Linear Programming Notes

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January 3, 1996
Latest Revision: Summer 2002

Contents

1 References

Four good references for linear programming are

1. Dimitris Bertsimas and John N. Tsitsiklis, *Introduction to Linear Optimization*, Athena Scientific.
2. Vašek Chvátal, *Linear Programming*, W.H. Freeman.
3. George L. Nemhauser and Laurence A. Wolsey, *Integer and Combinatorial Optimization*, Wiley.
4. Christos H. Papadimitriou and Kenneth Steiglitz, *Combinatorial Optimization: Algorithms and Complexity*, Prentice Hall.

I used some material from these sources in writing these notes. Also, some of the exercises were provided by Jon Lee and Francois Margot. Thanks in particular to Francois Margot for many useful suggestions for improving these notes.

Exercise 1.1 Find as many errors in these notes as you can and report them to me. □

2 Exercises: Linear Algebra

It is important to have a good understanding of the content of a typical one-semester undergraduate matrix algebra course. Here are some exercises to try. Note: Unless otherwise specified, all of my vectors are column vectors. If I want a row vector, I will transpose a column vector.

Exercise 2.1 Consider the product $C = AB$ of two matrices A and B. What is the formula for c_{ij}, the entry of C in row i, column j? Explain why we can regard the ith row of C as a linear combination of the rows of B. Explain why we can regard the jth column of C as a linear combination of the columns of A. Explain why we can regard the ith row of C as a sequence of inner products of the columns of B with a common vector. Explain why we can regard the jth column of C as a sequence of inner products of the rows of A with a common vector. Consider the block matrices

$$\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] \text{ and } \left[\begin{array}{c|c} E & F \\ \hline G & H \end{array}\right].$$

Assume that the number of columns of A and C equals the number of rows of E and F, and that the number of columns of B and D equals the number of rows of G and H. Describe the product of these two matrices. $\square$

Exercise 2.2 Associated with a matrix A are four vector spaces. What are they, how can you find a basis for each, and how are their dimensions related? Give a "natural" basis for the nullspace of the matrix $[A|I]$, where A is an $m \times n$ matrix and I is an $m \times m$ identity matrix concatenated onto A. $\square$

Exercise 2.3 Suppose V is a set of the form $\{Ax : x \in \mathbf{R}^k\}$, where A is an $n \times k$ matrix. Prove that V is also a set of the form $\{y \in \mathbf{R}^n : By = O\}$ where B is an $\ell \times n$ matrix, and explain how to find an appropriate matrix B. Conversely, suppose V is a set of the form $\{y \in \mathbf{R}^n : By = O\}$, where B is an $\ell \times n$ matrix. Prove that V is also a set of the form $\{Ax : x \in \mathbf{R}^k\}$, where A is an $n \times k$ matrix, and explain how to find an appropriate matrix A. $\square$

Exercise 2.4 Consider a linear system of equations, $Ax = b$. What are the various elementary row operations that can be used to obtain an equivalent system? What does it mean for two systems to be equivalent? $\square$

Exercise 2.5 Consider a linear system of equations, $Ax = b$. Describe the set of all solutions to this system. Explain how to use Gaussian elimination to determine this set. Prove that the system has no solution if and only if there is a vector y such that $y^T A = O^T$ and $y^T b \neq 0$. $\square$

Exercise 2.6 If $x \in \mathbf{R}^n$, what is the definition of $\|x\|_1$? Of $\|x\|_2$? Of $\|x\|_\infty$? For fixed matrix A (not necessarily square) and vector b, explain how to minimize $\|Ax - b\|_2$. Note: From now on in these notes, if no subscript appears in the notation $\|x\|$, then the norm $\|x\|_2$ is meant. $\square$

Exercise 2.7 Consider a square $n \times n$ matrix A. What is the determinant of A? How can it be expressed as a sum with $n!$ terms? How can it be expressed as an expansion by cofactors along an arbitrary row or column? How is it affected by the application of various elementary row operations? How can it be determined by Gaussian elimination? What does it mean for A to be singular? Nonsingular? What can you tell about the determinant of A from the dimensions of each of the four vector spaces associated with A? The determinant of A describes the volume of a certain geometrical object. What is this object? $\square$

Exercise 2.8 Consider a linear system of equations $Ax = b$ where A is square and nonsingular. Describe the set of all solutions to this system. What is Cramer's rule and how can it be used to find the complete set of solutions? $\square$

Exercise 2.9 Consider a square matrix A. When does it have an inverse? How can Gaussian elimination be used to find the inverse? How can Gauss-Jordan elimination be used to find the inverse? Suppose e_j is a vector of all zeroes, except for a 1 in the jth position. What does the solution to $Ax = e_j$ have to do with A^{-1}? What does the solution to $x^T A = e_j^T$ have to do with A^{-1}? Prove that if A is a nonsingular matrix with integer entries and determinant ± 1, then A^{-1} is also a matrix with integer entries. Prove that if A is a nonsingular matrix with integer entries and determinant ± 1, and b is a vector with integer entries, then the solution to $Ax = b$ is an integer vector. $\square$

Exercise 2.10 What is LU factorization? What is QR factorization, Gram-Schmidt orthogonalization, and their relationship? $\square$

Exercise 2.11 What does it mean for a matrix to be orthogonal? Prove that if A is orthogonal and x and y are vectors, then $\|x - y\|_2 = \|Ax - Ay\|_2$; i.e., multiplying two vectors by A does not change the Euclidean distance between them. $\square$

Exercise 2.12 What is the definition of an eigenvector and an eigenvalue of a square matrix? The remainder of the questions in this problem concern matrices over the real numbers, with real eigenvalues and eigenvectors. Find a square matrix with no eigenvalues. Prove that if A is a symmetric $n \times n$ matrix, there exists a basis for $\mathbf{R}^n$ consisting of eigenvectors of A. $\square$

Exercise 2.13 What does it mean for a symmetric matrix A to be positive semi-definite? Positive definite? If A is positive definite, describe the set $\{x : x^T Ax \leq 1\}$. What is the geometrical interpretation of the eigenvectors and eigenvalues of A with respect to this set?
□

Exercise 2.14 Suppose E is a finite set of vectors in $\mathbf{R}^n$. Let V be the vector space spanned by the vectors in E. Let $\mathcal{I} = \{S \subseteq E : S$ is linearly independent$\}$. Let $\mathcal{C} = \{S \subseteq E : S$ is linearly dependent, but no proper subset of S is linearly dependent$\}$. Let $\mathcal{B} = \{S \subseteq E : S$ is a basis for $V\}$. Prove the following:

1. $\emptyset \in \mathcal{I}$.
2. If $S_1 \in \mathcal{I}$, $S_2 \in \mathcal{I}$, and $\text{card}\, S_2 > \text{card}\, S_1$, then there exists an element $e \in S_2$ such that $S_1 \cup \{e\} \in \mathcal{I}$.
3. If $S \in \mathcal{I}$ and $S \cup \{e\}$ is dependent, then there is exactly one subset of $S \cup \{e\}$ that is in $\mathcal{C}$.
4. If $S_1 \in \mathcal{B}$ and $S_2 \in \mathcal{B}$, then $\text{card}\, S_1 = \text{card}\, S_2$.
5. If $S_1 \in \mathcal{B}$, $S_2 \in \mathcal{B}$, and $e_1 \in S_1$, then there exists an element $e_2 \in S_2$ such that $(S_1 \setminus \{e_1\}) \cup \{e_2\} \in \mathcal{B}$.
6. If $S_1 \in \mathcal{C}$, $S_2 \in \mathcal{C}$, $e \in S_1 \cap S_2$, and $e' \in S_1 \setminus S_2$, then there is a set $S_3 \in \mathcal{C}$ such that $S_3 \subseteq (S_1 \cup S_2) \setminus \{e\}$ and $e' \in S_3$.

□

3 Introduction

3.1 Example

Consider a hypothetical company that manufactures gadgets and gewgaws.

1. One kilogram of gadgets requires 1 hour of labor, 1 unit of wood, 2 units of metal, and yields a net profit of 5 dollars.

2. One kilogram of gewgaws requires 2 hours of labor, 1 unit of wood, 1 unit of metal, and yields a net profit of 4 dollars.

3. Available are 120 hours of labor, 70 units of wood, and 100 units of metal.

What is the company's optimal production mix? We can formulate this problem as the linear program

$$\begin{aligned} \max z = 5x_1 + 4x_2 \\ \text{s.t. } x_1 + 2x_2 \le 120 \\ x_1 + x_2 \le 70 \\ 2x_1 + x_2 \le 100 \\ x_1, x_2 \ge 0 \end{aligned}$$

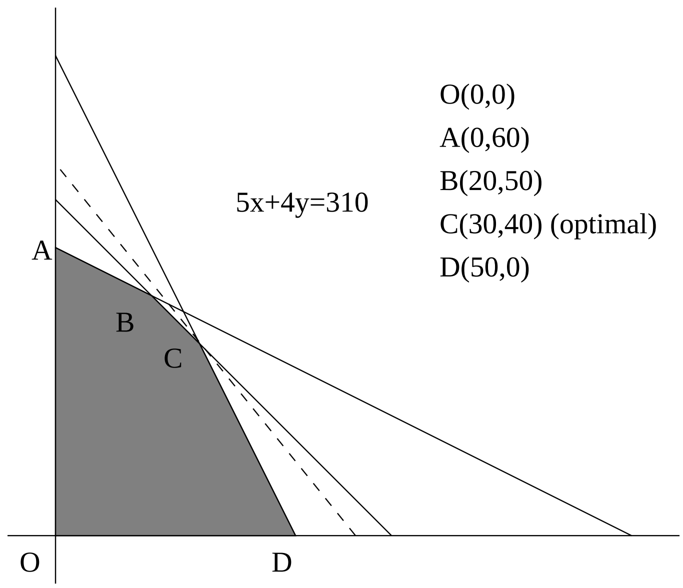
In matrix notation, this becomes

$$\begin{aligned} \max \begin{bmatrix} 5 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \\ \text{s.t. } \begin{bmatrix} 1 & 2 \\ 1 & 1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \le \begin{bmatrix} 120 \\ 70 \\ 100 \end{bmatrix} \\ \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \ge \begin{bmatrix} 0 \\ 0 \end{bmatrix} \end{aligned}$$

which is a problem of the form

$$\begin{aligned} \max c^T x \\ \text{s.t. } Ax \le b \\ x \ge O \end{aligned}$$

We can determine the solution of this problem geometrically. Graph the set of all points that satisfy the constraints. Draw some lines for which the objective function assumes a constant value (note that these are all parallel). Find the line with the highest value of z that has nonempty intersection with the set of feasible points. In this case the optimal solution is $(30, 40)$ with optimal value 310.



3.2 Definitions

A *linear function* is a function of the form $a_1x_1 + \cdots + a_nx_n$, where $a_1, \ldots, a_n \in \mathbf{R}$. A *linear equation* is an equation of the form $a_1x_1 + \cdots + a_nx_n = \beta$, where $a_1, \ldots, a_n, \beta \in \mathbf{R}$. If there exists at least one nonzero a_j, then the set of solutions to a linear equation is called a *hyperplane*. A *linear inequality* is an inequality of the form $a_1x_1 + \cdots + a_nx_n \leq \beta$ or $a_1x_1 + \cdots + a_nx_n \geq \beta$, where $a_1, \ldots, a_n, \beta \in \mathbf{R}$. If there exists at least one nonzero a_j, then the set of solutions to a linear inequality is called a *halfspace*. A *linear constraint* is a linear equation or linear inequality.

A *linear programming problem* is a problem in which a linear function is to be maximized (or minimized), subject to a finite number of linear constraints. A *feasible solution* or *feasible point* is a point that satisfies all of the constraints. If such a point exists, the problem is *feasible*; otherwise, it is *infeasible* . The set of all feasible points is called the *feasible region* or *feasible set*. The *objective function* is the linear function to be optimized. An *optimal solution* or *optimal point* is a feasible point for which the objective function is optimized. The value of the objective function at an optimal point is the *optimal value* of the linear program. In the case of a maximization (minimization) problem, if arbitrarily large (small) values of the objective function can be achieved, then the linear program is said to be *unbounded*. More precisely, the maximization (minimization) problem is unbounded if for all $M \in \mathbf{R}$ there exists a feasible point x with objective function value greater than (less than) M. Note: It is possible to have a linear program that has bounded objective function value but unbounded feasible region, so don't let this confusing terminology confuse you. Also note that an infeasible linear program has a bounded feasible region.

Exercise 3.1 Graphically construct some examples of each of the following types of two-variable linear programs:

1. Infeasible.
2. With a unique optimal solution.
3. With more than one optimal solution.
4. Feasible with bounded feasible region.
5. Feasible and bounded but with unbounded feasible region.
6. Unbounded.

□

A linear program of the form

$$\begin{aligned} \max \sum_{j=1}^{n} c_j x_j & \\ \text{s.t. } \sum_{j=1}^{n} a_{ij} x_j \le b_i, & \quad i = 1, \ldots, m \\ x_j \ge 0, & \quad j = 1, \ldots, n \end{aligned}$$

which, in matrix form, is

$$\begin{aligned} \max\, & c^T x \\ \text{s.t. } & Ax \le b \\ & x \ge O \end{aligned}$$

is said to be in *standard form*. For every linear program there is an equivalent one in standard form (begin thinking about this).

3.3 Back to the Example

Suppose someone approached the Gadget and Gewgaw Manufacturing Company (GGMC), offering to purchase the company's available labor hours, wood, and metal, at \$1.50 per hour of labor, \$1 per unit of wood, and \$1 per unit of metal. They are willing to buy whatever amount GGMC is willing to sell. Should GGMC sell everything? This is mighty tempting, because they would receive \$350, more than what they would gain by their current manufacturing plan. However, observe that if they manufactured some gadgets instead, for each kilogram of gadgets they would lose \$4.50 from the potential sale of their resources but gain \$5 from the sale of the gadgets. (Note, however, that it would be better to sell their resources than make gewgaws.) So they should not accept the offer to sell *all* of their resources at these prices.

