1 Axiomatic Systems

1.1 Features of Axiomatic Systems

One motivation for developing axiomatic systems is to determine precisely which properties of certain objects can be deduced from which other properties. The goal is to choose a certain fundamental set of properties (the *axioms*) from which the other properties of the objects can be deduced (e.g., as *theorems*). Apart from the properties given in the axioms, the objects are regarded as *undefined*.

As a powerful consequence, once you have shown that any particular collection of objects satisfies the axioms *however unintuitive or at variance with your preconceived notions these objects may be*, without any additional effort you may immediately conclude that all the theorems must also be true for these objects.

We want to choose our axioms wisely. We do not want them to lead to contradictions; i.e., we want the axioms to be *consistent*. We also strive for economy and want to avoid redundancy—not assuming any axiom that can be proved from the others; i.e., we want the axiomatic system to be *independent*. Finally, we may wish to insist that we be able to prove or disprove any statement about our objects from the axioms alone. If this is the case, we say that the axiomatic system is *complete*.

We can verify that an axiomatic system is consistent by finding a *model* for the axioms—a choice of objects that satisfy the axioms.

We can verify that a specified axiom is independent of the others by finding two models—one for which all of the axioms hold, and another for which the specified axiom is false but the other axioms are true.

We can verify that an axiomatic system is complete by showing that there is essentially only one model for it (all models are *isomorphic*); i.e., that the system is *categorical*.

Reference: Kay, Section 2.2.

1.2 Examples

Let's look at three examples of axiomatic systems for a collection of committees selected from a set of people. In each case, determine whether the axiomatic system is consistent or inconsistent. If it is consistent, determine whether the system is independent or redundant, complete or incomplete.

1. (a) There is a finite number of people.
   (b) Each committee consists of exactly two people.
   (c) Exactly one person is on an odd number of committees.

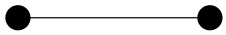
   This axiomatic system is inconsistent. In fact, we can prove that the first two axioms imply that the number of people on an odd number of committees must be even. This is sometimes called the *Handshaking Theorem* because it is often stated in the form of pairs of people shaking hands (rather than serving on committees).

   Handshaking Theorem: If various pairs of people among a finite number of people shake hands, the number of people who shake hands an odd number of times is even.

   Proof: Assume that there are n people. Let a_i denote the number of times individual i shakes hands, $i = 1, \ldots n$. Let K denote the total number of handshakes that take place. Then $a_1 + a_2 + \cdots + a_n = 2K$ because every handshake is counted twice when we add up the number of handshakes for each individual. Now $2K$ is an even number, so the sum $a_1 + a_2 + \cdots + a_n$ must be even. Therefore, the number of odd terms in this sum must be even.

2. (a) There is a finite number of people.
   (b) Each committee consists of exactly two people.
   (c) No person serves on more than two committees.
   (d) The number of people who serve on exactly one committee is even.

   We can think of a model satisfying four axioms; e.g., a set of two people and just one committee consisting of these two people:

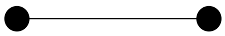


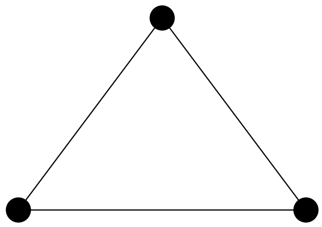
   Each dot represents a person and the line segment connecting the two dots indicates the committee. Therefore this axiomatic system is consistent.

   We can show that Axiom (2d) is a consequence of Axioms (2a), (2b) and (2c): Assume that Axioms (2a) and (2b) hold. Then by the Handshaking Theorem, the number of people serving on an odd number of committees is even. By Axiom (2c), if an individual serves on an odd number of committees, he/she must serve on exactly one committee. Therefore the number of people serving on exactly one committee is even.

   From this argument we can conclude that this axiomatic system is not independent.

