4 Distance and Betweenness

4.1 The Metric Axioms

This is an outline some of the main results of Section 2.5 of the text.

Axiom D-1 — Existence: Each pair of points A, B is associated with a unique real number $AB \geq 0$, called the *distance* from A to B.

Axiom D-2 — Positive Definiteness: For all points A and B, $AB > 0$ unless $A = B$.

Axiom D-3 — Symmetry: For all points A and B, $AB = BA$.

Definition of Betweenness: For any three points A, B, and C in space, we say that B is *between* A and C, and we write $A - B - C$, if and only if A, B, and C are distinct, collinear points, and $AC = AB + BC$.

Definition: If A, B, C, and D are four distinct collinear points, let the betweenness relations $A - B - C - D$ represent the composite of all four betweenness relations $A - B - C$, $A - B - D$, $A - C - D$, and $B - C - D$.

Theorem 3: If $A - B - C$, then $C - B - A$, and neither $A - C - B$ nor $B - A - C$. (This is Theorem 1 of Kay.)

Definition: Distance is said to satisfy the *Triangle Inequality* if $AB + BC \geq AC$ holds for all triples of points A, B, C. If Axioms D-1–D-3 and the Triangle Inequality hold, we have a *metric space*.

We won't require the Triangle Inequality to hold; it will be something we can prove later once we add some more axioms.

Theorem 4: If $A-B-C$, $A-C-D$, and the inequalities $AB+BD \geq AD$ and $BC+CD \geq BD$ hold, then $A-B-C-D$. (This is Theorem 2 of Kay.)

Definition:

Segment AB**:** If A and B are distinct points, the *segment* AB is $\overline{AB} = \{A,B\} \cup \{C : A-C-B\}$. Points A and B are called the *endpoints* of the segment.

Ray AB**:** If A and B are distinct points, the *ray* AB is $\overrightarrow{AB} = \{A,B\} \cup \{C : A-C-B\} \cup \{D : A-B-D\}$. Point A is called the *endpoint* or *origin* of the ray.

Line AB **:** If A, B, and C are distinct points such that $A-B-C$, then $\overleftrightarrow{AB} = \overrightarrow{BA} \cup \overrightarrow{BC}$. Actually, this must be proven to be equivalent to the original definition of $\overleftrightarrow{AB}$ as the unique line containing both A and B—we will do this later.

Angle ABC**:** If A, B, and C are noncollinear points, the *angle* ABC is $\angle ABC = \overrightarrow{BA} \cup \overrightarrow{BC}$. Point B is called the *vertex* of the angle. Note that our definition of angle explicitly excludes the possibility that A, B, and C are collinear.

Theorem 5:

1. $\overline{AB} = \overline{BA}$.
2. $\overline{AB} \subseteq \overrightarrow{AB}$.
3. $\overrightarrow{AB} \subseteq \overleftrightarrow{AB}$.

(This is Theorem 3 of Kay.)

Theorem 6: $\overrightarrow{AB} \cap \overrightarrow{BA} = \overline{AB}$. (This is Theorem 4 of Kay.)

Model for Axioms I-1–I-5, D-1–D-3:

Points: Space consists of six points: $S = \{A, B, C, D, E, F\}$.

Lines: There are ten lines, each being a certain finite set of points:

$$\begin{gathered}\{A, B, C, D\}\\ \{A, E\}\\ \{B, E\}\\ \{C, E\}\\ \{D, E\}\\ \{A, F\}\\ \{B, F\}\\ \{C, F\}\\ \{D, F\}\\ \{E, F\}\end{gathered}$$

Planes: There are six planes, each being a certain finite set of points:

$$\begin{gathered}\{A, B, C, D, E\}\\ \{A, B, C, D, F\}\\ \{A, E, F\}\\ \{B, E, F\}\\ \{C, E, F\}\\ \{D, E, F\}\end{gathered}$$

Distance: Distance between various pairs of points is defined by:

$$\begin{aligned}&AE = BE = CE = DE = EF = 2\\ &AF = BF = CF = DF = AC = 2\\ &AB = BC = CD = AD = BD = 1\\ &PQ = 0 \text{ if } P = Q\\ &PQ = QP \text{ for all points } P \text{ and } Q\end{aligned}$$

Observe that in this model, $A - B - C$ holds but $\overrightarrow{BA} \cup \overrightarrow{BC} \neq \overleftrightarrow{AB}$. This shows that the following axiom cannot be proved from the preceding axioms, since it does not necessarily hold for a model even when the preceding axioms hold for that model.

Axiom D-4: Given any three points A, B, and C on line ℓ such that $A - B - C$, $\overrightarrow{BA} \cup \overrightarrow{BC} = \ell$.

The following Axiom is equivalent to D-4:

Axiom D-4': Given any four distinct collinear points A, B, C, and D such that $A - B - C$, then either $D - A - B$, $A - D - B$, $B - D - C$, or $B - C - D$. (You can prove this is equivalent to Axiom D-4.)

1. Prove that Axiom D-4 implies Axiom D-4' and vice versa.

There is one more distance axiom, which we will encounter in Section 2.7 of Kay:

Axiom D-5 — Ruler Postulate: The points of each line ℓ may be assigned to real numbers x, $-\infty < x < \infty$, called *coordinates*, in such a manner that

1. Each point on ℓ is assigned to a unique coordinate.
2. Each coordinate is assigned to a unique point on ℓ.
3. Any two points on ℓ may be assigned to zero and a positive coordinate, respectively.
4. If points A and B on ℓ have coordinates a and b, respectively, then $AB = |a - b|$.

4.2 Distance in $\mathbf{E}^2$

4.2.1 The Distance Formula

Definition: The distance AB between the points $A = (x_1, y_1)$ and (x_2, y_2) in $\mathbf{E}^2$ is given by

$$AB = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

1. Given the points A and B above, consider a third point $C = (x_2, y_1)$ and use triangle ABC to prove the distance formula from the Pythagorean theorem.

   Points $A = (x_1, y_1)$, $B = (x_2, y_2)$, and $C = (x_2, y_1)$ form a right triangle (possibly degenerate) with hypotenuse AB. So $AB = \sqrt{AC^2 + CB^2} = \sqrt{|x_2 - x_1|^2 + |y_2 - y_1|^2} = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$.

2. Verify that Axioms D-1, D-2, and D-3 hold.

   The quantity $(x_2 - x_1)^2 + (y_2 - y_1)^2$ is nonnegative, so its square root is defined (and is also nonnegative). So D-1 holds. If $(x_1, y_1) \neq (x_2, y_2)$, then $(x_2 - x_1)^2 + (y_2 - y_1)^2$ is strictly positive, as is its square root. So D-2 holds. Finally, $\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$, so D-3 holds.

3. Given $A = (x_1, y_1)$, $B = (x_1, y_2)$, recall that we proved that

$$\overleftrightarrow{AB} = \{A + t(B - A) : t \in \mathbf{R}\}.$$

   Prove the following theorem:

   Assume that $C \in \overleftrightarrow{AB}$. Then $A - C - B$ if and only if $C = A + t(B - A)$ where $0 < t < 1$.

   Hint: First show that if $C = A + t(B - A)$ then $AC = |t| AB$.