Exercise 3.2 In the example above, GGMC wouldn't want to sell all of their resources at those prices. But they might want to sell some. What would be their best strategy? □

Exercise 3.3 Suppose now that GGMC is offered \$3 for each unit of wood and \$1 for each unit of metal that they are willing to sell, but no money for hours of labor. Explain why they would do just as well financially by selling all of their resources as by manufacturing their products. □

Exercise 3.4 In general, what conditions would proposed prices have to satisfy to induce GGMC to sell all of their resources? If you were trying to buy all of GGMC's resources as cheaply as possible, what problem would you have to solve? □

Exercise 3.5 If you want to purchase just one hour of labor, or just one unit of wood, or just one unit of metal, from GGMC, what price in each case must you offer to induce GGMC to sell? □

4 Exercises: Linear Programs

Exercise 4.1 Consider the following linear program (P):

$$\begin{array}{rl} \max z = x_1 + 2x_2 & \\ \text{s.t. } 3x_1 + x_2 \leq 3 & (1) \\ x_1 + x_2 \leq 3/2 & (2) \\ x_1 \geq 0 & (3) \\ x_2 \geq 0 & (4) \end{array}$$

1. Graph the feasible region.

2. Locate the optimal point(s).

3. Explain why the four constraints have the following respective outer normal vectors (an outer normal vector to a constraint is perpendicular to the defining line of the constraint and points in the opposite direction of the shaded side of the constraint):

   (1) $[3, 1]^T$.
   (2) $[1, 1]^T$.
   (3) $[-1, 0]^T$.
   (4) $[0, -1]^T$.

   Explain why the gradient of the objective function is the vector $[1, 2]^T$. For each corner point of the feasible region, compare the outer normals of the binding constraints at that point (the constraints satisfied with equality by that point) with the gradient of z. From this comparison, how can you tell geometrically if a given corner point is optimal or not?

4. Vary the objective function coefficients and consider the following linear program:

$$\begin{array}{c} \max z = c_1 x_1 + c_2 x_2 \\ \text{s.t. } 3x_1 + x_2 \leq 3 \\ x_1 + x_2 \leq 3/2 \\ x_1, x_2 \geq 0 \end{array}$$

   Carefully and completely describe the optimal value $z^*(c_1, c_2)$ as a function of the pair (c_1, c_2). What kind of function is this? Optional: Use some software such as Maple to plot this function of two variables.

5. Vary the right hand sides and consider the following linear program:

$$\begin{aligned} \max z &= x_1 + 2x_2 \\ \text{s.t. } 3x_1 + x_2 &\le b_1 \\ x_1 + x_2 &\le b_2 \\ x_1, x_2 &\ge 0 \end{aligned}$$

Carefully and completely describe the optimal value $z^*(b_1, b_2)$ as a function of the pair (b_1, b_2). What kind of function is this? Optional: Use some software such as Maple to plot this function of two variables.

6. Find the best nonnegative integer solution to (P). That is, of all feasible points for (P) having integer coordinates, find the one with the largest objective function value.

□

Exercise 4.2 Consider the following linear program (P):

$$\begin{aligned} \max z &= -x_1 - x_2 \\ \text{s.t. } x_1 &\le 1/2 && (1) \\ x_1 - x_2 &\le -1/2 && (2) \\ x_1 &\ge 0 && (3) \\ x_2 &\ge 0 && (4) \end{aligned}$$

Answer the analogous questions as in Exercise 4.1. □

Exercise 4.3 1. Consider the following linear program (P):

$$\begin{aligned} \max z &= 2x_1 + x_2 \\ \text{s.t. } x_1 &\le 2 && (1) \\ x_2 &\le 2 && (2) \\ x_1 + x_2 &\le 4 && (3) \\ x_1 - x_2 &\le 1 && (4) \\ x_1 &\ge 0 && (5) \\ x_2 &\ge 0 && (6) \end{aligned}$$

Associated with each of the 6 constraints is a line (change the inequality to equality in the constraint). Consider each pair of constraints for which the lines are not parallel, and examine the point of intersection of the two lines. Call this pair of constraints a *primal feasible pair* if the intersection point falls in the feasible region for (P). Call this pair of constraints a *dual feasible pair* if the gradient of the objective function can be expressed as a nonnegative linear combination of the two outer normal vectors of the two constraints. (The movitation for this terminology will become clearer later on.) List all primal-feasible pairs of constraints, and mark the intersection point for each pair. List all dual-feasible pairs of constraints (whether primal-feasible or not), and mark the intersection point for each pair. What do you observe about the optimal point(s)?

2. Repeat the above exercise for the GGMC problem.

□

Exercise 4.4 We have observed that any two-variable linear program appears to fall into exactly one of three categories: (1) those that are infeasible, (2) those that have unbounded objective function value, and (3) those that have a finite optimal objective function value. Suppose (P) is any two-variable linear program that falls into category (1). Into which of the other two categories can (P) be changed if we only alter the right hand side vector b? The objective function vector c? Both b and c? Are your answers true regardless of the initial choice of (P)? Answer the analogous questions if (P) is initially in category (2). In category (3). □

Exercise 4.5 Find a two-variable linear program

$$(P)\quad \begin{array}{c} \max c^T x \\ \text{s.t. } Ax \leq b \\ x \geq O \end{array}$$

with associated integer linear program

$$(IP)\quad \begin{array}{c} \max c^T x \\ \text{s.t. } Ax \leq b \\ x \geq O \text{ and integer} \end{array}$$

such that (P) has unbounded objective function value, but (IP) has a finite optimal objective function value. Note: "x integer" means that each coordinate x_j of x is an integer. □

Exercise 4.6 Prove the following: For each positive real number d there exists a two-variable linear program (P) with associated integer linear program (IP) such that the entries of A, b, and c are rational, (P) has a unique optimal solution x^*, (IP) has a unique optimal solution $\overline{x}^*$, and the Euclidean distance between x^* and $\overline{x}^*$ exceeds d. Can you do the same with a one-variable linear program? □

Exercise 4.7 Find a subset S of $\mathbf{R}^2$ and a linear objective function $c^T x$ such that the optimization problem

$$\begin{aligned} &\max c^T x \\ &\text{s.t. } x \in S \end{aligned}$$

is feasible, has no optimal objective function value, but yet does not have unbounded objective function value. □

Exercise 4.8 Find a quadratic objective function $f(x)$, a matrix A with two columns, and a vector b such that the optimization problem

$$\begin{aligned} &\max f(x) \\ &\text{s.t. } Ax \leq b \\ &\quad x \geq O \end{aligned}$$

has a unique optimal solution, but not at a corner point. □

Exercise 4.9 (Chvátal problem 1.5.) Prove or disprove: If the linear program

$$(P) \quad \begin{aligned} &\max c^T x \\ &\text{s.t. } Ax \leq b \\ &\quad x \geq O \end{aligned}$$

is unbounded, then there is a subscript k such that the linear program

$$\begin{aligned} &\max x_k \\ &\text{s.t. } Ax \leq b \\ &\quad x \geq O \end{aligned}$$

is unbounded. □

Exercise 4.10 (Bertsimas and Tsitsiklis problem 1.12.) Consider a set $S \subseteq \mathbf{R}^n$ described by the constraints $Ax \leq b$. The ball with center $y \in \mathbf{R}^n$ and radius $r \in \mathbf{R}_+$ is defined as $\{x \in \mathbf{R}^n : \|x - y\| \leq r\}$. Construct a linear program to solve the problem of finding a ball with the largest possible radius that is entirely contained within the set S. □

Exercise 4.11 Chvátal, problems 1.1–1.4. □

5 Theorems of the Alternatives

5.1 Systems of Equations

Let's start with a system of linear equations:

$$Ax = b.$$

Suppose you wish to determine whether this system is feasible or not. One reasonable approach is to use Gaussian elimination. If the system has a solution, you can find a particular one, $\overline{x}$. (You remember how to do this: Use elementary row operations to put the system in row echelon form, select arbitrary values for the independent variables and use back substitution to solve for the dependent variables.) Once you have a feasible $\overline{x}$ (no matter how you found it), it is straightforward to convince someone else that the system is feasible by verifying that $A\overline{x} = b$.

If the system is infeasible, Gaussian elimination will detect this also. For example, consider the system

$$\begin{aligned} x_1 + x_2 + x_3 + x_4 &= 1 \\ 2x_1 - x_2 + 3x_3 &= -1 \\ 8x_1 + 2x_2 + 10x_3 + 4x_4 &= 0 \end{aligned}$$

which in matrix form looks like

$$\left[\begin{array}{rrrr|r} 1 & 1 & 1 & 1 & 1 \\ 2 & -1 & 3 & 0 & -1 \\ 8 & 2 & 10 & 4 & 0 \end{array}\right].$$

Perform elementary row operations to arrive at a system in row echelon form:

$$\left[\begin{array}{rrr} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{array}\right] \left[\begin{array}{rrr} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -8 & 0 & 1 \end{array}\right] \left[\begin{array}{rrrr|r} 1 & 1 & 1 & 1 & 1 \\ 2 & -1 & 3 & 0 & -1 \\ 8 & 2 & 10 & 4 & 0 \end{array}\right] = \left[\begin{array}{rrrr|r} 1 & 1 & 1 & 1 & 1 \\ 0 & -3 & 1 & -2 & -3 \\ 0 & 0 & 0 & 0 & -2 \end{array}\right],$$

which implies

$$\left[\begin{array}{rrr} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -4 & -2 & 1 \end{array}\right] \left[\begin{array}{rrrr|r} 1 & 1 & 1 & 1 & 1 \\ 2 & -1 & 3 & 0 & -1 \\ 8 & 2 & 10 & 4 & 0 \end{array}\right] = \left[\begin{array}{rrrr|r} 1 & 1 & 1 & 1 & 1 \\ 0 & -3 & 1 & -2 & -3 \\ 0 & 0 & 0 & 0 & -2 \end{array}\right].$$

Immediately it is evident that the original system is infeasible, since the resulting equivalent system includes the equation $0x_1 + 0x_2 + 0x_3 + 0x_4 = -2$.

This equation comes from multiplying the matrix form of the original system by the third row of the matrix encoding the row operations: $[-4, -2, 1]$. This vector satisfies

$$\begin{bmatrix} -4 & -2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 2 & -1 & 3 & 0 \\ 8 & 2 & 10 & 4 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 \end{bmatrix}$$

and

$$\begin{bmatrix} -4 & -2 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} = -2.$$

In matrix form, we have found a vector $\overline{y}$ such that $\overline{y}^T A = O$ and $\overline{y}^T b \neq 0$. Gaussian elimination will always produce such a vector if the original system is infeasible. Once you have such a $\overline{y}$ (regardless of how you found it), it is easy to convince someone else that the system is infeasible.

Of course, if the system is feasible, then such a vector $\overline{y}$ cannot exist, because otherwise there would also be a feasible $\overline{x}$, and we would have

$$0 = O^T \overline{x} = (\overline{y}^T A)\overline{x} = \overline{y}^T (A\overline{x}) = \overline{y}^T b \neq 0,$$

which is impossible. (Be sure you can justify each equation and inequality in the above chain.) We have established our first Theorem of the Alternatives:

Theorem 5.1 *Either the system*

$$(I) \quad Ax = b$$

has a solution, or the system

$$(II) \quad \begin{array}{c} y^T A = O^T \\ y^T b \neq 0 \end{array}$$

has a solution, but not both.

As a consequence of this theorem, the following question has a "good characterization": Is the system (I) feasible? I will not give an exact definition of this concept, but roughly speaking it means that whether the answer is yes or no, there exists a "short" proof. In this case, if the answer is yes, we can prove it by exhibiting any particular solution to (I). And if the answer is no, we can prove it by exhibiting any particular solution to (II).

Geometrically, this theorem states that precisely one of the alternatives occurs:

1. The vector b is in the column space of A.

2. There is a vector y orthogonal to each column of A (and hence to the entire column space of A) but not orthogonal to b.

5.2 Fourier-Motzkin Elimination — A Starting Example

Now let us suppose we are given a system of linear inequalities

$$Ax \le b$$

and we wish to determine whether or not the system is feasible. If it is feasible, we want to find a particular feasible vector $\overline{x}$; if it is not feasible, we want hard evidence!

It turns out that there is a kind of analog to Gaussian elimination that works for systems of linear inequalities: Fourier-Motzkin elimination. We will first illustrate this with an example:

$$(I) \quad \begin{array}{c} x_1 - 2x_2 \le -2 \\ x_1 + x_2 \le 3 \\ x_1 \le 2 \\ -2x_1 + x_2 \le 0 \\ -x_1 \le -1 \\ 8x_2 \le 15 \end{array}$$

Our goal is to derive a second system (II) of linear inequalities with the following properties:

1. It has one fewer variable.

2. It is feasible if and only if the original system (I) is feasible.

3. A feasible solution to (I) can be derived from a feasible solution to (II).

(Do you see why Gaussian elimination does the same thing for systems of linear equations?) Here is how it works. Let's eliminate the variable x_1. Partition the inequalities in (I) into three groups, (I_-), (I_+), and (I_0), according as the coefficient of x_1 is negative, positive, or zero, respectively.

$$(I_-) \; \begin{array}{c} -2x_1 + x_2 \le 0 \\ -x_1 \le -1 \end{array} \qquad (I_+) \; \begin{array}{c} x_1 - 2x_2 \le -2 \\ x_1 + x_2 \le 3 \\ x_1 \le 2 \end{array} \qquad (I_0) \; 8x_2 \le 15$$

For each pair of inequalities, one from (I_-) and one from (I_+), multiply by positive numbers and add to eliminate x_1. For example, using the first inequality in each group,

$$\begin{array}{c} (\frac{1}{2})(-2x_1 + x_2 \le 0) \\ +(1)(x_1 - 2x_2 \le -2) \\ \hline -\frac{3}{2}x_2 \le -2 \end{array}$$

System (II) results from doing this for all such pairs, and then also including the inequalities in (I_0):

$$(II)\quad \begin{array}{c} -\frac{3}{2}x_2 \leq -2 \\ \frac{3}{2}x_2 \leq 3 \\ \frac{1}{2}x_2 \leq 2 \\ -2x_2 \leq -3 \\ x_2 \leq 2 \\ 0x_2 \leq 1 \\ 8x_2 \leq 15 \end{array}$$

The derivation of (II) from (I) can also be represented in matrix form. Here is the original system:

$$\left[\begin{array}{rr|r} 1 & -2 & -2 \\ 1 & 1 & 3 \\ 1 & 0 & 2 \\ -2 & 1 & 0 \\ -1 & 0 & -1 \\ 0 & 8 & 15 \end{array}\right]$$