   Finally there are many distinctly different models for this axiomatic system; e.g., one consisting of two people and one committee, and one consisting of three people and three committees:

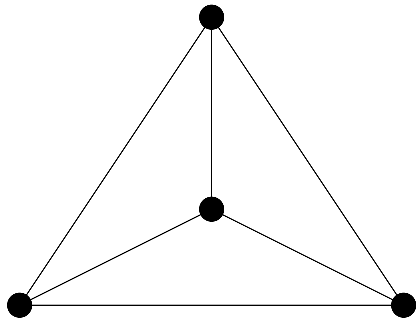




   Therefore this axiomatic system is not complete, since it does not determine essentially one model.

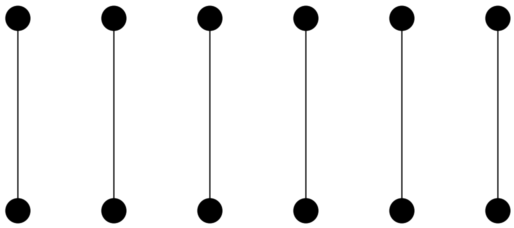
3. (a) Each committee consists exactly two people.
   (b) There are exactly six committees.
   (c) Each person serves on exactly three committees.

We can find a model for this axiomatic system:



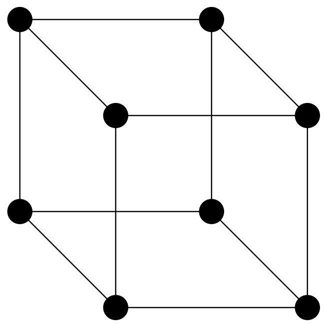
Therefore, this axiomatic system is consistent.

We can show that Axiom (3c) is independent of the other two axioms by looking at the following different model, in which Axioms (3a) and (3b) hold, but Axiom (3c) does not:



So we have one model in which all three axioms hold, and another in which the first two axioms hold but the third does not. This proves that the third axiom is independent of the other two.

We can show that Axiom (3b) is independent of the other two axioms by looking at the following model, in which Axioms (3a) and (3c) hold, but Axiom (3b) does not. Again, we are representing individuals by dots and committees by line segments.

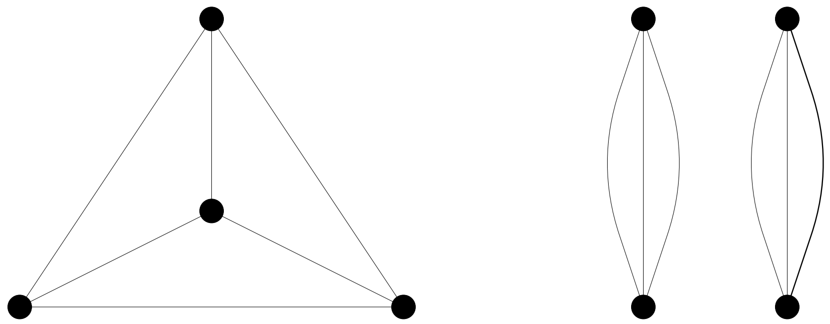


We can show that Axiom (3a) is independent of the other two axioms by looking the above figure in a different way, thinking of it as a drawing of a cube. If we use the six squares of the cube to represent six committees of four people each, then we have a model in which Axioms (3b) and (3c) hold, but Axiom (3a) does not.

To summarize, in each case we have proved that one of the axioms is independent of the other two by providing two models: one in which all three axioms hold, and another in which the particular axiom does not hold but the other two do.

Is our axiomatic system complete? Is there essentially only one model? We start by proving that there can only be four people: If we multiply the number of committees (6) by the number of people in each committee (2) we get the number 12. But since each person serves on exactly three committees, this counts each person three times. So the total number of people must be $12/3 = 4$.

However, even with this limitation, there are still essentially different models for this axiomatic system:



We can eliminate all but the first model above, and thereby make our axiomatic system complete, by adding one more axiom that is independent of the other three. For example, we could include as a fourth axiom that there do not exist two committees with exactly the same set of members.

1.3 Kirkman's Schoolgirl Problem

Consider the following axiomatic system for points and lines, where lines are certain subsets of points, but otherwise points and lines are undefined.

- Given any two distinct points, there is exactly one line containing both of them.
- Given any two distinct lines, they intersect in a single point.
- There exist four points, no three of which are contained in a common line.
- The total number of points is finite.

1. Try to come up with some models for this axiomatic system.

   The next section gives a model with 7 points and 7 lines.