4. Verify that Axiom D-4 (or Axiom D-4') holds.

   This follows readily from the parametric form of the line.

5. Verify that Axiom D-5 holds.

   Hint: Assume $C = A + t(B - A)$. Assign C the coordinate $t \cdot AB$.

6. For two ordered pairs $A = (x_1, y_1)$ and $B = (x_2, y_2)$, define

$$\boxed{A \cdot B = x_1 x_2 + y_1 y_2}$$

   For an ordered pair $A = (x_1, y_1)$, define

$$\boxed{\|A\| = \sqrt{x_1^2 + y_1^2} = \sqrt{A \cdot A}}$$

   Prove the following theorem directly from the definitions:

$$\boxed{A \cdot B \leq \|A\| \|B\|}$$

$$
\begin{aligned}
0 &\leq (x_1y_2 - x_2y_1)^2 \\
0 &\leq x_1^2y_2^2 - 2x_1x_2y_1y_2 + x_2^2y_1^2 \\
x_1^2x_2^2 + 2x_1x_2y_1y_2 + y_1^2y_2^2 &\leq x_1^2x_2^2 + x_1^2y_2^2 + x_2^2y_1^2 + y_1^2y_2^2 \\
(x_1x_2 + y_1y_2)^2 &\leq (x_1^2 + y_1^2)(x_2^2 + y_2^2) \\
|x_1x_2 + y_1y_2| &\leq \sqrt{x_1^2 + y_1^2}\sqrt{x_2^2 + y_2^2} \\
x_1x_2 + y_1y_2 &\leq \sqrt{x_1^2 + y_1^2}\sqrt{x_2^2 + y_2^2}
\end{aligned}
$$

7. Prove:

$$\boxed{(A + B) \cdot (A + B) = \|A\|^2 + 2A \cdot B + \|B\|^2}$$

$$
\begin{aligned}
(A + B) \cdot (A + B) &= A \cdot A + A \cdot B + B \cdot A + B \cdot B \\
&= \|A\|^2 + 2A \cdot B + \|B\|^2
\end{aligned}
$$

8. Observe the obvious fact that

$$\boxed{AB = \|B - A\| = \sqrt{(B - A) \cdot (B - A)}}$$

Prove the Triangle Inequality holds for any three points A, B, C:

$$\boxed{AC \leq AB + BC}$$

Suggestion: First prove that

$$\boxed{\|D + E\| \leq \|D\| + \|E\|}$$

Then let $D = B - A$ and $E = C - B$.

$$\begin{aligned}
D \cdot E &\leq \|D\|\|E\| \\
2D \cdot E &\leq 2\|D\|\|E\| \\
\|D\|^2 + 2D \cdot E + \|E\|^2 &\leq \|D\|^2 + 2\|D\|\|E\| + \|E\|^2 \\
(D + E) \cdot (D + E) &\leq \|D\|^2 + 2\|D\|\|E\| + \|E\|^2 \\
\|D + E\|^2 &\leq (\|D\| + \|E\|)^2 \\
\|D + E\| &\leq \|D\| + \|E\|
\end{aligned}$$

Let $D = B - A$ and $E = C - B$. Then

$$\begin{aligned}
\|B - A + C - B\| &\leq \|B - A\| + \|C - B\| \\
\|C - A\| &\leq \|B - A\| + \|C - B\| \\
AC &\leq AB + BC
\end{aligned}$$

4.2.2 The Snowflake Curve

Begin with an equilateral triangle. Let's assume that each side of the triangle has length one. Remove the middle third of each line segment and replace it with two sides of an "outward-pointing" equilateral triangle of side length 1/3. Now you have a six-pointed star formed from 12 line segments of length 1/3. Replace the middle third of each of these line segments with two sides of outward equilateral triangle of side length 1/9. Now you have a star-shaped figure with 48 sides. Continue to repeat this process, and the figure will converge to the "Snowflake Curve." Shown below are the first three stages in the construction of the Snowflake Curve.

1. In the limit, what is the length of the Snowflake Curve?

   At stage 0 there are 3 segments, each of length 1. At each stage, each segment is replaced by 4 segments, each 1/3 as long. So at stage n there are $3 \cdot (4)^n$ segments, each of length $(1/3)^n$, giving a perimeter of $3 \cdot (4/3)^n$. This tends to infinity as n tends to infinity since $4/3 > 1$.

2. In the limit, what is the area enclosed by the Snowflake Curve?

   At stage 0 there are 3 segments; and 1 triangle of area $A = \frac{1}{2}bh = \frac{1}{2}(1)\frac{\sqrt{3}}{2} = \frac{\sqrt{3}}{4}$. At stage 1 there are 12 segments; and each of the previous 3 segments yields a new triangle of area $\frac{1}{9}A$ (since each edge length is 1/3 as long). At stage 2 there are 48 segments; and each of the previous 12 segments yields a new triangle of area $\frac{1}{9}(\frac{1}{9}A) = \frac{1}{81}A$. At stage $n-1$ there are $3 \cdot 4^{n-1}$ segments. At stage n there are $3 \cdot 4^n$ segments; and each of the previous $3 \cdot 4^{n-1}$ segments yields a new triangle of area $(1/9)^n A$. So the total