Obtain the new system by multiplying on the left by the matrix that constructs the desired nonnegative combinations of the original inequalities:

$$\left[\begin{array}{cccccc} 1 & 0 & 0 & 1/2 & 0 & 0 \\ 0 & 1 & 0 & 1/2 & 0 & 0 \\ 0 & 0 & 1 & 1/2 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right] \left[\begin{array}{rr|r} 1 & -2 & -2 \\ 1 & 1 & 3 \\ 1 & 0 & 2 \\ -2 & 1 & 0 \\ -1 & 0 & -1 \\ 0 & 8 & 15 \end{array}\right]$$

$$= \left[\begin{array}{cc|c} 0 & -3/2 & -2 \\ 0 & 3/2 & 3 \\ 0 & 1/2 & 2 \\ 0 & -2 & -3 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \\ 0 & 8 & 15 \end{array}\right].$$

To see why the new system has the desired properties, let's break down this process a bit. First scale each inequality in the first two groups by positive numbers so that each coefficient of x_1 in (I_-) is -1 and each coefficient of x_1 in (I_+) is $+1$.

$$(I_-)\ \begin{array}{c} -x_1+\frac{1}{2}x_2 \le 0 \\ -x_1 \le -1 \end{array} \quad (I_+)\ \begin{array}{c} x_1-2x_2 \le -2 \\ x_1+x_2 \le 3 \\ x_1 \le 2 \end{array} \quad (I_0)\ 8x_2 \le 15$$

Isolate the variable x_1 in each of the inequalities in the first two groups.

$$(I_-)\ \begin{array}{c} \frac{1}{2}x_2 \le x_1 \\ 1 \le x_1 \end{array} \quad (I_+)\ \begin{array}{c} x_1 \le 2x_2-2 \\ x_1 \le -x_2+3 \\ x_1 \le 2 \end{array} \quad (I_0)\ 8x_2 \le 15$$

For each pair of inequalities, one from (I_-) and one from (I_+), create a new inequality by "sandwiching" and then eliminating x_1. Keep the inequalities in (I_0).

$$(IIa)\quad \begin{array}{ccc} \left\{ \begin{array}{c} \frac{1}{2}x_2 \\ 1 \end{array} \right\} & \le x_1 \le & \left\{ \begin{array}{c} 2x_2-2 \\ -x_2+3 \\ 2 \end{array} \right\} \\ 8x_2 & \le & 15 \end{array} \quad \longrightarrow \quad (IIb)\quad \begin{array}{c} \frac{1}{2}x_2 \le x_1 \le 2x_2-2 \\ \frac{1}{2}x_2 \le x_1 \le -x_2+3 \\ \frac{1}{2}x_2 \le x_1 \le 2 \\ 1 \le x_1 \le 2x_2-2 \\ 1 \le x_1 \le -x_2+3 \\ 1 \le x_1 \le 2 \\ 8x_2 \le 15 \end{array}$$

$$\longrightarrow \quad (IIc)\quad \begin{array}{c} \frac{1}{2}x_2 \le 2x_2-2 \\ \frac{1}{2}x_2 \le -x_2+3 \\ \frac{1}{2}x_2 \le 2 \\ 1 \le 2x_2-2 \\ 1 \le -x_2+3 \\ 1 \le 2 \\ 8x_2 \le 15 \end{array} \quad \longrightarrow \quad (II)\quad \begin{array}{c} -\frac{3}{2}x_2 \le -2 \\ \frac{3}{2}x_2 \le 3 \\ \frac{1}{2}x_2 \le 2 \\ -2x_2 \le -3 \\ x_2 \le 2 \\ 0x_2 \le 1 \\ 8x_2 \le 15 \end{array}$$

Observe that the system (II) does not involve the variable x_1. It is also immediate that if (I) is feasible, then (II) is also feasible. For the reverse direction, suppose that (II) is feasible. Set the variables (in this case, x_2) equal to any specific feasible values (in this case we choose a feasible value $\overline{x}_2$). From the way the inequalities in (II) were derived, it is evident that

$$\max \left\{ \begin{array}{c} \frac{1}{2}\overline{x}_2 \\ 1 \end{array} \right\} \le \min \left\{ \begin{array}{c} 2\overline{x}_2-2 \\ -\overline{x}_2+3 \\ 2 \end{array} \right\}.$$

So there exists a specific value $\overline{x}_1$ of x_1 such that

$$\begin{array}{ccc} \left\{ \begin{array}{c} \frac{1}{2}\overline{x}_2 \\ 1 \end{array} \right\} & \leq \overline{x}_1 \leq & \left\{ \begin{array}{c} 2\overline{x}_2 - 2 \\ -\overline{x}_2 + 3 \\ 2 \end{array} \right\} \\ 8\overline{x}_2 & \leq & 15 \end{array}$$

We will then have a feasible solution to (I).

5.3 Showing our Example is Feasible

From this example, we now see how to eliminate one variable (but at the possible considerable expense of increasing the number of inequalities). If we have a solution to the new system, we can determine a value of the eliminated variable to obtain a solution of the original system. If the new system is infeasible, then so is the original system.

From this we can tackle any system of inequalities: Eliminate all of the variables one by one until a system with no variables remains! Then work backwards to determine feasible values of all of the variables.

In our previous example, we can now eliminate x_2 from system (II):

$$\left[\begin{array}{ccccccc} 2/3 & 2/3 & 0 & 0 & 0 & 0 & 0 \\ 2/3 & 0 & 2 & 0 & 0 & 0 & 0 \\ 2/3 & 0 & 0 & 0 & 1 & 0 & 0 \\ 2/3 & 0 & 0 & 0 & 0 & 0 & 1/8 \\ 0 & 2/3 & 0 & 1/2 & 0 & 0 & 0 \\ 0 & 0 & 2 & 1/2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1/2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1/2 & 0 & 0 & 1/8 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{array} \right] \left[\begin{array}{cc|c} 0 & -3/2 & -2 \\ 0 & 3/2 & 3 \\ 0 & 1/2 & 2 \\ 0 & -2 & -3 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \\ 0 & 8 & 15 \end{array} \right]$$

$$= \left[\begin{array}{cc|c} 0 & 0 & 2/3 \\ 0 & 0 & 8/3 \\ 0 & 0 & 2/3 \\ 0 & 0 & 13/24 \\ 0 & 0 & 1/2 \\ 0 & 0 & 5/2 \\ 0 & 0 & 1/2 \\ 0 & 0 & 3/8 \\ 0 & 0 & 1 \end{array} \right].$$

Each final inequality, such as $0x_1 + 0x_2 \leq 2/3$, is feasible, since the left-hand side is zero and the right-hand side is nonnegative. Therefore the original system is feasible. To find one specific feasible solution, rewrite (II) as

$$\{4/3, 3/2\} \leq x_2 \leq \{2, 4, 15/8\}\,.$$

We can choose, for example, $\overline{x}_2 = 3/2$. Substituting into (I) (or (IIa)), we require

$$\{3/4, 1\} \leq x_1 \leq \{1, 3/2, 2\}\,.$$

So we could choose $\overline{x}_1 = 1$, and we have a feasible solution $(1, 3/2)$ to (I).

5.4 An Example of an Infeasible System

Now let's look at the system:

$$(I) \quad \begin{array}{c} x_1 - 2x_2 \leq -2 \\ x_1 + x_2 \leq 3 \\ x_1 \leq 2 \\ -2x_1 + x_2 \leq 0 \\ -x_1 \leq -1 \\ 8x_2 \leq 11 \end{array}$$

Multiplying by the appropriate nonnegative matrices to successively eliminate x_1 and x_2, we compute:

$$\left[\begin{array}{cccccc} 1 & 0 & 0 & 1/2 & 0 & 0 \\ 0 & 1 & 0 & 1/2 & 0 & 0 \\ 0 & 0 & 1 & 1/2 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right] \left[\begin{array}{rr|r} 1 & -2 & -2 \\ 1 & 1 & 3 \\ 1 & 0 & 2 \\ -2 & 1 & 0 \\ -1 & 0 & -1 \\ 0 & 8 & 11 \end{array}\right]$$

$$= \left[\begin{array}{cc|c} 0 & -3/2 & -2 \\ 0 & 3/2 & 3 \\ 0 & 1/2 & 2 \\ 0 & -2 & -3 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \\ 0 & 8 & 11 \end{array}\right] \quad (II)$$

and

$$\begin{bmatrix} 2/3 & 2/3 & 0 & 0 & 0 & 0 & 0 \\ 2/3 & 0 & 2 & 0 & 0 & 0 & 0 \\ 2/3 & 0 & 0 & 0 & 1 & 0 & 0 \\ 2/3 & 0 & 0 & 0 & 0 & 0 & 1/8 \\ 0 & 2/3 & 0 & 1/2 & 0 & 0 & 0 \\ 0 & 0 & 2 & 1/2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1/2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1/2 & 0 & 0 & 1/8 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix} \left[\begin{array}{cc|c} 0 & -3/2 & -2 \\ 0 & 3/2 & 3 \\ 0 & 1/2 & 2 \\ 0 & -2 & -3 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \\ 0 & 8 & 11 \end{array}\right]$$

$$= \left[\begin{array}{cc|c} 0 & 0 & 2/3 \\ 0 & 0 & 8/3 \\ 0 & 0 & 2/3 \\ 0 & 0 & 1/24 \\ 0 & 0 & 1/2 \\ 0 & 0 & 5/2 \\ 0 & 0 & 1/2 \\ 0 & 0 & -1/8 \\ 0 & 0 & 1 \end{array}\right] \quad (III)$$

Since one inequality is $0x_1 + 0x_2 \leq -1/8$, the final system (III) is clearly infeasible. Therefore the original system (I) is also infeasible. We can go directly from (I) to (III) by collecting together the two nonnegative multiplier matrices:

$$\begin{bmatrix} 2/3 & 2/3 & 0 & 0 & 0 & 0 & 0 \\ 2/3 & 0 & 2 & 0 & 0 & 0 & 0 \\ 2/3 & 0 & 0 & 0 & 1 & 0 & 0 \\ 2/3 & 0 & 0 & 0 & 0 & 0 & 1/8 \\ 0 & 2/3 & 0 & 1/2 & 0 & 0 & 0 \\ 0 & 0 & 2 & 1/2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1/2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1/2 & 0 & 0 & 1/8 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 1/2 & 0 & 0 \\ 0 & 1 & 0 & 1/2 & 0 & 0 \\ 0 & 0 & 1 & 1/2 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 2/3 & 2/3 & 0 & 2/3 & 0 & 0 \\ 2/3 & 0 & 2 & 4/3 & 0 & 0 \\ 2/3 & 1 & 0 & 1/3 & 1 & 0 \\ 2/3 & 0 & 0 & 1/3 & 0 & 1/8 \\ 1/2 & 2/3 & 0 & 1/3 & 1/2 & 0 \\ 1/2 & 0 & 2 & 1 & 1/2 & 0 \\ 1/2 & 1 & 0 & 0 & 3/2 & 0 \\ 1/2 & 0 & 0 & 0 & 1/2 & 1/8 \\ 0 & 0 & 1 & 0 & 1 & 0 \end{bmatrix} = M.$$

You can check that $M(I) = (III)$. Since M is a product of nonnegative matrices, it will itself be nonnegative. Since the infeasibility is discovered in the eighth inequality of (III), this comes from the eighth row of M, namely, $[1/2, 0, 0, 0, 1/2, 1/8]$. You can now demonstrate directly to anyone that (I) is infeasible using these nonnegative multipliers:

$$\begin{array}{c} (\frac{1}{2})(x_1 - 2x_2 \leq -2) \\ +(\frac{1}{2})(-x_1 \leq -1) \\ +(\frac{1}{8})(8x_2 \leq 11) \\ \hline 0x_1 + 0x_2 \leq -\frac{1}{8} \end{array}$$

In particular, we have found a nonnegative vector y such that $y^T A = O^T$ but $y^T b < 0$.

5.5 Fourier-Motzkin Elimination in General

Often I find that it is easier to understand a general procedure, proof, or theorem from a few good examples. Let's see if this is the case for you.

We begin with a system of linear inequalities

$$(I) \quad \sum_{j=1}^{n} a_{ij} x_j \leq b_i, \quad i = 1, \ldots, m.$$

Let's write this in matrix form as

$$Ax \leq b$$

or

$$A^i x \leq b_i, \quad i = 1, \ldots, m$$

where A^i represents the ith row of A.

Suppose we wish to eliminate the variable x_k. Define

$$\begin{array}{c} I_- = \{i : a_{ik} < 0\} \\ I_+ = \{i : a_{ik} > 0\} \\ I_0 = \{i : a_{ik} = 0\} \end{array}$$

For each $(p,q) \in I_- \times I_+$, construct the inequality

$$-\frac{1}{a_{pk}}(A^p x \le b_p) + \frac{1}{a_{qk}}(A^q x \le b_q).$$

By this I mean the inequality

$$\left(-\frac{1}{a_{pk}}A^p + \frac{1}{a_{qk}}A^q\right) x \le -\frac{1}{a_{pk}}b_p + \frac{1}{a_{qk}}b_q. \tag{1}$$

System (II) consists of all such inequalities, together with the original inequalities indexed by the set I_0.

It is clear that if we have a solution $(\overline{x}_1, \ldots, \overline{x}_n)$ to (I), then $(\overline{x}_1, \ldots, \overline{x}_{k-1}, \overline{x}_{k+1}, \ldots, \overline{x}_n)$ satisfies (II). Now suppose we have a solution $(\overline{x}_1, \ldots, \overline{x}_{k-1}, \overline{x}_{k+1}, \ldots, \overline{x}_n)$ to (II). Inequality (1) is equivalent to

$$\frac{1}{a_{pk}}(b_p - \sum_{j \ne k} a_{pj} x_j) \le \frac{1}{a_{qk}}(b_q - \sum_{j \ne k} a_{qj} x_j).$$

As this is satisfied by $(\overline{x}_1, \ldots, \overline{x}_{k-1}, \overline{x}_{k+1}, \ldots, \overline{x}_n)$ for all $(p,q) \in I_- \times I_+$, we conclude that

$$\max_{p \in I_-} \left\{ \frac{1}{a_{pk}}(b_p - \sum_{j \ne k} a_{pj} \overline{x}_j) \right\} \le \min_{q \in I_+} \left\{ \frac{1}{a_{qk}}(b_q - \sum_{j \ne k} a_{qj} \overline{x}_j) \right\}.$$

Choose $\overline{x}_k$ to be any value between these maximum and minimum values (inclusive). Then for all $(p,q) \in I_- \times I_+$,

$$\frac{1}{a_{pk}}(b_p - \sum_{j \ne k} a_{pj} \overline{x}_j) \le \overline{x}_k \le \frac{1}{a_{qk}}(b_q - \sum_{j \ne k} a_{qj} \overline{x}_j).$$

Now it is not hard to see that $(\overline{x}_1, \ldots, \overline{x}_{k-1}, \overline{x}_k, \overline{x}_{k+1}, \ldots, \overline{x}_n)$ satisfies all the inequalities in (I). Therefore (I) is feasible if and only if (II) is feasible.

