## Dehn's Theorem

First of all, I want to emphasize that the intent of the homework was *not* to make you figure out the details of the statement of Dehn's Theorem, nor even how to calculate the "Dehn invariant," but to realize that something pretty interesting is going on in three dimensions, in that it is not always the case (and in fact most often not the case) that two polyhedra with the same volume are equidissectible. Too bad—this could have led to a much larger collection of mathematical toys!

Having said that, I hear that many of you, despite these caveats, really *want* to know what in the world the theorem is saying. So I will try to explain how to calculate the Dehn invariant. My first encounter included a fair amount of head scratching!

We are going to look at ordered pairs of real numbers $(a, b)$ and define some operations with rules.

1. $(a, b) + (a, c) = (a, b + c)$. That is to say, if two ordered pairs have the same first coordinate, you can add them by keeping the first coordinate and adding the second coordinates. So $(1, 2) + (1, 3) = (1, 5)$.

2. $(a, b) + (c, b) = (a + c, b)$. That is to say, if two ordered pairs have the same second coordinate, you can add them by keeping the second coordinate and adding the first coordinates. So $(1, 2) + (3, 2) = (4, 2)$.

3. If $k$ is a *rational number*, $k(a, b) = (ka, b)$. That is to say, to multiply an ordered pair by a rational number, you can multiply the rational number by the first coordinate. So $\frac{2}{3}(3, 4) = (2, 4)$.

4. If $k$ is a *rational number*, $k(a, b) = (a, kb)$. That is to say, to multiply an ordered pair by a rational number, you can multiply the rational number by the second coordinate. So $\frac{2}{3}(3, 4) = (3, \frac{8}{3})$.

5. If $c - b$ is a *rational multiple of* $\pi$, then $(a, b) = (a, c)$. That is to say, if two ordered pairs share the same first coordinate, and the second coordinates differ by a rational number multiplied by $\pi$, then the ordered pairs are regarded as equal. So, for example, $(\sqrt{3}, 0) = (\sqrt{3}, 7\pi/12) = (\sqrt{3}, 47\pi)$. Also $(\sqrt{7}, \sqrt{3}) = (\sqrt{7}, \sqrt{3} - 8\pi/5)$. But $(\sqrt{6}, \sqrt{3}) \neq (\sqrt{6}, \pi)$.

Of course, we can combine all these rules at will, so we can do calculations like: $(12, 12) = 3(4, 12) = (4, 36) = (1, 36) + (3, 36) = (1, 25) + (1, 11) + (3, 36)$, etc.

It's "fun" to try to simplify expressions. For example:

$$\begin{aligned}
(12\sqrt{7}, \tfrac{5}{6}\pi) + (18\sqrt{7}, \sqrt{5}+\pi) &= 2(6\sqrt{7}, \tfrac{5}{6}\pi) + 3(6\sqrt{7}, \sqrt{5}+\pi) && \text{(Rule 3)} \\
&= (6\sqrt{7}, \tfrac{5}{3}\pi) + (6\sqrt{7}, 3\sqrt{5}+3\pi) && \text{(Rule 4)} \\
&= (6\sqrt{7}, \tfrac{14}{3}\pi + 3\sqrt{5}) && \text{(Rule 1)} \\
&= (6\sqrt{7}, 3\sqrt{5}) && \text{(Rule 5)} \\
&= 3(6\sqrt{7}, \sqrt{5}) && \text{(Rule 4)} \\
&= (18\sqrt{7}, \sqrt{5}) && \text{(Rule 3)}
\end{aligned}$$

Now for *Dehn's Theorem.* Take any polyhedron $P$. For each edge determine (1) its length $\ell(e)$ and (2) the (radian) measure $\alpha(e)$ of the dihedral angle formed by the two faces meeting at that edge. Form the ordered pair $(\ell(e), \alpha(e))$. Add up all of these ordered pairs (one for each edge). This is the *Dehn invariant* for $P$. Now repeat for another polyhedron $Q$. If the two Dehn invariants are equal via (possibly repeated applications of) Rules 1–5 *and* $P$ and $Q$ have the same volumes, then they are equidissectible. Otherwise they are not.