2. Find a model containing exactly 15 points.

   The solution to the following Kirkman problem gives such a model with 35 lines, each having exactly 3 points, but there are other solutions as well.

3. Solve the following famous puzzle proposed by T. P. Kirkman in 1847:

   > A school-mistress is in the habit of taking her girls for a daily walk. The girls are fifteen in number, and are arranged in five rows of three each, so that each girl might have two companions. The problem is to dispose them so that for seven consecutive days no girl will walk with any of her school-fellows in any triplet more than once.

   Here is one possible solution, where the children are labeled 0 through 14:

Sun	Mon	Tue	Wed	Thu	Fri	Sat
0,5,10	0,1,4	1,2,5	4,5,8	2,4,10	4,6,12	3,10,12
1,6,11	2,3,6	3,4,7	6,7,10	3,5,11	5,7,13	4,11,13
2,7,12	7,8,11	8,9,12	0,11,12	6,8,14	1,8,10	1,7,14
3,8,13	9,10,13	10,11,14	2,13,14	0,7,9	2,9,11	0,2,8
4,9,14	5,12,14	0,6,13	1,3,9	1,12,13	0,3,14	5,6,9

   Reference: Ball and Coxeter, Chapter X.

1.4 Finite Projective Planes

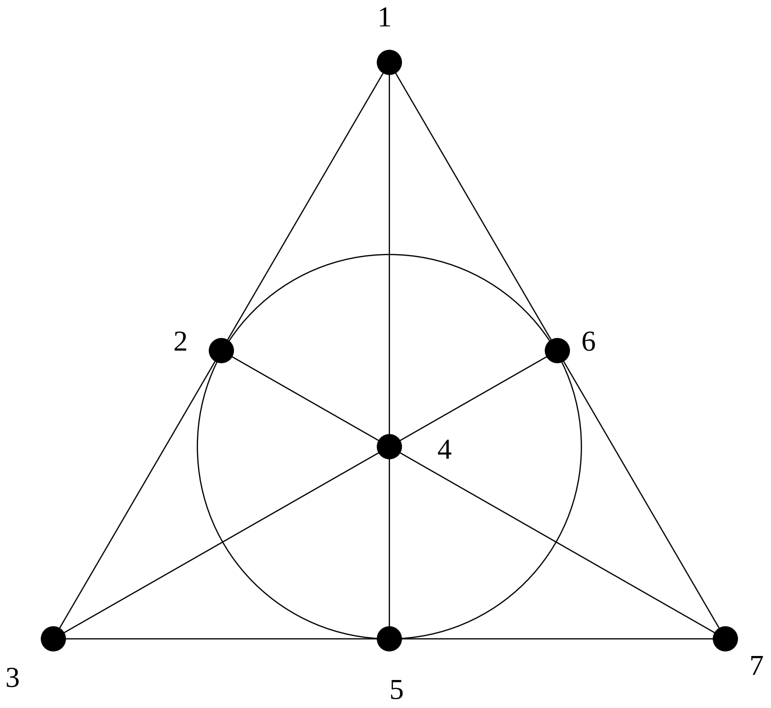
The axioms described in the previous section define structures called *finite projective planes*. I have a game called *Configurations* that is designed to introduce the players to the existence, construction , and properties of finite projective planes. When I checked in August 1997 the game was available from WFF 'N PROOF Learning Games Associates, 1490 South Boulevard, Ann Arbor, MI 48104, phone: (313) 665-2269, fax: (313) 764-9339, for a cost of $12.50.

Here are examples of some problems from this game:

1. In each box below write a number from 1 to 7, subject to the two rules: (1) The three numbers in each column must be different; (2) the same pair of numbers must not occur in two different columns.

	Col 1	Col 2	Col 3	Col 4	Col 5	Col 6	Col 7
Row 1	1	1	1	2	2	3	3
Row 2	2	4	6	4	5	4	5
Row 3	3	5	7	7	6	6	7

2. Use the solution to the above problem to label the seven points of the following diagram with the numbers 1 through 7 so that the columns of the above problem correspond to the triples of points in the diagram below that lie on a common line or circle.



This is called the *Fano plane*.