Observe that each inequality in (II) is a nonnegative combination of inequalities in (I), so there is a nonnegative matrix M_k such that (II) is expressible as $M_k(Ax \le b)$. If we start with a system $Ax \le b$ and eliminate all variables sequentially via nonnegative matrices $M_1, \ldots, M_n$, then we will arrive at a system of inequalities of the form $0 \le b'_i$, $i = 1, \ldots, m'$. This system is expressible as $M(Ax \le b)$, where $M = M_n \cdots M_1$. If no b'_i is negative, then the final system is feasible and we can work backwards to obtain a feasible solution to the original system. If b'_i is negative for some i, then let $\overline{y}^T = M^i$ (the ith row of M), and we have a nonnegative vector $\overline{y}$ such that $\overline{y}^T A = O^T$ and $\overline{y}^T b < 0$.

This establishes a Theorem of the Alternatives for linear inequalities:

Theorem 5.2 *Either the system*

$$(I) \quad Ax \leq b$$

has a solution, or the system

$$(II) \quad \begin{array}{c} y^T A = O^T \\ y^T b < 0 \\ y \geq O \end{array}$$

has a solution, but not both.

Note that the "not both" part is the easiest to verify. Otherwise, we would have a feasible $\overline{x}$ and $\overline{y}$ satisfying

$$0 = O^T \overline{x} = (\overline{y}^T A)\overline{x} = \overline{y}^T (A\overline{x}) \leq \overline{y}^T b < 0,$$

which is impossible.

As a consequence of this theorem, we have a good characterization for the question: Is the system (I) feasible? If the answer is yes, we can prove it by exhibiting any particular solution to (I). If the answer is no, we can prove it by exhibiting any particular solution to (II).

5.6 More Alternatives

There are many Theorems of the Alternatives, and we shall encounter more later. Most of the others can be derived from the one of the previous section and each other. For example,

Theorem 5.3 *Either the system*

$$(I) \quad \begin{array}{c} Ax \leq b \\ x \geq O \end{array}$$

has a solution, or the system

$$(II) \quad \begin{array}{c} y^T A \geq O^T \\ y^T b < 0 \\ y \geq O \end{array}$$

has a solution, but not both.

Proof. System (I) is feasible if and only if the following system is feasible:

$$(I') \quad \begin{bmatrix} A \\ -I \end{bmatrix} x \leq \begin{bmatrix} b \\ O \end{bmatrix}$$

System (II) is feasible if and only if the following system is feasible:

$$(II') \qquad \begin{array}{c} \begin{bmatrix} y^T & w^T \end{bmatrix} \begin{bmatrix} A \\ -I \end{bmatrix} = O^T \\ \begin{bmatrix} y^T & w^T \end{bmatrix} \begin{bmatrix} b \\ O \end{bmatrix} < 0 \\ \begin{bmatrix} y^T & w^T \end{bmatrix} \geq \begin{bmatrix} O^T & O^T \end{bmatrix} \end{array}$$

Equivalently,

$$\begin{array}{c} y^T A - w^T = O^T \\ y^T b < O \\ y, w \geq O \end{array}$$

Now apply Theorem 5.2 to the pair (I'), (II'). □

6 Exercises: Systems of Linear Inequalities

Exercise 6.1 Discuss the consequences of having one or more of I_-, I_+, or I_0 being empty during the process of Fourier-Motzkin elimination. Does this create any problems? □

Exercise 6.2 Fourier-Motzkin elimination shows how we can start with a system of linear inequalities with n variables and obtain a system with $n-1$ variables. Explain why the set of all feasible solutions of the second system is a projection of the set of all feasible solutions of the first system. Consider a few examples where $n = 3$ and explain how you can classify the inequalities into types I_-, I_+, and I_0 geometrically (think about eliminating the third coordinate). Explain geometrically where the new inequalities in the second system are coming from. □

Exercise 6.3 Consider a given system of linear constraints. A subset of these constraints is called *irredundant* if it describes the same feasible region as the given system and no constraint can be dropped from this subset without increasing the set of feasible solutions.

Find an example of a system $Ax \le b$ with three variables such that when x_3, say, is eliminated, the resulting system has a larger irredundant subset than the original system. That is to say, the feasible set of the resulting system requires more inequalities to describe than the feasible set of the original system. Hint: Think geometrically. Can you find such an example where the original system has two variables? □

Exercise 6.4 Use Fourier-Motzkin elimination to graph the set of solutions to the following system:

$$\begin{aligned}
+x_1 + x_2 + x_3 &\le 1\\
+x_1 + x_2 - x_3 &\le 1\\
+x_1 - x_2 + x_3 &\le 1\\
+x_1 - x_2 - x_3 &\le 1\\
-x_1 + x_2 + x_3 &\le 1\\
-x_1 + x_2 - x_3 &\le 1\\
-x_1 - x_2 + x_3 &\le 1\\
-x_1 - x_2 - x_3 &\le 1
\end{aligned}$$

What is this geometrical object called? □

Exercise 6.5 Prove the following Theorem of the Alternatives: Either the system

$$Ax \ge b$$

has a solution, or the system

$$\begin{aligned} y^T A &= O^T \\ y^T b &> 0 \\ y &\geq O \end{aligned}$$

has a solution, but not both. □

Exercise 6.6 Prove the following Theorem of the Alternatives: Either the system

$$\begin{aligned} Ax &\geq b \\ x &\geq O \end{aligned}$$

has a solution, or the system

$$\begin{aligned} y^T A &\leq O^T \\ y^T b &> 0 \\ y &\geq O \end{aligned}$$

has a solution, but not both. □

Exercise 6.7 Prove or disprove: The system

$$(I) \quad Ax = b$$

has a solution if and only if each of the following systems has a solution:

$$(I') \quad Ax \leq b \qquad (I'') \quad Ax \geq b$$

□

Exercise 6.8 (The Farkas Lemma). Derive and prove a Theorem of the Alternatives for the following system:

$$\begin{aligned} Ax &= b \\ x &\geq O \end{aligned}$$

Give a geometric interpretation of this theorem when A has two rows. When A has three rows. □

Exercise 6.9 Give geometric interpretations to other Theorems of the Alternatives that we have discussed. □

Exercise 6.10 Derive and prove a Theorem of the Alternatives for the system

$$\begin{aligned}\sum_{j=1}^{n} a_{ij}x_j \le b_i, &\quad i \in I_1\\ \sum_{j=1}^{n} a_{ij}x_j = b_i, &\quad i \in I_2\\ x_j \ge 0, &\quad j \in J_1\\ x_j \text{ unrestricted}, &\quad j \in J_2\end{aligned}$$

where (I_1, I_2) is a partition of $\{1, \ldots, m\}$ and (J_1, J_2) is a partition of $\{1, \ldots, n\}$. □

Exercise 6.11 Derive and prove a Theorem of the Alternatives for the system

$$Ax < b.$$

□

Exercise 6.12 Chvátal, problem 16.6. □

7 Duality

In this section we will learn that associated with a given linear program is another one, its dual, which provides valuable information about the nature of the original linear program.

7.1 Economic Motivation

The dual linear program can be motivated economically, algebraically, and geometrically. You have already seen an economic motivation in Section 3.3. Recall that GGMC was interested in producing gadgets and gewgaws and wanted to solve the linear program

$$\begin{aligned} \max z = 5x_1 + 4x_2 \\ \text{s.t. } x_1 + 2x_2 \leq 120 \\ x_1 + x_2 \leq 70 \\ 2x_1 + x_2 \leq 100 \\ x_1, x_2 \geq 0 \end{aligned}$$

Another company (let's call it the Knickknack Company, KC) wants to offer money for GGMC's resources. If they are willing to buy whatever GGMC is willing to sell, what prices should be set so that GGMC will end up selling all of its resources? What is the minimum that KC must spend to accomplish this? Suppose y_1, y_2, y_3 represent the prices for one hour of labor, one unit of wood, and one unit of metal, respectively. The prices must be such that GGMC would not prefer manufacturing any gadgets or gewgaws to selling all of their resources. Hence the prices must satisfy $y_1 + y_2 + 2y_3 \geq 5$ (the income from selling the resources needed to make one kilogram of gadgets must not be less than the net profit from making one kilogram of gadgets) and $2y_1 + y_2 + y_3 \geq 4$ (the income from selling the resources needed to make one kilogram of gewgaws must not be less than the net profit from making one kilogram of gewgaws). KC wants to spend as little as possible, so it wishes to minimize the total amount spent: $120y_1 + 70y_2 + 100y_3$. This results in the linear program

$$\begin{aligned} \min 120y_1 + 70y_2 + 100y_3 \\ \text{s.t. } y_1 + y_2 + 2y_3 \geq 5 \\ 2y_1 + y_2 + y_3 \geq 4 \\ y_1, y_2, y_3 \geq 0 \end{aligned}$$

In matrix form, this is

$$\min \begin{bmatrix} 120 & 70 & 100 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$
$$\text{s.t.} \quad \begin{bmatrix} 1 & 1 & 2 \\ 2 & 1 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} \geq \begin{bmatrix} 5 \\ 4 \end{bmatrix}$$
$$\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} \geq \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

or

$$\min \begin{bmatrix} y_1 & y_2 & y_3 \end{bmatrix} \begin{bmatrix} 120 \\ 70 \\ 100 \end{bmatrix}$$
$$\text{s.t.} \quad \begin{bmatrix} y_1 & y_2 & y_3 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 1 & 1 \\ 2 & 1 \end{bmatrix} \geq \begin{bmatrix} 5 & 4 \end{bmatrix}$$
$$\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} \geq \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

If we represent the GGMC problem and the KC problem in the following compact forms, we see that they are "transposes" of each other.

1	2	120
1	1	70
2	1	100
5	4	max

GGMC

1	1	2	5
2	1	1	4
120	70	100	min

KC

7.2 The Dual Linear Program

Given any linear program (P) in standard form

$$(P) \quad \begin{array}{c} \max c^T x \\ \text{s.t. } Ax \leq b \\ x \geq O \end{array}$$

or

$$\max \sum_{j=1}^{n} c_j x_j$$
$$\text{s.t.} \sum_{j=1}^{n} a_{ij} x_j \le b_i, \quad i = 1, \ldots, m$$
$$x_j \ge 0, \quad j = 1, \ldots, n$$

its *dual* is the LP

$$(D) \quad \begin{aligned} &\min y^T b \\ &\text{s.t. } y^T A \ge c^T \\ &y \ge O \end{aligned}$$

or

$$\begin{aligned} &\min b^T y \\ &\text{s.t. } A^T y \ge c \\ &y \ge O \end{aligned}$$

or

$$\min \sum_{i=1}^{m} b_i y_i$$
$$\text{s.t.} \sum_{i=1}^{m} a_{ij} y_i \ge c_j, \quad j = 1, \ldots, n$$
$$y_i \ge 0, \quad i = 1, \ldots, m$$

Note the change from maximization to minimization, the change in the direction of the inequalities, the interchange in the roles of objective function coefficients and right-hand sides, the one-to-one correspondence between the inequalities in $Ax \le b$ and the variables in (D), and the one-to-one correspondence between the inequalities in $y^T A \ge c^T$ and the variables in (P). In compact form, the two problems are transposes of each other:

A	b
c^T	max

(P)

A^T	c
b^T	min

(D)

By the way, the problem (P) is called the *primal* problem. It has been explained to me that George Dantzig's father made two contributions to the theory of linear programming: the word "primal," and George Dantzig. Dantzig had already decided to use the word "dual" for the second LP, but needed a term for the original problem.

7.3 The Duality Theorems

One algebraic motivation for the dual is given by the following theorem, which states that any feasible solution for the dual LP provides an upper bound for the value of the primal LP:

Theorem 7.1 (Weak Duality) *If $\overline{x}$ is feasible for (P) and $\overline{y}$ is feasible for (D), then $c^T\overline{x} \leq \overline{y}^T b$.*

PROOF. $c^T\overline{x} \leq (\overline{y}^T A)\overline{x} = \overline{y}^T(A\overline{x}) \leq \overline{y}^T b$. □

Example 7.2 The prices $(1, 2, 3)$ are feasible for KC's problem, and yield an objective function value of 560, which is ≥ 310. □

As an easy corollary, if we are fortunate enough to be given $\overline{x}$ and $\overline{y}$ feasible for (P) and (D), respectively, with equal objective function values, then they are each optimal for their respective problems:

Corollary 7.3 *If $\overline{x}$ and $\overline{y}$ are feasible for (P) and (D), respectively, and if $c^T\overline{x} = \overline{y}^T b$, then $\overline{x}$ and $\overline{y}$ are optimal for (P) and (D), respectively.*

PROOF. Suppose $\hat{x}$ is any feasible solution for (P). Then $c^T\hat{x} \leq \overline{y}^T b = c^T\overline{x}$. Similarly, if $\hat{y}$ is any feasible solution for (D), then $\hat{y}^T b \geq \overline{y}^T b$. □

Example 7.4 The prices $(0, 3, 1)$ are feasible for KC's problem, and yield an objective function value of 310. Therefore, $(30, 40)$ is an optimal solution to GGMC's problem, and $(0, 3, 1)$ is an optimal solution to KC's problem. □

Weak Duality also immediately shows that if (P) is unbounded, then (D) is infeasible:

Corollary 7.5 *If (P) has unbounded objective function value, then (D) is infeasible. If (D) has unbounded objective function value, then (P) is infeasible.*

PROOF. Suppose (D) is feasible. Let $\overline{y}$ be a particular feasible solution. Then for all $\overline{x}$ feasible for (P) we have $c^T\overline{x} \leq \overline{y}^T b$. So (P) has bounded objective function value if it is feasible, and therefore cannot be unbounded. The second statement is proved similarly. □

Suppose (P) is feasible. How can we verify that (P) is unbounded? One way is if we discover a vector $\overline{w}$ such that $A\overline{w} \leq O$, $\overline{w} \geq O$, and $c^T\overline{w} > 0$. To see why this is the case, suppose that $\overline{x}$ is feasible for (P). Then we can add a positive multiple of $\overline{w}$ to $\overline{x}$ to get another feasible solution to (P) with objective function value as high as we wish.