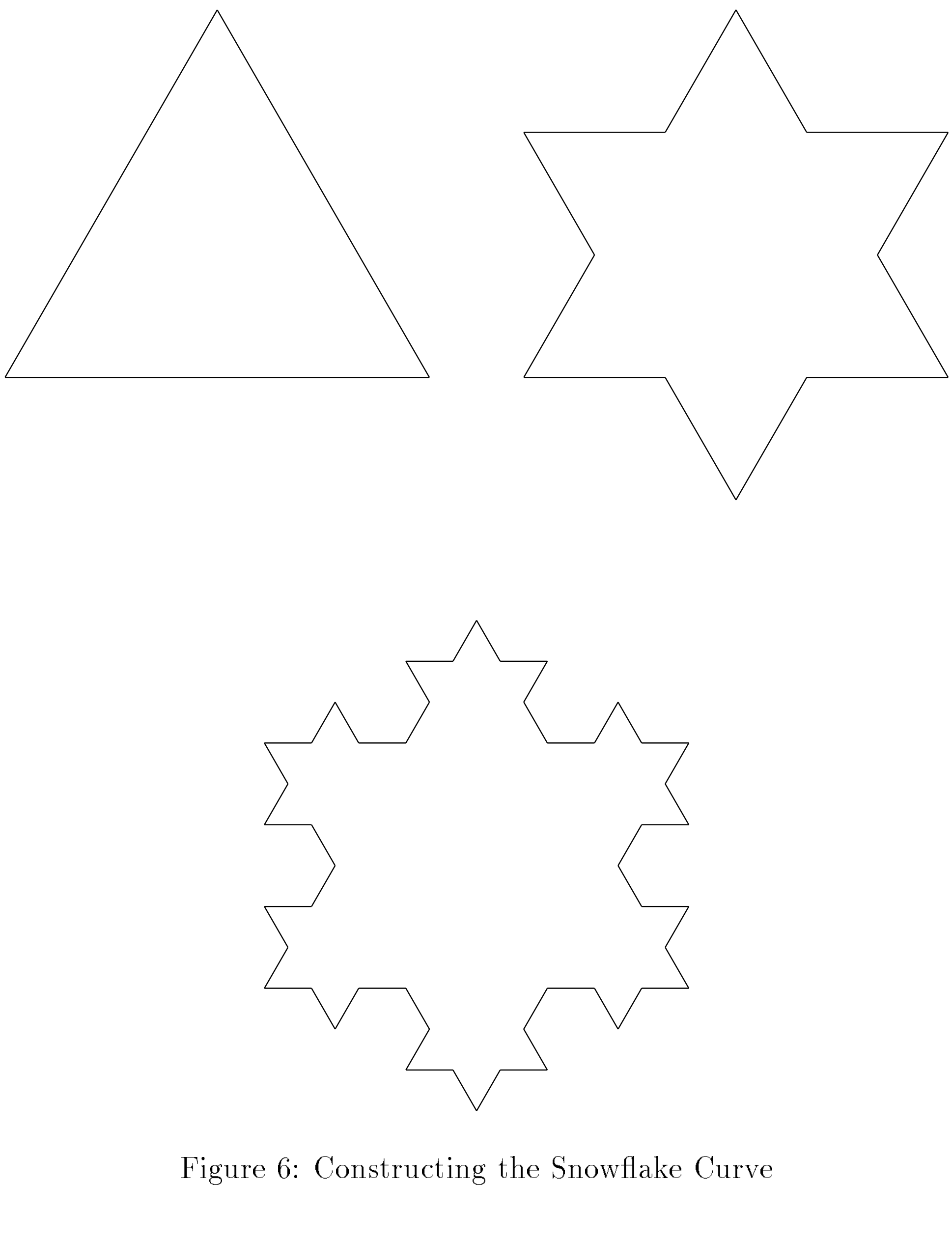


Figure 6: Constructing the Snowflake Curve

area in the limit is:

$$
\begin{aligned}
& A + 3\tfrac{1}{9}A + 12\tfrac{1}{81}A + \cdots + 3 \cdot 4^{n-1}(\tfrac{1}{9})^n A + \cdots \\
= \;& A + \tfrac{3}{4}A[\tfrac{4}{9} + (\tfrac{4}{9})^2 + \cdots + (\tfrac{4}{9})^n + \cdots] \\
= \;& A + \tfrac{3}{4}A\left[\frac{\frac{4}{9}}{1-\frac{4}{9}}\right] \\
= \;& A + \tfrac{3}{4}A\tfrac{4/9}{5/9} \\
= \;& A + \tfrac{3}{4}A\tfrac{4}{5} \\
= \;& A + \tfrac{3}{5}A \\
= \;& \tfrac{8}{5}A \\
= \;& \tfrac{8}{5}\tfrac{\sqrt{3}}{4} \\
= \;& \tfrac{2\sqrt{3}}{5}
\end{aligned}
$$

4.2.3 The Longimeter

How can we measure the lengths of curves in “real life?” There are devices consisting of wheels with some sort of dial that you can roll over a map to estimate distances, and larger versions that you can roll in front of you on, e.g., paths, to measure distance (what are these things called?). You can also estimate the distance that you walk by wearing a pedometer.

Here is another way to estimate the length of a curve on a map, using a simple device called a *longimeter*. On a transparent sheet of plastic create a square grid, each square having side length of, say 1 mm. Superimpose this grid your curve in three different orientations, differing one from the other by a rotation of 30°. In each of the three cases, count how many squares the curve passes through. Let the sum of these three numbers be S. Then an estimate of the length of the curve is $S/3.82$ mm.

In the example below, I rotated the figure rather than the grid. Each square has side length 0.25 in. The sum S is $16 + 16 + 15 = 47$, so the estimate of the length of the curve is $47/3.82 \approx 12.30$ units of length 0.25 in, or 3.07 in.

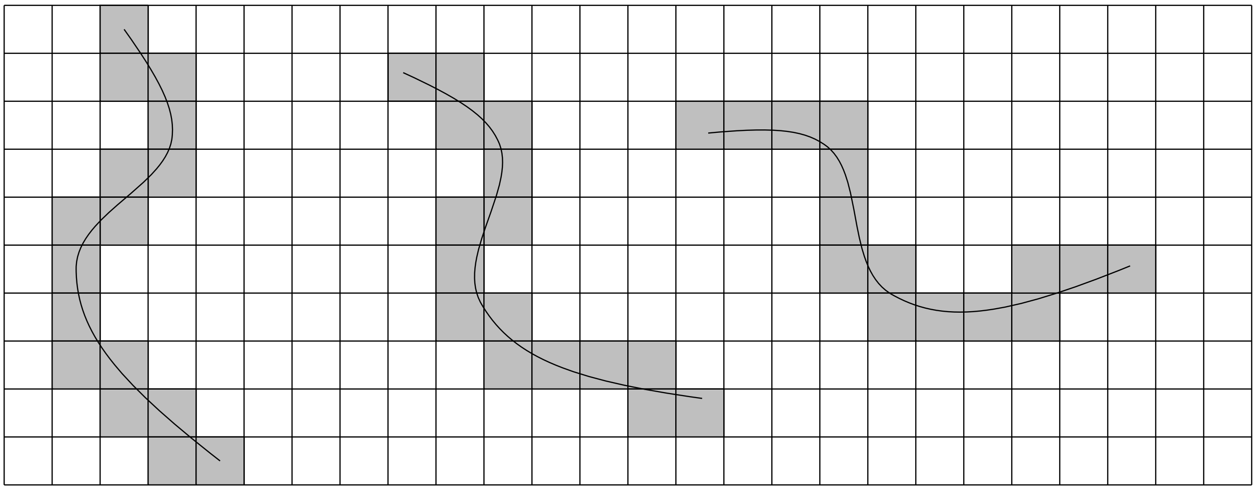


Figure 7: Using a Longimeter

Research question: Read the reference below and write up an explanation of why this method works. In particular, where does the number 3.82 come from? (This is not explicitly explained in the book.)

Reference: H. Steinhaus, *Mathematical Snapshots*, Oxford University Press, New York, 1989, pp. 105–107.

4.2.4 Fractals

The notion of "length" of certain naturally occurring objects can, however, be tricky, and can lead one into the notion of fractals. The following quote comes from a book by Mandelbrot:

> *To introduce a first category of fractals, namely curves whose fractal dimension is greater than* 1*, consider a stretch of coastline. It is evident that its length is at least equal to the distance measured along a straight line between its beginning and its end. However, the typical coastline is irregular and winding, and there is no question it is much longer than the straight line between its end points.*
>
> *There are various ways of evaluating its length more accurately... The result is most peculiar: coastline length turns out to be an elusive notion that slips between the fingers of one who wants to grasp it. All measurement methods ultimately lead to the conclusion that the typical coastline's length is very large and so ill determined that it is best considered infinite....*
>
> *Set dividers to a prescribed opening* ϵ*, to be called the yardstick length, and walk these dividers along the coastline, each new step starting where the previous step leaves off. The number of steps multipled by* ϵ *is an approximate length* $L(\epsilon)$*. As the dividers' opening becomes smaller and smaller, and as we repeat the operation, we have been taught to expect* $L(\epsilon)$ *to settle rapidly to a well-defined value called the* true length*. But in fact what we expect does not happen. In the typical case, the observed* $L(\epsilon)$ *tends to increase without limit.*
>
> *The reason for this behavior is obvious: When a bay or peninsula noticed on a maa scaled to* $1/100,000$ *is reexamined on a map at* $1/10,000$*, subbays and subpeninsulas become visible. On a* $1/1,000$ *scale map, sub-subbays and sub-subpeninsulas appear, and so forth. Each adds to the measured length.*
>
> —B.B. Mandelbrot, "How Long is the Coast of Britain," *The Fractal Geometry of Nature*, W.H. Freeman and Company, New York, 1983, Chapter 5, p. 25.

