ 1 & -1 & 1 \end{vmatrix} + \begin{vmatrix} 1 & -1 & -1 \\ -1 & -1 & 1 \\ -1 & -1 & -1 \end{vmatrix} + \begin{vmatrix} -1 & 1 & 1 \\ -1 & -1 & -1 \\ -1 & -1 & 1 \end{vmatrix} + \begin{vmatrix} -1 & -1 & -1 \\ -1 & 1 & 1 \\ -1 & 1 & -1 \end{vmatrix} \\
& \left. + \begin{vmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & 1 \end{vmatrix} + \begin{vmatrix} 1 & -1 & 1 \\ -1 & 1 & 1 \\ -1 & -1 & 1 \end{vmatrix} + \begin{vmatrix} -1 & 1 & -1 \\ 1 & -1 & -1 \\ -1 & -1 & -1 \end{vmatrix} + \begin{vmatrix} 1 & -1 & -1 \\ -1 & 1 & -1 \\ 1 & 1 & -1 \end{vmatrix}\right) \\
& = \tfrac{1}{6}(4+4+4+4+4+4+4+4+4+4+4+4) = 8.
\end{aligned}
$$