Perhaps surprisingly, the converse is also true, and the proof shows some of the value of Theorems of the Alternatives.

Theorem 7.6 *Assume (P) is feasible. Then (P) is unbounded (has unbounded objective function value) if and only if the following system is feasible:*

$$(UP) \quad \begin{aligned} Aw &\le O \\ c^T w &> 0 \\ w &\ge O \end{aligned}$$

Proof. Suppose $\overline{x}$ is feasible for (P).

First assume that $\overline{w}$ is feasible for (UP) and $t \ge 0$ is a real number. Then

$$\begin{aligned} A(\overline{x} + t\overline{w}) &= A\overline{x} + tA\overline{w} \le b + O = b \\ \overline{x} + t\overline{w} &\ge O + tO = O \\ c^T(\overline{x} + t\overline{w}) &= c^T\overline{x} + tc^T\overline{w} \end{aligned}$$

Hence $\overline{x}+t\overline{w}$ is feasible for (P), and by choosing t appropriately large, we can make $c^T(\overline{x}+t\overline{w})$ as large as desired since $c^T\overline{w}$ is a positive number.

Conversely, suppose that (P) has unbounded objective function value. Then by Corollary 7.5, (D) is infeasible. That is, the following system has no solution:

$$\begin{aligned} y^T A &\ge c^T \\ y &\ge O \end{aligned}$$

or

$$\begin{aligned} A^T y &\ge c \\ y &\ge O \end{aligned}$$

By the Theorem of the Alternatives proved in Exercise 6.6, the following system is feasible:

$$\begin{aligned} w^T A^T &\le O^T \\ w^T c &> 0 \\ w &\ge O \end{aligned}$$

or

$$\begin{aligned} Aw &\le O \\ c^T w &> 0 \\ w &\ge O \end{aligned}$$

Hence (UP) is feasible. □

Example 7.7 Consider the LP:

$$(P) \quad \begin{aligned} \max\ & 100x_1 + x_2 \\ \text{s.t.}\ & -2x_1 + 3x_2 \le 1 \\ & x_1 - 2x_2 \le 2 \\ & x_1, x_2 \ge 0 \end{aligned}$$

The system (UP) in this case is:

$$\begin{aligned} -2w_1 + 3w_2 &\le 0 \\ w_1 - 2w_2 &\le 0 \\ 100w_1 + w_2 &> 0 \\ w_1, w_2 &\ge 0 \end{aligned}$$

One feasible point for (P) is $\overline{x} = (1, 0)$. One feasible solution to (UP) is $\overline{w} = (2, 1)$. So (P) is unbounded, and we can get points with arbitrarily high objective function values by $\overline{x} + t\overline{w} = (1 + 2t, t)$, $t \ge 0$, which has objective function value $100 + 201t$. □

There is an analogous theorem for the unboundedness of (D) that is proved in the obviously similar way:

Theorem 7.8 *Assume (D) is feasible. Then (D) is unbounded if and only if the following system is feasible:*

$$(UD) \qquad \begin{aligned} v^T A &\ge O^T \\ v^T b &< 0 \\ v &\ge O \end{aligned}$$

The following highlights an immediate corollary of the proof:

Corollary 7.9 *(P) is feasible if and only if (UD) is infeasible. (D) is feasible if and only if (UP) is infeasible.*

Let's summarize what we now know in a slightly different way:

Corollary 7.10 *If (P) is infeasible, then either (D) is infeasible or (D) is unbounded. If (D) is infeasible, then either (P) is infeasible or (P) is unbounded.*

We now turn to a very important theorem, which is part of the strong duality theorem, that lies at the heart of linear programming. This shows that the bounds on each other's objective function values that the pair of dual LP's provides are always tight.

Theorem 7.11 *Suppose (P) and (D) are both feasible. Then (P) and (D) each have finite optimal objective function values, and moreover these two values are equal.*

Proof. We know by Weak Duality that if $\overline{x}$ and $\overline{y}$ are feasible for (P) and (D), respectively, then $c^T\overline{x} \le \overline{y}^T b$. In particular, neither (P) nor (D) is unbounded. So it suffices to show that the following system is feasible:

$$(I) \quad \begin{array}{c} Ax \le b \\ x \ge O \\ y^T A \ge c^T \\ y \ge O \\ c^T x \ge y^T b \end{array}$$

For if $\overline{x}$ and $\overline{y}$ are feasible for this system, then by Weak Duality in fact it would have to be the case that $c^T\overline{x} = \overline{y}^T b$.

Let's rewrite this system in matrix form:

$$\begin{bmatrix} A & O \\ O & -A^T \\ -c^T & b^T \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \le \begin{bmatrix} b \\ -c \\ 0 \end{bmatrix}$$
$$x, y \ge O$$

We will assume that this system is infeasible and derive a contradiction. If it is not feasible, then by Theorem 5.3 the following system has a solution $\overline{v}, \overline{w}, \overline{t}$:

$$(II) \quad \begin{array}{c} \begin{bmatrix} v^T & w^T & t \end{bmatrix} \begin{bmatrix} A & O \\ O & -A^T \\ -c^T & b^T \end{bmatrix} \ge \begin{bmatrix} O^T & O^T \end{bmatrix} \\ \begin{bmatrix} v^T & w^T & t \end{bmatrix} \begin{bmatrix} b \\ -c \\ 0 \end{bmatrix} < 0 \\ v, w, t \ge O \end{array}$$

So we have

$$\begin{array}{c} \overline{v}^T A - \overline{t} c^T \ge O^T \\ -\overline{w}^T A^T + \overline{t} b^T \ge O^T \\ \overline{v}^T b - \overline{w}^T c < 0 \\ \overline{v}, \overline{w}, \overline{t} \ge O \end{array}$$

Case 1: Suppose $\overline{t} = 0$. Then

$$\begin{array}{c} \overline{v}^T A \ge O^T \\ A\overline{w} \le O \\ \overline{v}^T b < c^T \overline{w} \\ \overline{v}, \overline{w} \ge O \end{array}$$

Now we cannot have both $c^T\overline{w} \leq 0$ and $\overline{v}^T b \geq 0$; otherwise $0 \leq \overline{v}^T b < c^T\overline{w} \leq 0$, which is a contradiction.

Case 1a: Suppose $c^T\overline{w} > 0$. Then $\overline{w}$ is a solution to (UP), so (D) is infeasible by Corollary 7.9, a contradiction.

Case 1b: Suppose $\overline{v}^T b < 0$. Then $\overline{v}$ is a solution to (UD), so (P) is infeasible by Corollary 7.9, a contradiction.

Case 2: Suppose $\overline{t} > 0$. Set $\overline{x} = \overline{w}/\overline{t}$ and $\overline{y} = \overline{v}/\overline{t}$. Then

$$\begin{array}{c} A\overline{x} \leq b \\ \overline{x} \geq O \\ \overline{y}^T A \geq c^T \\ \overline{y} \geq O \\ c^T\overline{x} > \overline{y}^T b \end{array}$$

Hence we have a pair of feasible solutions to (P) and (D), respectively, that violates Weak Duality, a contradiction.

We have now shown that (II) has no solution. Therefore, (I) has a solution. □

Corollary 7.12 *Suppose (P) has a finite optimal objective function value. Then so does (D), and these two values are equal. Similarly, suppose (D) has a finite optimal objective function value. Then so does (P), and these two values are equal.*

PROOF. We will prove the first statement only. If (P) has a finite optimal objective function value, then it is feasible, but not unbounded. So (UP) has no solution by Theorem 7.6. Therefore (D) is feasible by Corollary 7.9. Now apply Theorem 7.11. □

We summarize our results in the following central theorem, for which we have already done all the hard work:

Theorem 7.13 (Strong Duality) *Exactly one of the following holds for the pair (P) and (D):*

1. *They are both infeasible.*
2. *One is infeasible and the other is unbounded.*
3. *They are both feasible and have equal finite optimal objective function values.*

Corollary 7.14 *If $\overline{x}$ and $\overline{y}$ are feasible for (P) and (D), respectively, then $\overline{x}$ and $\overline{y}$ are optimal for (P) and (D), respectively, if and only if $c^T\overline{x} = \overline{y}^T b$.*

Corollary 7.15 *Suppose $\overline{x}$ is feasible for (P). Then $\overline{x}$ is optimal for (P) if and only if there exists $\overline{y}$ feasible for (D) such that $c^T\overline{x} = \overline{y}^T b$. Similarly, suppose $\overline{y}$ is feasible for (D). Then $\overline{y}$ is optimal for (D) if and only if there exists $\overline{x}$ feasible for (P) such that $c^T\overline{x} = \overline{y}^T b$.*

7.4 Comments on Good Characterization

The duality theorems show that the following problems for (P) have "good characterizations." That is to say, whatever the answer, there exists a "short" proof.

1. Is (P) feasible? If the answer is yes, you can prove it by producing a particular feasible solution to (P). If the answer is no, you can prove it by producing a particular feasible solution to (UD).

2. Assume that you know that (P) is feasible. Is (P) unbounded? If the answer is yes, you can prove it by producing a particular feasible solution to (UP). If the answer is no, you can prove it by producing a particular feasible solution to (D).

3. Assume that $\overline{x}$ is feasible for (P). Is $\overline{x}$ optimal for (P)? If the answer is yes, you can prove it by producing a particular feasible solution to (D) with the same objective function value. If the answer is no, you can prove it by producing a particular feasible solution to (P) with higher objective function value.

7.5 Complementary Slackness

Suppose $\overline{x}$ and $\overline{y}$ are feasible for (P) and (D), respectively. Under what conditions will $c^T\overline{x}$ equal $\overline{y}^Tb$? Recall the chain of inequalities in the proof of Weak Duality:

$$c^T\overline{x} \le (\overline{y}^TA)\overline{x} = \overline{y}^T(A\overline{x}) \le \overline{y}^Tb.$$

Equality occurs if and only if both $c^T\overline{x} = (\overline{y}^TA)\overline{x}$ and $\overline{y}^T(A\overline{x}) = \overline{y}^Tb$. Equivalently,

$$\overline{y}^T(b - A\overline{x}) = 0$$

and

$$(\overline{y}^TA - c^T)\overline{x} = 0.$$

In each case, we are requiring that the inner product of two nonnegative vectors (for example, $\overline{y}$ and $b - A\overline{x}$) be zero. The only way this can happen is if these two vectors are never both positive in any common component. This motivates the following definition: Suppose $\overline{x} \in \mathbf{R}^n$ and $\overline{y} \in \mathbf{R}^m$. Then $\overline{x}$ and $\overline{y}$ satisfy *complementary slackness* if

1. For all j, either $\overline{x}_j = 0$ or $\sum_{i=1}^m a_{ij}\overline{y}_i = c_j$ or both; and

2. For all i, either $\overline{y}_i = 0$ or $\sum_{j=1}^n a_{ij}\overline{x}_j = b_i$ or both.

Theorem 7.16 *Suppose $\overline{x}$ and $\overline{y}$ are feasible for (P) and (D), respectively. Then $c^T\overline{x} = \overline{y}^Tb$ if and only if $\overline{x}, \overline{y}$ satisfy complementary slackness.*

Corollary 7.17 *If $\overline{x}$ and $\overline{y}$ are feasible for (P) and (D), respectively, then $\overline{x}$ and $\overline{y}$ are optimal for (P) and (D), respectively, if and only if they satisfy complementary slackness.*

Corollary 7.18 *Suppose $\overline{x}$ is feasible for (P). Then $\overline{x}$ is optimal for (P) if and only if there exists $\overline{y}$ feasible for (D) such that $\overline{x}, \overline{y}$ satisfy complementary slackness. Similarly, suppose $\overline{y}$ is feasible for (D). Then $\overline{y}$ is optimal for (D) if and only if there exists $\overline{x}$ feasible for (P) such that $\overline{x}, \overline{y}$ satisfy complementary slackness.*

Example 7.19 Consider the optimal solution $(30, 40)$ of GGMC's problem, and the prices $(0, 3, 1)$ for KC's problem. You can verify that both solutions are feasible for their respective problems, and that they satisfy complementary slackness. But let's exploit complementary slackness a bit more. Suppose you only had the feasible solution $(30, 40)$ and wanted to verify optimality. Try to find a feasible solution to the dual satisfying complementary slackness. Because the constraint on hours is not satisfied with equality, we must have $y_1 = 0$. Because both x_1 and x_2 are positive, we must have both dual constraints satisfied with equality. This results in the system:

$$
\begin{aligned}
y_1 &= 0 \\
y_2 + 2y_3 &= 5 \\
y_2 + y_3 &= 4
\end{aligned}
$$

which has the unique solution $(0, 3, 1)$. Fortunately, all values are also nonnegative. Therefore we have a feasible solution to the dual that satisfies complementary slackness. This proves that $(30, 40)$ is optimal and produces a solution to the dual in the bargain. □

7.6 Duals of General LP's

What if you want a dual to an LP not in standard form? One approach is first to transform it into standard form somehow. Another is to come up with a definition of a general dual that will satisfy all of the duality theorems (weak and strong duality, correspondence between constraints and variables, complementary slackness, etc.). Both approaches are related.

Here are some basic transformations to convert an LP into an equivalent one:

1. Multiply the objective function by -1 and change "max" to "min" or "min" to "max."

2. Multiply an inequality constraint by -1 to change the direction of the inequality.

3. Replace an equality constraint

$$\sum_{j=1}^{n} a_{ij}x_j = b_i$$

with two inequality constraints

$$\sum_{j=1}^{n} a_{ij}x_j \leq b_i$$
$$-\sum_{j=1}^{n} a_{ij}x_j \leq -b_i$$

4. Replace a variable that is nonpositive with a variable that is its negative. For example, if x_j is specified to be nonpositive by $x_j \leq 0$, replace every occurrence of x_j with $-\hat{x}_j$ and require $\hat{x}_j \geq 0$.

5. Replace a variable that is unrestricted in sign with the difference of two nonnegative variables. For example, if x_j is unrestricted (sometimes called *free*), replace every occurrence of x_j with $x_j^+ - x_j^-$ and require that x_j^+ and x_j^- be nonnegative variables.