4.3 Distance in $\mathbf{E}^3$

4.3.1 Formulas

You can easily verify that the analogues of the formulas and theorems in $\mathbf{E}^2$ hold for $\mathbf{E}^3$ as well:

Definition: The distance AB between the points $A = (x_1, y_1, z_1)$ and (x_2, y_2, z_2) in $\mathbf{E}^3$ is given by

$$AB = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

Assume that $C \in \overleftrightarrow{AB}$. Then $A - C - B$ if and only if $C = A + t(B - A)$ where $0 < t < 1$.

$$A \cdot B = x_1 x_2 + y_1 y_2 + z_1 z_2$$

$$\|A\| = \sqrt{x_1^2 + y_1^2 + z_1^2} = \sqrt{A \cdot A}$$

$$A \cdot B \le \|A\| \|B\|$$

$$\|D + E\| \le \|D\| + \|E\|$$

$$AC \le AB + BC$$

Assume that you have triangle ABC such that the coordinates of the three (distinct) points A, B, and C are $(0,0,0)$, (x_1, y_1, z_1), and (x_2, y_2, z_2), respectively. Then

$$\cos A = \frac{x_1x_2 + y_1y_2 + z_1z_2}{\sqrt{x_1^2 + y_1^2 + z_1^2}\sqrt{x_2^2 + y_2^2 + z_2^2}}$$

4.3.2 The Platonic Solids

There are exactly five different three-dimensional polyhedra that satisfy the following conditions:

- Each face is a regular polygon.
- All faces are congruent to each other.
- The same number of faces meets at each vertex (corner).
- No two neighboring faces are coplanar.
- The plane determined by any of the faces does not intersect the rest of the object anywhere else (the object is convex).

These polyhedra are called the *Platonic solids*.

Exercise 4.1 By considering the angles of regular polygons, confirm that there cannot be more than five such polyhedra:

- Four triangles, with three meeting at each vertex (*tetrahedron*).
- Eight triangles, with four meeting at each vertex (*octahedron*).
- Twenty triangles, with five meeting at each vertex (*icosahedron*).
- Six squares, with three meeting at each vertex (*cube*).
- Twelve pentagons, with three meeting at each vertex (*dodecahedron*).

To prove that all five objects can, in fact, be constructed, we can find coordinates for the vertices.

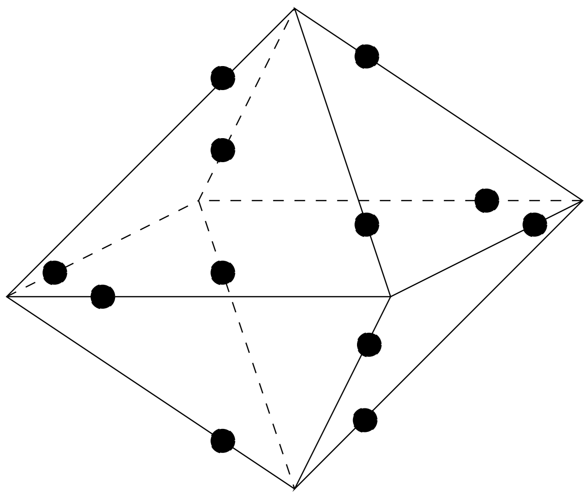
For the cube we have seen that we can use the points $(\pm 1, \pm 1, \pm 1)$.

For the octahedron we have seen that we can use the points $(\pm 1, 0, 0)$, $(0, \pm 1, 0)$, $(0, 0, \pm 1)$.

Exercise 4.2 Prove that the following four points form the vertices of a tetrahedron:

$$\begin{array}{c}(0,0,0)\\(1,1,0)\\(1,0,1)\\(0,1,1)\end{array}$$

Exercise 4.3 Determine coordinates for the icosahedron as follows. Begin with an octahedron with coordinates as above. Divide each of the 12 edges in the same ratio by 12 new points, as shown in the following figure.



Prove that if the ratio is properly chosen, these points will be the vertices of an icosahedron. Suggestion: Given the two end points of an edge of the octahedron, write the parametric formula for the line passing through them. Then the goal is to find the correct value of "t" (necessarily between 0 and 1) to locate the appropriate point on the edge. Also, a physical model of the octahedron can help you visualize the construction.

Exercise 4.4 Place a point at the center of each of the 20 triangles of the icosahedron. Verify that these 20 points will be the vertices of a dodecahedron.

4.3.3 The Global Positioning System (GPS)

Orbiting the earth is a collection of 24 satellites broadcasting signals that can be picked up and analyzed by portable receivers. You can buy such a receiver from stores like Radio Shack for a few hundred dollars. Using the signals this receiver is able to determine its distance from several of the satellites. Then, since the position in space of each of these satellites is also known, the receiver is able to compute its location in three-dimensional space (e.g., latitude, longitude, altitude).

For any given position determination you do not need to know the distance to all 24 of the satellites. How many distances do you need at any one time? How can you use these distances to calculate your position? Specifically, assume that your (unknown) position is (x_0, y_0, z_0) and that you know the distance d_i to satellite i at known location (x_i, y_i, z_i) for $i = 1, \ldots, n$. What value of n is necessary to determine your position, and what specific calculations must you perform? Suggestion: Try solving this problem first in $\mathbf{E}^1$, then in $\mathbf{E}^2$.

Reference: Thomas A. Herring, "The Global Positioning System," *Scientific American*, February 1996, pp. 44–50. See also the websites listed on the course webpage.

4.3.4 Earthquake Location

Suppose an earthquake occurs somewhere at unknown location (x_0, y_0, z_0) and unknown time t_0. You have several earthquake detectors; detector i is at location (x_i, y_i, z_i) and the tremor arrives at that position at time t_i. Let's make the simplifying assumption that the speed of the tremor v as it travels outward from the earthquake is constant. How can you calculate the location and time of the earthquake from the readings of a set of detectors? Do you need to know the speed v in advance or can you solve for it? What is the minimum number of dectectors needed to make the calculations? Do you need to make any special assumptions about the dectectors (e.g., do they need to be noncollinear or noncoplanar)? Suggestion: Try solving this problem first in $\mathbf{E}^1$, then in $\mathbf{E}^2$.

4.4 Distance in $\mathbf{S}^2$

A good book about "living" in a spherical world is *Sphereland* by Burger.

