2 Incidence

2.1 Incidence Axioms

Here are the Incidence Axioms, slightly reworded from Kay, Section 2.4. Lines and planes are certain subsets of points. We know nothing further about points, lines and planes beyond that which is specified in the axioms; i.e., they are the undefined terms.

Axiom I-1: Given two distinct points, there is exactly one line containing both of them.

Axiom I-2: Given three distinct noncollinear points (three points not contained in a common line), there is exactly one plane containing all three of them.

Axiom I-3: If two distinct points are contained in a plane, then any line containing both of these points is contained in that plane.

Axiom I-4: If two planes have a nonempty intersection, then their intersection is a line.

Axiom I-5: Space contains at least four noncoplanar points (four points not contained in a common plane) and contains at least three noncollinear points. Each plane contains at least three noncollinear points. Each line contains at least two distinct points.

2.2 Incidence Theorems

Notation: If A and B are two distinct points, then $\overleftrightarrow{AB}$ denotes the unique line containing both A and B.

Theorem 1: If $C \in \overleftrightarrow{AB}$, $D \in \overleftrightarrow{AB}$ and $C \neq D$, then $\overleftrightarrow{CD} = \overleftrightarrow{AB}$. (This is Theorem 1 of Kay, Section 2.4.)

Theorem 2: If two distinct lines ℓ and m meet (have nonempty intersection), then their intersection is a single point. If a line meets a plane and is not contained in that plane, their intersection is a single point. (This is Theorem 2 of Kay, Section 2.4.)

The proof of the Theorem 1 and the first part of Theorem 2 can be found in the book.

2.3 A Tetrahedron Model

The Incidence Axioms do not force the existence of an infinite number of points. We can verify this (and at the same time demonstrate the consistency of the axioms) with the tetrahedron model shown on page 92 of the textbook, which has four points, six lines, and four planes. Try building a physical model of it using a construction kit, paper or marshmallows and toothpicks.

Examine the Maple worksheet "tetra.ms" to see how to display a model of this tetrahedron. Try to run Maple and type in the commands yourself. Once you have a picture, use the mouse to drag the picture to different orientations. (In older versions of Maple you need to redisplay the picture by clicking one or more mouse buttons, perhaps simultaneously. In newer versions of Maple you need to click on "R".)

2.4 Geometrical Worlds

Here are some geometrical "worlds." In each case we make certain choices on what we will call POINTS, LINES and PLANES. (I capitalize these words as a reminder these may not appear to be our "familiar" points, lines and planes.) In each case you should begin thinking about which of the incidence axioms hold for our choice of POINTS, LINES and PLANES. In particular, does Axiom I-1 hold? It would be helpful for experimentation to have some spherical surfaces to draw on, such as (very smooth) tennis balls, ping-pong balls, oranges or Lénárt spheres.

Also, consider the question of whether or not the following property holds:

For a given LINE and a given POINT not on that LINE, there is a unique LINE containing the given POINT that does not intersect the given LINE.

2.4.1 The Analytical Euclidean Plane: $\mathbf{E}^2$

POINTS: Ordered pairs (x, y) of real numbers; i.e., elements of $\mathbf{R}^2$.

LINES: Sets of points that satisfy an equation of the form $ax + by + c = 0$, where a, b and c are real numbers; and further a and b are not both zero.

PLANES: There is only one PLANE; namely, the set of all of the POINTS.

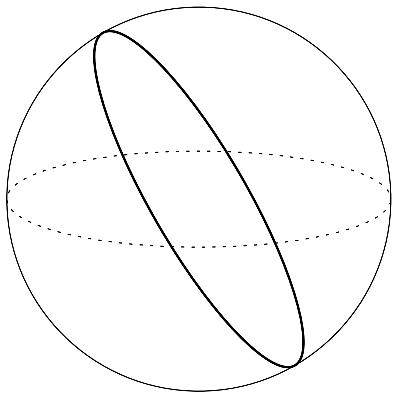
We will verify the Axiom I-1 for this model later.

2.4.2 The Sphere: $\mathbf{S}^2$

POINTS: All points in $\mathbf{R}^2$ that lie on a sphere of radius 1 centered at the origin.

LINES: Circles on the sphere that divide the sphere into two equal hemispheres. (Such circles are called *great circles.*)

PLANES: There is only one PLANE; namely, the set of all of the POINTS.



Axiom I-1 does not hold for this model. If two POINTS are chosen that are not opposite each other, then indeed there is one and only one LINE (great circle) containing both of them. But if the two POINTS are opposite each other, like the North and South Poles, then there is an infinite number of LINES containing both of them.