Using these transformations, every LP can be converted into an equivalent one in standard form. By *equivalent* I mean that a feasible (respectively, optimal) solution to the original problem can be obtained from a feasible (respectively, optimal) solution to the new problem. The dual to the equivalent problem can then be determined. But you can also apply the inverses of the above transformations to the dual and get an appropriate dual to the original problem.

Try some concrete examples for yourself, and then dive into the proof of the following theorem:

Theorem 7.20 *The following is a pair of dual LP's:*

$$(P) \quad \begin{array}{rl} \max \sum_{j=1}^{n} c_j x_j & \\ \text{s.t.} \ \sum_{j=1}^{n} a_{ij}x_j \leq b_i, & i \in I_1 \\ \sum_{j=1}^{n} a_{ij}x_j \geq b_i, & i \in I_2 \\ \sum_{j=1}^{n} a_{ij}x_j = b_i, & i \in I_3 \\ x_j \geq 0, & j \in J_1 \\ x_j \leq 0, & j \in J_2 \\ x_j \ \text{unrestricted in sign}, & j \in J_3 \end{array}$$

$$(D) \quad \begin{array}{rl} \min \sum_{i=1}^{m} b_i y_i & \\ \text{s.t.} \ \sum_{i=1}^{m} a_{ij}y_i \geq c_j, & j \in J_1 \\ \sum_{i=1}^{m} a_{ij}y_i \leq c_j, & j \in J_2 \\ \sum_{i=1}^{m} a_{ij}y_i = c_j, & j \in J_3 \\ y_i \geq 0, & i \in I_1 \\ y_i \leq 0, & i \in I_2 \\ y_i \ \text{unrestricted in sign}, & i \in I_3 \end{array}$$

where (I_1, I_2, I_3) *is a partition of* $\{1, \ldots, m\}$ *and* (J_1, J_2, J_3) *is a partition of* $\{1, \ldots, n\}$.

PROOF. Rewrite (P) in matrix form:

$$\begin{array}{c} \max c^{1T}x^1 + c^{2T}x^2 + c^{3T}x^3 \\ \text{s.t.} \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix} \begin{bmatrix} x^1 \\ x^2 \\ x^3 \end{bmatrix} \begin{array}{c} \leq \\ \geq \\ = \end{array} \begin{bmatrix} b^1 \\ b^2 \\ b^3 \end{bmatrix} \\ x^1 \geq O \\ x^2 \leq O \\ x^3 \text{ unrestricted} \end{array}$$

Now make the substitutions $\hat{x}^1 = x^1$, $\hat{x}^2 = -x^2$ and $\hat{x}^3 - \hat{x}^4 = x^3$:

$$\begin{array}{c} \max c^{1T}\hat{x}^1 - c^{2T}\hat{x}^2 + c^{3T}\hat{x}^3 - c^{3T}\hat{x}^4 \\ \text{s.t.} \begin{bmatrix} A_{11} & -A_{12} & A_{13} & -A_{13} \\ A_{21} & -A_{22} & A_{23} & -A_{23} \\ A_{31} & -A_{32} & A_{33} & -A_{33} \end{bmatrix} \begin{bmatrix} \hat{x}^1 \\ \hat{x}^2 \\ \hat{x}^3 \\ \hat{x}^4 \end{bmatrix} \begin{array}{c} \leq \\ \geq \\ = \end{array} \begin{bmatrix} b^1 \\ b^2 \\ b^3 \end{bmatrix} \\ \hat{x}^1, \hat{x}^2, \hat{x}^3, \hat{x}^4 \geq O \end{array}$$

Transform the constraints:

$$\begin{array}{c} \max c^{1T}\hat{x}^1 - c^{2T}\hat{x}^2 + c^{3T}\hat{x}^3 - c^{3T}\hat{x}^4 \\ \text{s.t.} \begin{bmatrix} A_{11} & -A_{12} & A_{13} & -A_{13} \\ -A_{21} & A_{22} & -A_{23} & A_{23} \\ A_{31} & -A_{32} & A_{33} & -A_{33} \\ -A_{31} & A_{32} & -A_{33} & A_{33} \end{bmatrix} \begin{bmatrix} \hat{x}^1 \\ \hat{x}^2 \\ \hat{x}^3 \\ \hat{x}^4 \end{bmatrix} \leq \begin{bmatrix} b^1 \\ -b^2 \\ b^3 \\ -b^3 \end{bmatrix} \\ \hat{x}^1, \hat{x}^2, \hat{x}^3, \hat{x}^4 \geq O \end{array}$$

Take the dual:

$$\begin{array}{c} \min b^{1T}\hat{y}^1 - b^{2T}\hat{y}^2 + b^{3T}\hat{y}^3 - b^{3T}\hat{y}^4 \\ \text{s.t.} \begin{bmatrix} A_{11}^T & -A_{21}^T & A_{31}^T & -A_{31}^T \\ -A_{12}^T & A_{22}^T & -A_{32}^T & A_{32}^T \\ A_{13}^T & -A_{23}^T & A_{33}^T & -A_{33}^T \\ -A_{13}^T & A_{23}^T & -A_{33}^T & A_{33}^T \end{bmatrix} \begin{bmatrix} \hat{y}^1 \\ \hat{y}^2 \\ \hat{y}^3 \\ \hat{y}^4 \end{bmatrix} \geq \begin{bmatrix} c^1 \\ -c^2 \\ c^3 \\ -c^3 \end{bmatrix} \\ \hat{y}^1, \hat{y}^2, \hat{y}^3, \hat{y}^4 \geq O \end{array}$$

Transform the constraints:

$$\begin{array}{c} \min b^{1T}\hat{y}^1 - b^{2T}\hat{y}^2 + b^{3T}\hat{y}^3 - b^{3T}\hat{y}^4 \\ \text{s.t.} \begin{bmatrix} A_{11}^T & -A_{21}^T & A_{31}^T & -A_{31}^T \\ A_{12}^T & -A_{22}^T & A_{32}^T & -A_{32}^T \\ A_{13}^T & -A_{23}^T & A_{33}^T & -A_{33}^T \end{bmatrix} \begin{bmatrix} \hat{y}^1 \\ \hat{y}^2 \\ \hat{y}^3 \\ \hat{y}^4 \end{bmatrix} \begin{array}{c} \geq \\ \leq \\ = \end{array} \begin{bmatrix} c^1 \\ c^2 \\ c^3 \end{bmatrix} \\ \hat{y}^1, \hat{y}^2, \hat{y}^3, \hat{y}^4 \geq O \end{array}$$

Transform the variables by setting $y^1 = \hat{y}^1$, $y^2 = -\hat{y}^2$, and $y^3 = \hat{y}^3 - \hat{y}^4$:

$$\begin{array}{c} \min b^{1T}y^1 + b^{2T}y^2 + b^{3T}y^3 \\ \text{s.t.} \begin{bmatrix} A_{11}^T & A_{21}^T & A_{31}^T \\ A_{12}^T & A_{22}^T & A_{32}^T \\ A_{13}^T & A_{23}^T & A_{33}^T \end{bmatrix} \begin{bmatrix} y^1 \\ y^2 \\ y^3 \end{bmatrix} \begin{array}{c} \geq \\ \leq \\ = \end{array} \begin{bmatrix} c^1 \\ c^2 \\ c^3 \end{bmatrix} \\ y^1 \geq O \\ y^2 \leq O \\ y^3 \text{ unrestricted} \end{array}$$

Write this in summation form, and you have (D). □

Whew! Anyway, this pair of dual problems will satisfy all of the duality theorems, so it was probably worth working through this generalization at least once. We say that (D) is the dual of (P), and also that (P) is the dual of (D). Note that there is still a one-to-one correspondence between the variables in one LP and the "main" constraints (not including the variable sign restrictions) in the other LP. Hillier and Lieberman (*Introduction to Operations Research*) suggest the following mnemonic device. Classify variables and constraints of linear programs as *standard* (S), *opposite* (O), or *bizarre* (B) as follows:

Maximization Problems

	Variables	Constraints
S	≥ 0	$\leq$
O	≤ 0	$\geq$
B	unrestricted in sign	$=$

Minimization Problems

	Variables	Constraints
S	≥ 0	$\geq$
O	≤ 0	$\leq$
B	unrestricted in sign	$=$

Then in the duality relationship, standard variables are paired with standard constraints, opposite variables are paired with opposite constraints, and bizarre variables are paired with bizarre constraints. If we express a pair of dual linear programs in compact form, labeling columns according to the type of variable and rows according to the type of constraint, we see that they are still transposes of each other:

	S	O	B	
S	A_{11}	A_{12}	A_{13}	b^1
O	A_{21}	A_{22}	A_{23}	b^2
B	A_{31}	A_{32}	A_{33}	b^3
	c^{1T}	c^{2T}	c^{3T}	max

(P)

	S	O	B	
S	A_{11}	A_{21}	A_{31}	c^1
O	A_{12}	A_{22}	A_{32}	c^2
B	A_{13}	A_{23}	A_{33}	c^3
	b^{1T}	b^{2T}	b^{3T}	min

(D)

Example 7.21 The following is a pair of dual linear programs:

(P)

$$\begin{array}{lrrrrrcr}
\max & 3x_1 & -2x_2 & & +4x_4 & +5x_5 & & \\
\text{s.t.} & x_1 & +x_2 & & & & \geq & 3 \\
& x_1 & -x_2 & +x_3 & -x_4 & +x_5 & = & 10 \\
& -6x_1 & & +2x_3 & +4x_4 & +x_5 & \leq & 2 \\
& & 9x_2 & & -11x_4 & & \geq & 0 \\
\end{array}$$

$$x_1, x_5 \geq 0$$
$$x_2, x_3 \leq 0$$
$$x_4 \text{ unrestricted in sign}$$

(D)

$$\begin{array}{lrrrrcr}
\min & 3y_1 & +10y_2 & +2y_3 & & & \\
\text{s.t.} & y_1 & +y_2 & -6y_3 & & \geq & 3 \\
& y_1 & -y_2 & & +9y_4 & \leq & -2 \\
& & y_2 & +2y_3 & & \leq & 0 \\
& & -y_2 & +4y_3 & -11y_4 & = & 4 \\
& & y_2 & +y_3 & & \geq & 5 \\
\end{array}$$

$$y_1, y_4 \leq 0$$
$$y_2 \text{ unrestricted in sign}$$
$$y_3 \geq 0$$

	S	O	O	B	S	
O	1	1	0	0	0	3
B	1	−1	1	−1	1	10
S	−6	0	2	4	1	2
O	0	9	0	−11	0	0
	3	−2	0	4	5	max

	O	B	S	O	
S	1	1	−6	0	3
O	1	−1	0	9	−2
O	0	1	2	0	0
B	0	−1	4	−11	4
S	0	1	1	0	5
	3	10	2	0	min

□

Here are some special cases of pairs of dual LP's:

$$(P)\quad \begin{array}{c}\max c^T x \\ \text{s.t. } Ax \leq b\end{array} \qquad (D)\quad \begin{array}{c}\min y^T b \\ \text{s.t. } y^T A = c^T \\ y \geq O\end{array}$$

and

$$
(P)\quad \begin{array}{c}\max c^T x\\ \text{s.t. } Ax = b\\ x \geq O\end{array} \qquad (D)\quad \begin{array}{c}\min y^T b\\ \text{s.t. } y^T A \geq c^T\end{array}
$$

Exercise 7.22 Suppose (P) and (D) are as given in Theorem 7.20. Show that the appropriate general forms of (UP) and (UD) are:

$$
(UP)\quad \begin{array}{c}
\sum_{j=1}^n a_{ij}w_j \leq 0, \quad i \in I_1\\
\sum_{j=1}^n a_{ij}w_j \geq 0, \quad i \in I_2\\
\sum_{j=1}^n a_{ij}w_j = 0, \quad i \in I_3\\
\sum_{j=1}^n c_j w_j > 0\\
w_j \geq 0, \quad j \in J_1\\
w_j \leq 0, \quad j \in J_2\\
w_j \text{ unrestricted in sign}, \quad j \in J_3
\end{array}
\qquad
(UD)\quad \begin{array}{c}
\sum_{i=1}^m a_{ij}v_i \geq 0, \quad j \in J_1\\
\sum_{i=1}^m a_{ij}v_i \leq 0, \quad j \in J_2\\
\sum_{i=1}^m a_{ij}v_i = 0, \quad j \in J_3\\
\sum_{i=1}^m b_i v_i < 0\\
v_i \geq 0, \quad i \in I_1\\
v_i \leq 0, \quad i \in I_2\\
v_i \text{ unrestricted in sign}, \quad i \in I_3
\end{array}
$$

□

7.7 Geometric Motivation of Duality

We mentioned in the last section that the following is a pair of dual LP's:

$$
(P)\quad \begin{array}{c}\max c^T x\\ \text{s.t. } Ax \leq b\end{array} \qquad (D)\quad \begin{array}{c}\min y^T b\\ \text{s.t. } y^T A = c^T\\ y \geq O\end{array}
$$

What does it mean for $\overline{x}$ and $\overline{y}$ to be feasible and satisfy complementary slackness for this pair of LP's? The solution $\overline{y}$ to (D) gives a way to write the objective function vector of (P) as a nonnegative linear combination of the outer normals of the constraints of (P). In effect, (D) is asking for the "cheapest" such expression. If $\overline{x}$ does not satisfy a constraint of (P) with equality, then the corresponding dual variable must be zero by complementary slackness. So the only outer normals used in the nonnegative linear combination are those for the binding constraints (the constraints satisfied by $\overline{x}$ with equality).

We have seen this phenomenon when we looked at two-variable linear programs earlier. For example, look again at Exercise 4.3. Every dual-feasible pair of constraints corresponds to a particular solution to the dual problem (though there are other solutions to the dual as well), and a pair of constraints that is both primal-feasible and dual feasible corresponds to a pair of solutions to (P) and (D) that satisfy complementary slackness and hence are optimal.

8 Exercises: Duality

Note: By e is meant a vector consisting of all 1's.