4.4.1 Distance Around a Circle

Suppose you have a circle of radius 1. Its circumference is $C = 2\pi r = 2\pi$, which is a bit bigger than 6.2. The central angle that cuts off a piece of the circumference of length 1 is called a *radian*. Therefore, there are 2π radians around the center of a circle and we can convert back and forth between degrees and radians by

$$\theta(\text{in radians}) = \frac{\pi}{180^\circ}\theta(\text{in degrees})$$

$$\theta(\text{in degrees}) = \frac{180^\circ}{\pi}\theta(\text{in radians})$$

Using radians makes many formulas look "nicer." For example,

Suppose C is a circle of radius r. The length ℓ of an arc intercepted by a central angle θ is given by

$$\ell = r\theta \quad (\text{if } \theta \text{ is measured in radians})$$

$$\ell = \frac{\pi}{180^\circ}r\theta \quad (\text{if } \theta \text{ is measured in degrees})$$

Another motivation for expressing angles is radians are the Taylor series formulas for sine and cosine:

For angle x measured in radians:

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$$

Exercise 4.5 Derive these Taylor series.

Let $f(x) = \sin x$. Then

$$\begin{aligned}
& f(0) + f'(0)x + f''(0)x^2/2! + f'''(0)x^3/3! + f^{(4)}(0)x^4/4! + f^{(5)}(0)x^5/5! + f^{(6)}(0)x^6/6! + \cdots \\
= \; & \sin(0) + \cos(0)x - \sin(0)x^2/2! - \cos(0)x^3/3! + \sin(0)x^4/4! + \cos(0)x^5/5! - \sin(0)x^6/6! - \cdots \\
= \; & x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots
\end{aligned}$$

Let $f(x) = \cos x$. Then

$$\begin{aligned}
& f(0) + f'(0)x + f''(0)x^2/2! + f'''(0)x^3/3! + f^{(4)}(0)x^4/4! + f^{(5)}(0)x^5/5! + f^{(6)}(0)x^6/6! + \cdots \\
= \; & \cos(0) - \sin(0)x - \cos(0)x^2/2! + \sin(0)x^3/3! + \cos(0)x^4/4! - \sin(0)x^5/5! - \cos(0)x^6/6! + \cdots \\
= \; & 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots
\end{aligned}$$

Exercise 4.6 Sum the squares of the above series to verify that $\sin^2 x + \cos^2 x = 1$.

This might remind you of the Taylor series for e^x:

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \frac{x^6}{6!} + \frac{x^7}{7!} + \cdots$$

Exercise 4.7 Derive this Taylor series.

Let $f(x) = e^x$. Then

$$\begin{aligned} & f(0) + f'(0)x + f''(0)x^2/2! + f'''(0)x^3/3! + f^{(4)}(0)x^4/4! + f^{(5)}(0)x^5/5! + f^{(6)}(0)x^6/6! + \cdots \\ = \; & e^0 + e^0 x + e^0 x^2/2! + e^0 x^3/3! + e^0 x^4/4! + e^0 x^5/5! + e^0 x^6/6! + \cdots \\ = \; & 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \frac{x^6}{6!} + \frac{x^7}{7!} + \cdots \end{aligned}$$

Exercise 4.8 Use the above series to show that $e^a e^b = e^{a+b}$.

From substitution (and some observations about convergence), one gets the beautiful formula for all complex numbers x:

$$e^{ix} = \cos x + i \sin x$$

In particular, setting $x = \pi$ yields an expression containing the perhaps five most important constants in mathematics:

$$e^{i\pi} + 1 = 0$$

These formulas provide a connection between two representations for complex numbers on the one hand, and Cartesian and polar coordinates on the other.

Any complex number $z = a + bi$ can be represented by a point (a, b) in the Cartesian plane. But by the above formula, you can set $r = \sqrt{a^2 + b^2}$ and find θ such that $\cos\theta = a/r$ and $\sin\theta = b/r$. That is, (r, θ) are polar coordinates for the point (a, b). Then $z = r(\cos\theta + i\sin\theta) = re^{i\theta}$.

Exercise 4.9 Suppose $z_1 = a_1 + ib_1$ and $z_2 = a_2 + ib_2$, corresponding to the points $P_1 = (a_1, b_1)$ and $P_2 = (a_2, b_2)$, respectively, in the Cartesian plane. Explain how to find $z = z_1 + z_2$ geometrically.

Draw segments OP_1 and OP_2, and "complete the parallelogram." The fourth point of the parallelogram will be the point for $z_1 + z_2$.

Exercise 4.10 Suppose $z_1 = r_1e^{i\theta_1}$ and $z_2 = r_2e^{i\theta_2}$, corresponding to the points P_1, P_2 in the Cartesian plane with polar coordinates (r_1, θ_1), (r_2, θ_2), respectively. Explain how to find $z = z_1z_2$ geometrically.

Draw segments OP_1 and OP_2. Since $z_1z_2 = r_1r_2e^{i(\theta_1+\theta_2)}$, you need to construct a segment OP such that its angle with respect to the positive x-axis is the sum $\theta_1 + \theta_2$ of the angles associated with OP_1 and OP_2, and the distance of P from O is the product r_1r_2 of the distances OP_1 and OP_2.

Exercise 4.11 From what you learned in the previous exercise,

1. Show geometrically that $i^2 = -1$.

   The point $i = 0 + 1i$ corresponds to the point P with Cartesian coordinates $(1, 0)$. The angle of OP with respect to the positive x-axis is $\pi/2$, and the distance OP is 1. Multiplying i by itself thus results in a point Q such that the angle of OQ with respective to the positive x-axis is π, and the distance OQ is 1. So $Q = (-1, 0)$, which corresponds to the complex number $-1 + 0i = -1$.

2. Find three complex numbers such that $z^3 = 1$.
3. Find three complex numbers such that $z^3 = i$.
4. Explain how to find all solutions to any equation of the form $z^n = z_0$ where n is a positive integer and z_0 is a particular complex number.

Exercise 4.12 Suppose $z = z_1 z_2$ where $z_1 = e^{i\theta_1}$ and $z_2 = e^{i\theta_2}$. Use this to prove the angle sum formulas for sine and cosine.

$$\begin{aligned} e^{i(\theta_1+\theta_2)} &= e^{i\theta_1} e^{i\theta_2} \\ \cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2) &= (\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 + i\sin\theta_2) \\ &= \cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2 + i(\cos\theta_1\sin\theta_2 + \sin\theta_1\cos\theta_2) \end{aligned}$$

Equating the real and the imaginary parts yields:

$$\begin{aligned} \cos(\theta_1+\theta_2) &= \cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2 \\ \sin(\theta_1+\theta_2) &= \sin\theta_1\cos\theta_2 + \cos\theta_1\sin\theta_2 \end{aligned}$$

4.4.2 Distance Axioms in $\mathbf{S}^2$

Recall that $\mathbf{S}^2$ is a sphere of radius 1 centered at the origin. Let A and B be any two points on $\mathbf{S}^2$. Find a great circle containing both A and B (remember that great circles are "lines" in $\mathbf{S}^2$). This circle is divided into two arcs by the points A and B. Define the distance between A and B to be the length of the shorter of these arcs. Note that if A and B are not exactly opposite one another (antipodal), then there is a unique great circle containing both of them, so the distance between them is well-defined. If, on the other hand, A and B are antipodal, then there is an infinite number of great circles containing them, but the lengths of all the great-circular arcs joining A and B are the same. So even in this case the distance AB is well-defined.