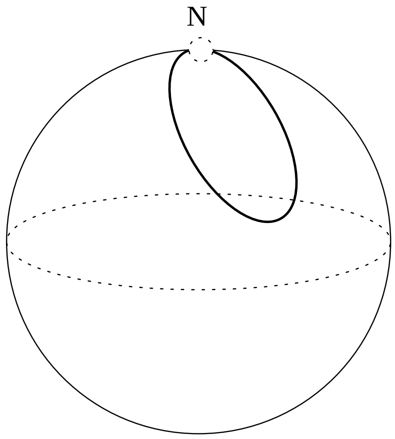
Given a LINE and a POINT not on the LINE, does there exist exactly one LINE passing through that POINT that does not intersect the given LINE? No, because every great circle intersects every other great circle, so it is impossible to have two nonintersecting LINES of any sort in this model.

2.4.3 The Punctured Sphere: $\mathbf{U}^2$

POINTS: All points in $\mathbf{R}^2$ that lie on a sphere of radius 1 centered at the origin, with the exception of the point $N = (0, 0, 1)$ (the "North Pole"), which is excluded.

LINES: Circles on the sphere that pass through N, excluding the point N itself.

PLANES: There is only one PLANE; namely, the set of all of the POINTS.



Axiom I-1 does hold for this model. If any two POINTS are chosen, then these two points, together with the point N, uniquely determine a circle (the LINE) containing these three points. Alternatively, these three points uniquely determine a plane, which, when intersected with the sphere, gives the desired LINE.

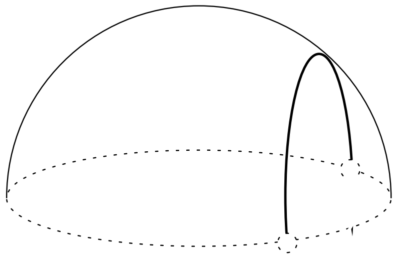
Given a LINE and a POINT not on the LINE, does there exist exactly one LINE passing through that POINT that does not intersect the given LINE? Yes, because in order for two LINES not to intersect, they must correspond to two circles on the sphere that have a common tangent line ℓ at N. Such circles are determined by intersecting the punctured sphere with planes containing that common tangent line ℓ. So if a LINE (circle through N) is given, and a point not on that line, there is exactly one plane through that given point and containing the tangent line ℓ to the circle at N.

2.4.4 The Open Hemisphere: $\mathbf{H}^2$

POINTS: All points in $\mathbf{R}^2$ that lie on the upper hemisphere of radius 1 centered at the origin with strictly positive z-coordinate. (So the "equator" of points with z-coordinate equalling 0 is excluded.)

LINES: Semicircles (not including endpoints) on this hemisphere that are perpendicular to the "equator".

PLANES: There is only one PLANE; namely, the set of all of the POINTS.



Axiom I-1 does hold for this model. If any two POINTS are chosen, then these two points uniquely determine a plane that contains them and is perpendicular to the equator. This plane intersects the hemisphere in the desired LINE (semicircle).

Given a LINE and a POINT not on the LINE, does there exist exactly one LINE passing through that POINT that does not intersect the given LINE? No, because a few minutes of experimentation will convince you that in fact that in every case there are an infinite number of LINES passing through the given POINT that do not intersect the given LINE.

2.4.5 The Projective Plane: $\mathbf{P}^2$

POINTS: All ordinary lines in $\mathbf{R}^3$ that pass through the origin.

LINES: All ordinary planes in $\mathbf{R}^3$ that pass through the origin.

PLANES: There is only one PLANE; namely, the set of all of the POINTS.

Axiom I-1 does hold for this model. If any two POINTS (ordinary lines containing the the origin) are chosen, then these two ordinary lines uniquely determine an ordinary plane through the origin that contains them. This plane is the unique LINE that contains the given POINTS.

Given a LINE and a POINT not on the LINE, does there exist exactly one LINE passing through that POINT that does not intersect the given LINE? No, because every pair of LINES (ordinary planes containing the origin) intersect in a POINT (ordinary line containing the origin).

2.4.6 Analytical Euclidean Space: $\mathbf{E}^3$

POINTS: Ordered triples (x, y, z) of real numbers.

LINES: Sets of points of the form...

We will fill this in later.

PLANES: Sets of points that satisfy an equation of the form $ax + by + cz + d = 0$, where a, b, c and d are real numbers; and further a, b and c are not all zero.

2.4.7 Analytical Euclidean 4-Space: $\mathbf{E}^4$

POINTS:

We will fill this in later.

LINES:

We will fill this in later.

PLANES:

We will fill this in later.

2.4.8 Analytical Euclidean n-Space: $\mathbf{E}^n$

Here n is an integer greater than 3.

POINTS:

We will fill this in later.

LINES:

We will fill this in later.

PLANES:

We will fill this in later.