Exercise 8.1 Consider the classic diet problem: Various foods are available, each unit of which contributes a certain amount toward the minimum daily requirements of various nutritional needs. Each food has a particular cost. The goal is to choose how many units of each food to purchase to meet the minimum daily nutritional requirements, while minimizing the total cost. Formulate this as a linear program, and give an "economic interpretation" of the dual problem. □

Exercise 8.2 Find a linear program (P) such that both (P) and its dual (D) are infeasible. □

Exercise 8.3 Prove that the *set* $S = \{x : Ax \le b,\ x \ge O\}$ is unbounded if and only if $S \ne \emptyset$ and the following system is feasible:

$$\begin{aligned} Aw &\le O \\ w &\ge O \\ w &\ne O \end{aligned}$$

Note: By $w \ge O$, $w \ne O$ is meant that every component of w is nonnegative, and at least one component is positive. A solution to the above system is called a *feasible direction* for S. Draw some examples of two variable regions to illustrate how you can find the set of feasible directions geometrically. □

Exercise 8.4 Prove that if the LP

$$\begin{aligned} &\max c^T x \\ &\text{s.t. } Ax \le b \\ &\quad x \ge O \end{aligned}$$

is unbounded, then the LP

$$\begin{aligned} &\max e^T x \\ &\text{s.t. } Ax \le b \\ &\quad x \ge O \end{aligned}$$

is unbounded. What can you say about the converse of this statement? □

Exercise 8.5 Suppose you use Lagrange multipliers to solve the following problem:

$$\begin{aligned} &\max c^T x \\ \text{s.t. } &Ax = b \end{aligned}$$

What is the relationship between the Lagrange multipliers and the dual problem? □

Exercise 8.6 Suppose that the linear program

$$\begin{aligned} &\max c^T x \\ \text{s.t. } &Ax \le b \\ &x \ge O \end{aligned}$$

is unbounded. Prove that, for any $\hat{b}$, the following linear program is either infeasible or unbounded:

$$\begin{aligned} &\max c^T x \\ \text{s.t. } &Ax \le \hat{b} \\ &x \ge O \end{aligned}$$

□

Exercise 8.7 Consider the following linear programs:

$$(P) \quad \begin{aligned} &\max c^T x \\ \text{s.t. } &Ax \le b \\ &x \ge 0 \end{aligned} \qquad (\overline{P}) \quad \begin{aligned} &\max c^T x \\ \text{s.t. } &Ax \le b + u \\ &x \ge O \end{aligned} \qquad (D) \quad \begin{aligned} &\min y^T b \\ \text{s.t. } &y^T A \ge c^T \\ &y \ge 0 \end{aligned}$$

Here, u is a vector the same size as b. (u is a vector of real numbers, not variables.) Assume that (P) has a finite optimal objective function value z^*. Let y^* be any optimal solution to (D). Prove that $c^T x \le z^* + u^T y^*$ for every feasible solution x of $(\overline{P})$. What does this mean economically when applied to the GGMC problem? □

Exercise 8.8 Consider the following pair of linear programs:

$$(P) \quad \begin{aligned} &\max c^T x \\ \text{s.t. } &Ax \le b \\ &x \ge 0 \end{aligned} \qquad (D) \quad \begin{aligned} &\min y^T b \\ \text{s.t. } &y^T A \ge c^T \\ &y \ge 0 \end{aligned}$$

For all nonnegative x and y, define the function $\phi(x, y) = c^T x + y^T b - y^T Ax$. Assume that $\overline{x}$ and $\overline{y}$ are nonnegative. Prove that $\overline{x}$ and $\overline{y}$ are feasible and optimal for the above two linear programs, respectively, if and only if

$$\phi(\overline{x}, y) \ge \phi(\overline{x}, \overline{y}) \ge \phi(x, \overline{y})$$

for all nonnegative x and y (whether x and y are feasible for the above linear programs or not). (This says that $(\overline{x}, \overline{y})$ is a *saddlepoint* of ϕ.) □

Exercise 8.9 Consider the *fractional linear program*

$$(FP) \quad \begin{array}{c} \max \dfrac{c^T x + \alpha}{d^T x + \beta} \\ \text{s.t. } Ax \le b \\ x \ge O \end{array}$$

and the associated linear program

$$(P) \quad \begin{array}{c} \max c^T w + \alpha t \\ \text{s.t. } Aw - bt \le O \\ d^T w + \beta t = 1 \\ w \ge O, t \ge 0 \end{array}$$

where A is an $m \times n$ matrix, b is an $m \times 1$ vector, c and d are $n \times 1$ vectors, and α and β are scalars. The variables x and w are $n \times 1$ vectors, and t is a scalar variable.

Suppose that the feasible region for (FP) is nonempty, and that $d^T x + \beta > 0$ for all x that are feasible to (FP). Let (w^*, t^*) be an optimal solution to (P).

1. Suppose that the feasible region of (FP) is a bounded set. Prove that $t^* > 0$.

2. Given that $t^* > 0$, demonstrate how an optimal solution of (FP) can be recovered from (w^*, t^*) and prove your assertion.

□

Exercise 8.10

1. Give a geometric interpretation of complementary slackness for the LP

$$\begin{array}{c} \max c^T x \\ \text{s.t. } Ax \le b \\ x \ge O \end{array}$$

   and its dual.

2. Now give an economic interpretation of complementary slackness.

□

Exercise 8.11 Consider the linear program

$$(P) \quad \begin{array}{l} \min c^T x \\ \text{s.t. } Ax = b \\ \ell \le x \le u \end{array}$$

where ℓ and u are vectors of constants and $\ell_i < u_i$ for all i. Suppose that x is feasible for (P). Prove that x is optimal for (P) if and only if there exists a vector y such that, for all i,

$$\begin{array}{ll} (A^T y)_i \ge c_i & \text{if } x_i > \ell_i \\ (A^T y)_i \le c_i & \text{if } x_i < u_i. \end{array}$$

□

Exercise 8.12 There are algorithmic proofs using the simplex method of Theorem 7.13 that do not explicitly rely upon Theorem 5.3—see the discussion leading up to Theorem 9.24. Assume that Theorem 7.13 has been proved some other way. Now reprove Theorem 5.3 using Theorem 7.13 and the fact that (I) is feasible if and only if the following LP is feasible (and thus has optimal value 0):

$$(P) \quad \begin{array}{l} \max O^T x \\ \text{s.t. } Ax \le b \\ x \ge O \end{array}$$

□

Exercise 8.13 Derive and prove a Theorem of the Alternatives for the system

$$(I) \quad Ax < b$$

in the following way: Introduce a scalar variable t and a vector e of 1's, and consider the LP

$$(P) \quad \begin{array}{l} \max t \\ \text{s.t. } Ax + et \le b \end{array}$$

Begin by noting that (P) is always feasible, and proving that (I) is infeasible if and only if (P) has a nonpositive optimal value. □

Exercise 8.14 Consider the pair of dual LP's

$$(P) \quad \begin{array}{l} \max c^T x \\ \text{s.t. } Ax \le b \\ x \ge O \end{array} \qquad (D) \quad \begin{array}{l} \min y^T b \\ \text{s.t. } y^T A \ge c^T \\ y \ge O \end{array}$$

Suppose $\overline{x}$ and $\overline{y}$ are feasible for (P) and (D), respectively. Then $\overline{x}$ and $\overline{y}$ satisfy *strong complementary slackness* if

1. For all j, either $\overline{x}_j = 0$ or $\sum_{i=1}^{m} a_{ij}\overline{y}_i = c_j$, but not both; and

2. For all i, either $\overline{y}_i = 0$ or $\sum_{j=1}^{n} a_{ij}\overline{x}_j = b_i$, but not both.

Prove that if (P) and (D) are both feasible, then there exists a pair $\overline{x}, \overline{y}$ of optimal solutions to (P) and (D), respectively, that satisfies strong complementary slackness. Illustrate with some examples of two variable LP's. Hint: One way to do this is to consider the following LP:

$$\begin{array}{c} \max t \\ \text{s.t. } Ax \le b \\ Ax - Iy + et \le b \\ -A^T y \le -c \\ -Ix - A^T y + ft \le -c \\ -c^T x + b^T y \le 0 \\ x, y, t \ge O \end{array}$$

Here, both e and f are vectors of all 1's, and t is a scalar variable. □

Exercise 8.15 Consider the quadratic programming problem

$$(P) \qquad \begin{array}{c} \min Q(x) = c^T x + \frac{1}{2} x^T D x \\ \text{s.t. } Ax \ge b \\ x \ge O \end{array}$$

where A is an $m \times n$ matrix and D is a symmetric $n \times n$ matrix.

1. Assume that $\overline{x}$ is an optimal solution of (P). Prove that $\overline{x}$ is an optimal solution of the following linear program:

$$(P') \qquad \begin{array}{c} \min (c^T + \overline{x}^T D)x \\ \text{s.t. } Ax \ge b \\ x \ge O \end{array}$$

   Suggestion: Let $\hat{x}$ be any other feasible solution to (P'). Then $\lambda\hat{x} + (1-\lambda)\overline{x}$ is also a feasible solution to (P') for any $0 < \lambda < 1$.

2. Assume that $\overline{x}$ is an optimal solution of (P). Prove that there exist nonnegative vectors $\overline{y} \in \mathbf{R}^m$, $\overline{u} \in \mathbf{R}^n$, and $\overline{v} \in \mathbf{R}^m$ such that

$$\begin{bmatrix} \overline{u} \\ \overline{v} \end{bmatrix} - \begin{bmatrix} D & -A^T \\ A & O \end{bmatrix} \begin{bmatrix} \overline{x} \\ \overline{y} \end{bmatrix} = \begin{bmatrix} c \\ -b \end{bmatrix}$$

   and such that $\overline{u}^T \overline{x} + \overline{v}^T \overline{y} = 0$.

□

Exercise 8.16 Consider a $p \times q$ chessboard. Call a subset of cells *independent* if no pair of cells are adjacent to each other via a single knight's move. Call any line segment joining the centers of two cells that are adjacent via a single knight's move a *knight line*. A knight line is said to *cover* its two endpoint cells. A *knight line cover* is a set of knight lines such that every cell on a chessboard is covered by at least one knight line. Consider the problem (P) of finding the maximum size k^* of an independent set. Consider the problem (D) of finding the minimum size ℓ^* of a knight lines cover. Prove that if k is the size of any independent set and ℓ is the size of any knight line cover, then $k \leq \ell$. Conclude that $k^* \leq \ell^*$. Use this result to solve both (P) and (D) for the 8×8 chessboard. For the 2×6 chessboard. □

Exercise 8.17 Look up the definitions and some theorems about Eulerian graphs. Explain why the question: "Is a given graph G Eulerian?" has a good characterization. □

Exercise 8.18 Chvátal, 5.1, 5.3, 5.8, 9.1–9.3, 9.5, 16.4, 16.5, 16.9–16.12, 16.14. □

9 The Simplex Method

9.1 Bases and Tableaux

In this section we finally begin to discuss how to solve linear programs. Let's start with a linear program in standard form

$$(\hat{P})\quad \begin{array}{c} \max z = \hat{c}^T\hat{x} \\ \text{s.t. } \hat{A}\hat{x} \leq b \\ \hat{x} \geq O \end{array}$$

where $\hat{A}$ is an $m \times n$ matrix.

The dual of $(\hat{P})$ is

$$(\hat{D})\quad \begin{array}{c} \min \hat{y}^T b \\ \text{s.t. } \hat{y}^T\hat{A} \geq \hat{c}^T \\ \hat{y} \geq O \end{array}$$

In summation notation, $(\hat{P})$ is of the form

$$\begin{array}{c} \max z = \sum_{j=1}^{n} c_j x_j \\ \text{s.t. } \sum_{j=1}^{n} a_{ij} x_j \leq b_i, \quad i = 1, \ldots, m \\ x_j \geq 0, \quad j = 1, \ldots, n \end{array}$$

The first step will be to turn this system into a system of equations by introducing m nonnegative *slack* variables, one for each inequality in $\hat{A}\hat{x} \leq b$:

$$\begin{array}{c} \max z = \sum_{j=1}^{n} c_j x_j \\ \text{s.t. } (\sum_{j=1}^{n} a_{ij} x_j) + x_{n+i} = b_i, \quad i = 1, \ldots, m \\ x_j \geq 0, \quad j = 1, \ldots, n+m \end{array}$$

Now we have a problem of the form

$$(P)\quad \begin{array}{c} \max c^T x \\ \text{s.t. } Ax = b \\ x \geq O \end{array}$$

where $x = (\hat{x}, x_{n+1}, \ldots, x_{n+m})$, $c = (\hat{c}, 0, \ldots, 0)$, and $A = [\hat{A}|I]$. In particular, the rows of A are linearly independent (A has *full row rank*).

The dual of (P) is

$$(D) \quad \begin{array}{c} \min y^T b \\ \text{s.t. } y^T A \geq c^T \end{array}$$

We can write (P) as a tableau:

A	O	b
c^T	1	0

which represents the system

$$\begin{bmatrix} A & O \\ c^T & 1 \end{bmatrix} \begin{bmatrix} x \\ -z \end{bmatrix} = \begin{bmatrix} b \\ 0 \end{bmatrix}.$$

Example 9.1 With the addition of slack variables, the GGMC problem becomes

$$\begin{array}{c} \max z = 5x_1 + 4x_2 \\ \text{s.t. } x_1 + 2x_2 + x_3 = 120 \\ x_1 + x_2 + x_4 = 70 \\ 2x_1 + x_2 + x_5 = 100 \\ x_1, \ldots, x_5 \geq 0 \end{array}$$

which in tableau form is

x_1	x_2	x_3	x_4	x_5	$-z$	
1	2	1	0	0	0	120
1	1	0	1	0	0	70
2	1	0	0	1	0	100
5	4	0	0	0	1	0

.

□

Definition 9.2 By A_j we mean the jth column of A. Let $S = (j_1, \ldots, j_k)$ be an ordered subset of $\{1, \ldots, n+m\}$. By x_S we mean $(x_{j_1}, \ldots, x_{j_k})$. Similarly, $c_S = (c_{j_1}, \ldots, c_{j_k})$, and A_S is the $m \times k$ matrix $[A_{j_1} \cdots A_{j_k}]$.

An ordered set $B \subseteq \{1, \ldots, n+m\}$ is a *basis* if $\operatorname{card} B = m$ and A_B is a nonsingular $m \times m$ submatrix of A. If B is a basis, then the variables in x_B are called the *basic variables*, and the variables in x_N are called the *nonbasic variables*, where $N = (1, \ldots, n+m) \setminus B$. We will follow a time-honored traditional abuse of notation and write B instead of A_B, and N instead of A_N. Whether B or N stands for a subset or a matrix should be clear from context.