Exercise 4.13 Which of the distance axioms D-1 – D-5 hold?

Axioms D-1 – D-4 are ok, but Axiom D-5 doesn't hold because the distance between pairs of points is never greater than half the circumference of the sphere.

Exercise 4.14 Suppose $A = (x_1, y_1, z_1)$ and $B = (x_2, y_2, z_2)$ are two points on $\mathbf{S}^2$. Determine an explicit formula for the distance AB.

AB equals the length of the circular arc, so we need to find the measure of the central angle AOB. The cosine of this angle is $(A \cdot B)/(\|A\|\|B\|) = A \cdot B$, so the measure of this angle is $\cos^{-1} A \cdot B$. Then the length of the arc is the measure of the angle multiplied by the radius of the great circle (which is 1). This gives the formula:

$$\boxed{AB = \cos^{-1}(x_1x_2 + y_1y_2 + z_1z_2)}$$

Exercise 4.15 Look up the latitude and longitude of Lexington, KY and Tokyo, Japan. Look up the diameter, radius, or circumference of the earth. Use this information to determine the distance between these two cities.

Exercise 4.16 Does the triangle inequality hold?

Yes, the triangle inequality holds. To prove this with algebra, assume that we have three points on the sphere: (x_1, y_1, x_1), (x_2, y_2, z_2), (x_3, y_3, z_3). We want to show

$$\cos^{-1}(x_1x_2 + y_1y_2 + z_1z_2) + \cos^{-1}(x_2x_3 + y_2y_3 + z_2z_3) \geq \cos^{-1}(x_1x_3 + y_1y_3 + z_1z_3)$$

Define

$$\begin{aligned}\theta_{12} = \cos^{-1}(x_1x_2 + y_1y_2 + z_1z_2)\\ \theta_{23} = \cos^{-1}(x_2x_3 + y_2y_3 + z_2z_3)\\ \theta_{13} = \cos^{-1}(x_1x_3 + y_1y_3 + z_1z_3)\end{aligned}$$

Then we need to show

$$\theta_{12} + \theta_{23} \geq \theta_{13}.$$

Take the cosine of both sides, and remember that the cosine function decreases for $0 \leq \theta \leq \pi$. So we need to show

$$\cos(\theta_{12} + \theta_{23}) \leq \cos\theta_{13}$$

or, by the angle addition formula,

$$\cos\theta_{12}\cos\theta_{23} - \sin\theta_{12}\sin\theta_{23} \leq \cos\theta_{13}. \tag{1}$$

Now

$$\cos\theta_{12} = x_1x_2 + y_1y_2 + z_1z_2$$

So

$$\begin{aligned}\sin\theta_{12} &= \sqrt{1 - (x_1x_2 + y_1y_2 + z_1z_2)^2}\\ &= \sqrt{(x_1y_2 - x_2y_1)^2 + (x_1z_2 - x_2z_1)^2 + (y_1z_2 - y_2z_1)^2}\end{aligned}$$

after replacing 1 by $(x_1^2 + y_1^2 + z_1^2)(x_2^2 + y_2^2 + z_2^2)$ and using algebra. (Remember, the points are on a sphere of radius 1.)

Similarly

$$\begin{aligned}\cos\theta_{23} &= x_2x_3 + y_2y_3 + z_2z_3,\\ \sin\theta_{23} &= \sqrt{(x_2y_3 - x_3y_2)^2 + (x_2z_3 - x_3z_2)^2 + (y_2z_3 - y_3z_2)^2}.\end{aligned}$$

Also

$$\cos\theta_{13} = x_1x_3 + y_1y_3 + z_1z_3.$$

Substituting into equation (1), we need to show

$$\begin{aligned} & (x_1x_2 + y_1y_2 + z_1z_2)(x_2x_3 + y_2y_3 + z_2z_3) \\ - & \sqrt{(x_1y_2 - x_2y_1)^2 + (x_1z_2 - x_2z_1)^2 + (y_1z_2 - y_2z_1)^2} \\ \times & \sqrt{(x_2y_3 - x_3y_2)^2 + (x_2z_3 - x_3z_2)^2 + (y_2z_3 - y_3z_2)^2} \\ \leq & \; x_1x_3 + y_1y_3 + z_1z_3. \end{aligned}$$

Multiply the right-hand side by $x_2^2 + y_2^2 + z_2^2$ (which equals 1) and use algebra to get the following equivalent expression:

$$\begin{aligned} & (x_1y_2 - x_2y_1)(x_3y_2 - x_2y_3) + (x_1z_2 - x_2z_1)(x_3z_2 - x_2z_3) + (y_1z_2 - y_2z_1)(y_3z_2 - y_2z_3) \\ \leq & \sqrt{(x_1y_2 - x_2y_1)^2 + (x_1z_2 - x_2z_1)^2 + (y_1z_2 - y_2z_1)^2} \\ \times & \sqrt{(x_2y_3 - x_3y_2)^2 + (x_2z_3 - x_3z_2)^2 + (y_2z_3 - y_3z_2)^2}. \end{aligned}$$

Defining

$$\begin{array}{ll} p_1 = x_1y_2 - x_2y_1 & p_2 = x_3y_2 - x_2y_3 \\ q_1 = x_1z_2 - x_2z_1 & q_2 = x_3z_2 - x_2z_3 \\ r_1 = y_1z_2 - y_2z_1 & r_2 = y_3z_2 - y_2z_3 \end{array}$$

we see that what we have to prove is

$$p_1p_2 + q_1q_2 + r_1r_2 \leq \sqrt{p_1^2 + q_1^2 + r_1^2}\sqrt{p_2^2 + q_2^2 + r_2^2}.$$

But we have already proved this inequality in Section 4.3.1, so we are done.

4.4.3 What is the Size of the Earth?

The kilometer was first defined as $\frac{1}{10{,}000}$ of the distance from the North Pole to the equator of the Earth. (Can you find a reference that verifies this statement?)

Exercise 4.17 What is the circumference of the Earth in kilometers? In miles? Use the conversion 1 mile equals 1.6 kilometers.

The circumference is $4 \times 10,000 = 40,000$ kilometers, which is approximately $40,000/1.6 \approx 25,000$ miles.

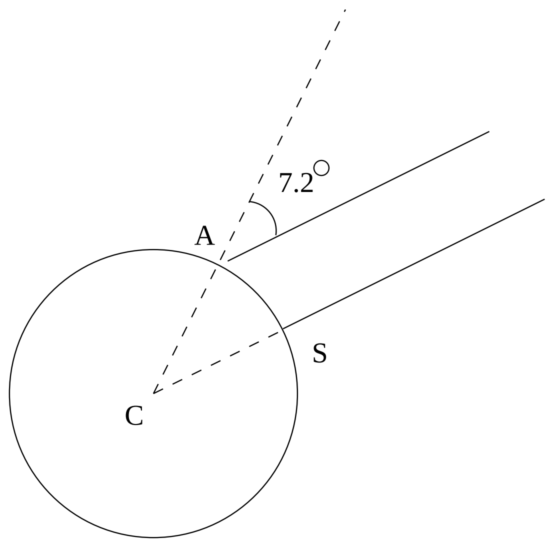
Exercise 4.18 What is the radius of the Earth in kilometers? In miles?

The radius of the Earth is $40,000/(2\pi) \approx 6400$ kilometers, which (upon dividing by 1.6) is approximately 4000 miles.

Exercise 4.19 Research question: What is the current official definition of a kilometer and a mile?

4.4.4 Eratosthenes' Estimate of the Size of the Earth

Eratosthenes in about 200 B.C. observed that on a certain day the sun was directly overhead in the city of Syene, but rays from the sun struck the ground at an angle of $7.2°$ from the vertical in the city of Alexandria, which lay 5000 stadia due north of Syene. Assuming that the Earth was spherical, he used this to estimate its size.



Exercise 4.20 Use this information to determine the radius and circumference of the Earth, in stadia.

Angle ACS has measure $7.2°$. Dividing by $360°$, we see that this is $\frac{7.2}{360} = \frac{1}{50}$th of $360°$. So the distance from Syene to Alexandria is $\frac{1}{50}$th of the circumference of the Earth. This implies that the circumference is $50 \times 5000 = 250,000$ stadia.

Exercise 4.21 While we do not know the length of a stadium precisely, one estimate is that 1 stadium equals 157.5 meters. Use this to estimate the radius and circumference of the Earth, in kilometers. How do these figures compare with current estimates?

Converting to meters, the circumference is approximately $250,000 \times 157.5 \approx 39,000,000$ meters, or $39,000$ kilometers. The radius is then $39,000/(2\pi) \approx 6200$ kilometers. Current estimates of circumference and radius are 40,000 kilometers and 6400 kilometers, respectively.

4.5 Distance in $\mathbf{U}^2$

4.5.1 The Distance Axioms

Recall that POINTS in $\mathbf{U}^2$ are points on the unit sphere centered at the origin, except for the point $N = (0, 0, 1)$. For two points $A = (x_1, y_1, z_1)$ and $B = (x_2, y_2, z_2)$ in $\mathbf{U}^2$, define the distance AB to be

$$AB = \sqrt{(\frac{x_2}{1 - z_2} - \frac{x_1}{1 - z_1})^2 + (\frac{y_2}{1 - z_2} - \frac{y_1}{1 - z_1})^2}$$

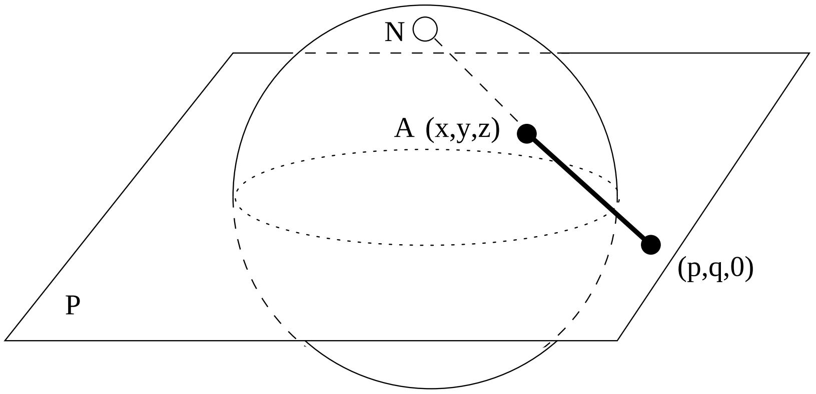
Note that, peculiar that this definition may appear to be, it is well-defined because neither z_1 or z_2 equals 1.

Exercise 4.22 Verify that Axioms D-1 – D-3 hold.

Observing that both z_1 and z_2 are strictly less than 1, it is readily seen that distance is well-defined and nonnegative, so D-1 holds. From the definition, D-3 is clear. That two distinct points determine a strictly positive distance can be seen from the discussion in the next section.

4.5.2 Stereographic Projection

The easiest way to explain what is going on in $\mathbf{U}^2$ is to define a mapping from $\mathbf{U}^2$ to the analytic Cartesian plane $\mathbf{E}^2$ in the following way. Let P be the plane given by the equation $z = 0$; i.e., the plane containing the "equator" of the unit sphere. For each point $A = (x, y, z)$ in $\mathbf{U}^2$, consider the ordinary ray $\overrightarrow{NA}$ in $\mathbf{E}^3$. Let $(p, q, 0)$ be the point where this ray intersects P.



Then you can prove that

$$\boxed{\begin{aligned} p &= \frac{x}{1-z} \\ q &= \frac{y}{1-z} \end{aligned}}$$

Exercise 4.23 Prove the above formulas.

The points on $\overleftrightarrow{NA}$ are given parametrically by

$$\begin{gathered} N + t(A - N) \\ (0,0,1) + t(x-0, y-0, z-1) \\ (tx, ty, 1 + t(z-1)) \end{gathered}$$

To determine where this ray intersects the plane given by the equation $z = 0$, we need to find t such that $1 + t(z-1) = 0$. This is easy; $t = 1/(1-z)$. Substituting into the parametric equation gives $p = tx = x/(1-z)$ and $q = ty = y/(1-z)$.

We use these formulas to define the map $\phi : (x, y, z) \to (p, q)$ from $\mathbf{U}^2$ to $\mathbf{E}^2$. This map is called *stereographic projection.*

The inverse $\phi^{-1} : (p, q) \to (x, y, z)$ is given by

$$\boxed{\begin{aligned} x &= \frac{2p}{p^2+q^2+1} \\ y &= \frac{2q}{p^2+q^2+1} \\ z &= \frac{p^2+q^2-1}{p^2+q^2+1} \end{aligned}}$$

Exercise 4.24 Prove that (x, y, z) given by the above formulas is a point on the unit sphere centered at the origin.

$$\begin{aligned} x^2+y^2+z^2 &= \frac{4p^2+4q^2+p^4+q^4+1+2p^2q^2-2p^2-2q^2}{(p^2+q^2+1)^2} \\ &= \frac{p^4+q^4+1+2p^2+2q^2+2p^2q^2}{(p^2+q^2+1)^2} \\ &= \frac{(p^2+q^2+1)^2}{(p^2+q^2+1)^2} \\ &= 1 \end{aligned}$$

Exercise 4.25 Prove that ϕ^{-1} is the inverse of ϕ by showing that $\phi \circ \phi^{-1}$ and $\phi^{-1} \circ \phi$ are each the identify map.

We check $\phi \circ \phi^{-1}$ by substituting the formulas for x, y, z into those for p, q:

$$\begin{aligned} 1-z &= 1-\frac{p^2+q^2-1}{p^2+q^2+1} \\ &= \frac{p^2+q^2+1-(p^2+q^2-1)}{p^2+q^2+1} \\ &= \frac{2}{p^2+q^2+1} \end{aligned}$$

$$\begin{aligned} p &= \frac{x}{1-z} \\ &= \frac{(2p)/(p^2+q^2+1)}{(2)/(p^2+q^2+1)} \\ &= p, \text{ as required.} \end{aligned}$$

$$\begin{aligned} q &= \frac{y}{1-z} \\ &= \frac{(2q)/(p^2+q^2+1)}{(2)/(p^2+q^2+1)} \\ &= q, \text{ as required.} \end{aligned}$$

So $\phi \circ \phi^{-1}$ is the identity map.

We check $\phi^{-1}\phi$ by substituting the formulas for p, q into those for x, y, z:

$$\begin{aligned} p^2+q^2+1 &= \frac{x^2+y^2+(1-z)^2}{(1-z)^2} \\ &= \frac{x^2+y^2+1-2z+z^2}{(1-z)^2} \\ &= \frac{2-2z}{(1-z)^2} \\ &= \frac{2}{1-z} \end{aligned}$$

$$\begin{aligned} x &= \frac{2p}{p^2+q^2+1} \\ &= \frac{(2x)/(1-z)}{(2)/(1-z)} \\ &= x, \text{ as required.} \\ y &= \frac{2q}{p^2+q^2+1} \\ &= \frac{(2y)/(1-z)}{(2)/(1-z)} \\ &= y, \text{ as required.} \end{aligned}$$

Now we know that ϕ maps $\mathbf{U}^2$ bijectively onto $\mathbf{E}^2$. What do LINES in $\mathbf{U}^2$ map to? A LINE in $\mathbf{U}^2$ is a circle on the unit sphere passing through N (with the point N itself deleted). It is the intersection of the unit sphere with a plane Q through N. Each of the rays $\overrightarrow{NA}$ for points A on the circle lie in this plane, so the intersections of these rays with the plane P equal the intersection $P \cup Q$, which is an ordinary line. So LINES in $\mathbf{U}^2$ map to ordinary lines in $\mathbf{E}^2$.

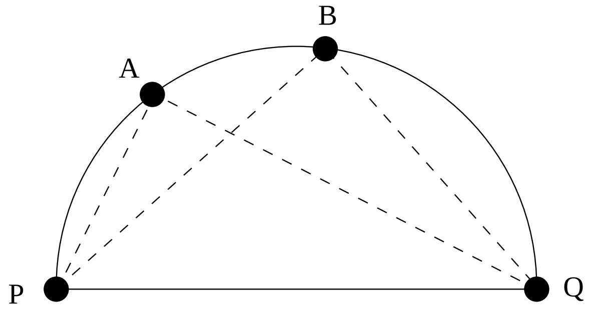
Conversely, suppose you are given an ordinary line ℓ in $\mathbf{E}^2$. Regard this line as sitting in P. There is a unique plane containing this line and passing through N. This plane intersects the unit sphere in a circle passing through N. Deleting N from this circle gives $\phi^{-1}(\ell)$.

Looking at the formula for ϕ, the distance between two points A, B in $\mathbf{U}^2$ is the same as the ordinary distance between the points $\phi(A), \phi(B)$ in $\mathbf{E}^2$. So all the distance axioms in $\mathbf{U}^2$ hold because they hold in $\mathbf{E}^2$. For the same reason, the triangle inequality also holds.

Exercise 4.26 We know that ϕ maps circles on the unit sphere through N (but not including N) to ordinary lines in $\mathbf{E}^2$, and conversely that ϕ^{-1} maps ordinary lines in $\mathbf{E}^2$ to circles on the unit sphere through N (but not including N). Let C be a circle on the unit sphere that does not pass through N. Describe $\phi(C)$.

4.6 Distance in $\mathbf{H}^2$

Recall that POINTS in $\mathbf{H}^2$ are points (x, y, z) on the unit sphere centered at the origin such that $z > 1$; i.e, points in the upper hemisphere, excluding the equator. LINES in $\mathbf{H}^2$ are open half-circles perpendicular to the equator. For two points A, B, consider the unique semicircle that contains both of them, and let P and Q be the endpoints of the semicircle on the equator as shown below:



Define the distance AB to be

$$\boxed{AB = \ln(\frac{AQ}{AP}\frac{BP}{BQ})}$$

where AP, AQ, BP, and BQ are the ordinary lengths of line segments.

Exercise 4.27 Verify that Axioms D-1 – D-3 hold for this model.

First observe that triangle PAQ is a right triangle with right angle at A (this is a theorem about triangles inscribed in semicircles) and also triangle PBQ is a right triangle with right angle at B. So $AP = PQ \sin \angle AQP$ and $BP = PQ \sin \angle BQP$. Now $m\angle BQP \geq m\angle AQP$, and both angles are less than right angles, so $\sin \angle BQP \geq \sin \angle AQP$. From this we conclude that $BP \geq AP$. Similarly, $AQ \geq BQ$. Since $BP \geq AP$ and $AQ \geq BQ$, then the quantity that we are taking the logarithm of is greater than or equal to 1. So its logarithm is nonnegative. Therefore D-1 holds. If $A \neq B$ then $BP > AP$ and $AQ > BQ$, so we are taking the logarithm of a number that is strictly greater than 1, resulting in a logarithm that is strictly positive. Therefore D-2 holds. Finally, for D-3,

$$AB = \ln \frac{AQ}{AP}\frac{BP}{BQ} = \frac{BP}{BQ}\frac{AQ}{AP} = BA.$$

Exercise 4.28 For two points A, C, consider the unique perpendicular semicircle that contains both of them, and let B be a point on the arc of the semicircle between A and C. Prove that $A - B - C$.

It is given in this case that A, B, C are collinear, so it remains to show that $AB + BC = AC$. Using the same points P and Q as in the above figure,

$$\begin{aligned} AB + BC &= \ln(\tfrac{AQ}{AP}\tfrac{BP}{BQ}) + \ln(\tfrac{BQ}{BP}\tfrac{CP}{CQ}) \\ &= \ln(\tfrac{AQ}{AP}\tfrac{BP}{BQ}\tfrac{BQ}{BP}\tfrac{CP}{CQ}) \\ &= \ln(\tfrac{AQ}{AP}\tfrac{CP}{CQ}) \\ &= AC \end{aligned}$$

Exercise 4.29 Prove that if A remains fixed and B moves toward Q, then AB tends to infinity.

As B moves toward Q, the expressions AP and AQ do not change, BP converges to the finite number PQ, and BQ converges to 0. Therefore the fraction $(AQ \cdot AP)/(BP \cdot BQ)$ tends to positive infinity, and so does its logarithm.

Exercise 4.30 Prove that Axioms D-4 and D-5 hold for this model.

Exercise 4.31 Suppose $A = (x_1, y_1, z_1)$ and $B = (x_2, y_2, z_2)$ in the figure above.

1. Determine the equation of the plane containing both of them that is perpendicular to the equator.
2. Determine the coordinates of the points P and Q.
3. Write a formula for AB in terms of the coordinates of A and B.

Exercise 4.32 Does the triangle inequality hold for this model?

4.7 Space-Time Distance

Consider a space-time model in which points are given by four coordinates (x, y, z, t), where (x, y, z) is the location of an event and t is the time of the event. For two points (events) $A = (x_1, y_1, z_1, t_1)$ and $B = (x_2, y_2, z_2, t_2)$, define the distance between them to be

$$\boxed{AB = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2 - c^2(t_1 - t_2)^2}}$$

where c is the speed of light.

Define the line $\overleftrightarrow{AB}$ to be the set of all points C such that $CA + AB = AB$ or $AC + CB = AB$ or $AB + BC = AC$.

Exercise 4.33 Explore which incidence and distance axioms hold.