Given a basis B, we can perform elementary row operations on the tableau so that the columns associated with B and $-z$ form an identity matrix within the tableau. (Arrange it so that the last row continues to contain the entry 1 in the $-z$ column.) If the ordered basis is $(j_i, \ldots, j_m)$, then the columns of the identity matrix appear in the same order, finishing up with the $-z$ column. The resulting tableau is called a *basic tableau.* Let us denote it by

$\overline{A}$	O	$\overline{b}$
$\overline{c}^T$	1	$-\overline{b}_0$

The tableau represents a set of equations

$$\begin{aligned} \overline{A}x &= \overline{b} \\ \overline{c}^T x - z &= -\overline{b}_0 \end{aligned} \tag{2}$$

that is equivalent to the original set of equations

$$\begin{aligned} Ax &= b \\ c^T x - z &= 0 \end{aligned} \tag{3}$$

since it was obtained by elementary (invertible) row operations from the original set. That is to say, (x, z) satisfies (2) if and only if it satisfies (3).

Example 9.3 Some bases and basic tableaux for the GGMC problem.

1. $B = (3, 4, 5)$

x_1	x_2	x_3	x_4	x_5	$-z$	
1	2	1	0	0	0	120
1	1	0	1	0	0	70
2	1	0	0	1	0	100
5	4	0	0	0	1	0

2. $B = (3, 2, 5)$

x_1	x_2	x_3	x_4	x_5	$-z$	
-1	0	1	-2	0	0	-20
1	1	0	1	0	0	70
1	0	0	-1	1	0	30
1	0	0	-4	0	1	-280

3. $B = (3, 2, 1)$

x_1	x_2	x_3	x_4	x_5	$-z$	
0	0	1	−3	1	0	10
0	1	0	2	−1	0	40
1	0	0	−1	1	0	30
0	0	0	−3	−1	1	−310

4. $B = (4, 2, 1)$

x_1	x_2	x_3	x_4	x_5	$-z$	
0	0	−1/3	1	−1/3	0	−10/3
0	1	2/3	0	−1/3	0	140/3
1	0	−1/3	0	2/3	0	80/3
0	0	−1	0	−2	1	−320

5. $B = (5, 2, 1)$

x_1	x_2	x_3	x_4	x_5	$-z$	
0	0	1	−3	1	0	10
0	1	1	−1	0	0	50
1	0	−1	2	0	0	20
0	0	1	−6	0	1	−300

□

By the way, most people do not include the $-z$ column, since it is always the same. I kept it in so that the full identity matrix would be evident.

Definition 9.4 For every basic tableau there is a natural solution to $Ax = b$; namely, set $\overline{x}_N = O$ and read off the values of $\overline{x}_B$ from the last column. The resulting point is called a *basic point*. What we are doing is equivalent to solving $Bx_B + Nx_N = b$ by setting $\overline{x}_N = O$ and solving $Bx_B = b$ to get $\overline{x}_B = B^{-1}b$. We can also find the value $-\overline{z}$ of $-z = -c^T\overline{x}$ associated with the basic point $\overline{x}$ in the lower right-hand entry.

Example 9.5 The basic points for the basic tableaux in the previous example are:

1. $B = (3, 4, 5)$: $\overline{x} = (0, 0, 120, 70, 100)$, $c^T\overline{x} = 0$.

2. $B = (3, 2, 5)$: $\overline{x} = (0, 70, -20, 0, 30)$, $c^T\overline{x} = 280$.

3. $B = (3, 2, 1)$: $\overline{x} = (30, 40, 10, 0, 0)$, $c^T\overline{x} = 310$.

4. $B = (4, 2, 1)$: $\overline{x} = (80/3, 140/3, 0, -10/3, 0)$, $c^T\overline{x} = 320$.

5. $B = (5,2,1)$: $\overline{x} = (20,50,0,0,10)$, $c^T\overline{x} = 300$.

Examine the graph for the GGMC problem. In each case the first two coordinates give a point that is the intersection of two of the lines corresponding to the five original constraints. The reason for this is that setting a variable equal to zero is equivalent to enforcing a constraint with equality. In particular, setting one of the two original variables x_1, x_2 to zero enforces the respective constraint $x_1 \geq 0$ or $x_2 \geq 0$ with equality; whereas setting one of the three slack variables x_3, x_4, x_5 to zero enforces one of the respective constraints $x_1 + 2x_2 \leq 120$, $x_1 + x_2 \leq 70$, $2x_1 + x_2 \leq 100$ with equality. Since in this example every constraint corresponds to a halfplane and there are always two nonbasic variables, the point is the intersection of two lines. Think about Exercise 4.3 during the following discussion. □

Definition 9.6 If B is a basis such that the corresponding basic point $\overline{x}$ is nonnegative, then $\overline{x}$ is feasible for the linear program (P), and dropping the slack variables yields a feasible solution for the linear program $(\hat{P})$. In such a case, B is called a *(primal) feasible basis*, the tableau is called a *(primal) feasible tableau*, and $\overline{x}$ is called a *(primal) basic feasible solution* (BFS).

Suppose T is our initial tableau and $\overline{T}$ is a tableau associated with a basis B. Let's try to understand the entries of $\overline{T}$. There exists a square matrix M such that $MT = \overline{T}$. You can find this matrix within $\overline{T}$ by looking where the identity matrix was originally in T; namely, it is the submatrix of $\overline{T}$ determined by the columns for the slack variables and $-z$. It has the form

$$M = \begin{bmatrix} M' & O \\ -\overline{y}^T & 1 \end{bmatrix}.$$

(I am writing $-\overline{y}^T$ because I know what is coming!) Now multiplying T by M creates an identity matrix in the basic columns of $\overline{T}$, so

$$\begin{bmatrix} M' & O \\ -\overline{y}^T & 1 \end{bmatrix} \begin{bmatrix} B & O \\ c_B^T & 1 \end{bmatrix} = \begin{bmatrix} I & O \\ O^T & 1 \end{bmatrix}.$$

From this we conclude that $M' = B^{-1}$ and $-\overline{y}^T B + c_B^T = O^T$, so

$$M = \begin{bmatrix} B^{-1} & O \\ -\overline{y}^T & 1 \end{bmatrix}$$

where $\overline{y}^T = c_B^T B^{-1}$, and

$$\overline{T} = \begin{array}{|c|c|c|} \hline B^{-1}A & O & B^{-1}b \\ \hline c^T - c_B^T B^{-1}A & 1 & -c_B^T B^{-1}b \\ \hline \end{array} = \begin{array}{|c|c|c|} \hline B^{-1}A & O & B^{-1}b \\ \hline c^T - \overline{y}^T A & 1 & -\overline{y}^T b \\ \hline \end{array}.$$

Summarizing the formulas for the entries of $\overline{T}$:

$$\begin{aligned}
\overline{y}^T &= c_B^T B^{-1} \\
\overline{A} &= B^{-1}A \\
\overline{b} &= B^{-1}b \\
\overline{c}^T &= c^T - \overline{y}^T A \\
\overline{b}_0 &= c_B^T B^{-1} b \quad = \overline{y}^T b
\end{aligned}$$

Observe that while the ordered basis $(j_1, \ldots, j_m)$ indexes the *columns* of B, it indexes the *rows* of B^{-1}.

Exercise 9.7 For each of the bases in Example 9.3, determine the matrices B and B^{-1} (pay attention to the ordering of the rows and columns), find the vector $\overline{y}^T$, and check some of the formulas for $\overline{c}^T$, $\overline{b}$, and $\overline{b}_0$. For example, if the ordered basis is $(4, 2, 1)$, then

$$B = \begin{bmatrix} 0 & 2 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 2 \end{bmatrix},$$

$$B^{-1} = \begin{bmatrix} -1/3 & 1 & -1/3 \\ 2/3 & 0 & -1/3 \\ -1/3 & 0 & 2/3 \end{bmatrix},$$

$c_B^T = (0, 4, 5)$ and $\overline{y}^T = c_B^T B^{-1} = (1, 0, 2)$. □

Note that $-\overline{y}^T$ itself can be found in the last row beneath the columns for the slack variables, and that the lower right-hand entry equals $-\overline{y}^T b$. The above calculations also confirm that the lower right-hand entry equals $-c^T \overline{x}$ for the associated basic point $\overline{x}$, since $c^T \overline{x} = c_B^T \overline{x}_B + c_N^T \overline{x}_N = c_B^T B^{-1} b$.

Example 9.8 The vectors $\overline{y}$ for the basic tableaux in the previous example are:

1. $B = (3, 4, 5)$: $\overline{y} = (0, 0, 0)$, $\overline{y}^T b = 0$.

2. $B = (3, 2, 5)$: $\overline{y} = (0, 4, 0)$, $\overline{y}^T b = 280$.

3. $B = (3, 2, 1)$: $\overline{y} = (0, 3, 1)$, $\overline{y}^T b = 310$.

4. $B = (4, 2, 1)$: $\overline{y} = (1, 0, 2)$, $\overline{y}^T b = 320$.

5. $B = (5, 2, 1)$: $\overline{y} = (-1, 6, 0)$, $\overline{y}^T b = 300$.

□

Now we can also see a connection with the dual problem $(\hat{D})$. For suppose the last row of $\overline{T}$ contains nonpositive entries in the first $n + m$ columns. Then $\overline{c}^T = c^T - \overline{y}^T A \leq O^T$, so $\overline{y}^T A \geq c^T$. Hence $\overline{y}$ is feasible for (D). Recalling that $A = [\hat{A}|I]$ and $c^T = (\hat{c}^T, O^T)$, we have

$$\begin{array}{c} \overline{y}^T \hat{A} \geq \hat{c}^T \\ \overline{y}^T I \geq O^T \end{array}$$

Therefore $\overline{y}$ is also feasible for $(\hat{D})$.

Definition 9.9 Suppose a basic tableau $\overline{T}$ is given in which $\overline{y} = c_B^T B^{-1}$ is feasible for (D). Then the basis B is called a *dual feasible basis*, the tableau is called a *dual feasible tableau*, and $\overline{y}$ is called a *dual basic feasible solution*.

Another way to derive the entries in $\overline{T}$ is to solve for x_B in terms of x_N and substitute into $z = c^T x$:

$$\begin{array}{c} Bx_B + Nx_n = b \\ x_B + B^{-1}Nx_N = B^{-1}b \end{array}$$

This accounts for the upper portion of $\overline{T}$.

$$\begin{array}{rcl} x_B & = & B^{-1}b - B^{-1}Nx_N \\ z & = & c_B^T x_B + c_N^T x_N \\ & = & c_B^T(B^{-1}b - B^{-1}Nx_N) + c_N^T x_N \\ & = & c_B^T B^{-1}b + (c_N^T - c_B^T B^{-1}N)x_N \end{array}$$

So setting $\overline{y}^T = c_B^T B^{-1}$, we have

$$\begin{array}{rcl} z & = & \overline{y}^T b + (c_N^T - \overline{y}^T N)x_N \\ & = & \overline{y}^T b + (c^T - \overline{y}^T A)x \end{array}$$

since $c_B^T - \overline{y}^T B = O^T$. This accounts for the last row of $\overline{T}$.

Definition 9.10 Now suppose a basic tableau $\overline{T}$ is both primal and dual feasible. Then we know that the associated basic feasible solutions $\overline{x}$ and $\overline{y}$ are feasible for (P) and (D), respectively, and have equal objective function values, since $c^T\overline{x} = c_B^T(B^{-1}b) = (c_B^TB^{-1})b = \overline{y}^Tb$. So Weak Duality implies that $\overline{x}$ and $\overline{y}$ are optimal for (P) and (D), respectively. In this case, B is called an *optimal basis*, the tableau is called an *optimal tableau*, $\overline{x}$ is called an *optimal (primal) basic feasible solution* or *basic optimal solution*, and $\overline{y}$ is called an *optimal dual basic feasible solution* or *dual basic optimal solution*. Note in this case that dropping the slack variables from $\overline{x}$ gives a feasible solution to $(\hat{P})$ which has the same objective function value as $\overline{y}$, which is feasible for $(\hat{D})$. So we also have a pair of optimal solutions to $(\hat{P})$ and $(\hat{D})$.

Example 9.11 Classifying the tableaux in the previous example, we have:

1. $B = (3, 4, 5)$: Primal feasible, but not dual feasible.

2. $B = (3, 2, 5)$: Neither primal nor dual feasible.

3. $B = (3, 2, 1)$: Both primal and dual feasible, hence optimal.

4. $B = (4, 2, 1)$: Dual feasible, but not primal feasible.

5. $B = (5, 2, 1)$: Primal feasible, but not dual feasible.

□

9.2 Pivoting

The simplex method solves the linear program (P) by attempting to find an optimal tableau. One can move from a basic tableau to an "adjacent" one by *pivoting*.

Given a matrix M and a nonzero entry m_{rs}, a pivot is carried out in the following way:

1. Multiply row r by m_{rs}^{-1}.

2. For each $i \neq r$, add the necessary multiple of row r to row i so that the (i, s)th entry becomes zero. This is done by adding $-m_{is}/m_{rs}$ times row r to row i.

Row r is called the *pivot row*, column s is called the *pivot column*, and m_{rs} is called the *pivot entry*. Note that if M is $m \times n$ and contains an $m \times m$ identity matrix before the pivot, then it will contain an identity matrix after the pivot. Column s will become e_r (the vector with all components equal 0 except for a 1 in the rth position). Any column of M that equals e_i with $i \neq r$ will remain unchanged by the pivot.

Example 9.12 Pivoting on the entry in row 1, column 4 of Tableau 5 of Example 9.3 results in Tableau 4. □

Suppose we have a feasible basis B with associated primal feasible tableau $\overline{T}$. It is convenient to label the rows of $\overline{T}$ (and the entries of $\overline{b}$) by the elements of the basis B, since each basic variable appears with nonzero coefficient in exactly one row. For example, the rows of Tableau 5 in Example 9.3 would be labeled 5, 2 and 1, in that order.

Suppose the tableau is not optimal and we want to find a potentially better primal feasible tableau. Some $\overline{c}_s$ is positive.

	0	$\oplus$
$+^\circ$	$\vdots$	$\vdots$
	0	$\oplus$
$* \ \cdots \ * \ + \ * \ \cdots \ *$	1	$-\overline{b}_0$

My tableau notation is:

$+$	positive entry
$-$	negative entry
0	zero entry
$\oplus$	nonnegative entry
$\ominus$	nonpositive entry
$*$	sign of entry unknown or irrelevant
$\circ$	pivot entry

Pivoting on any positive entry of $\overline{A}_s$ will not cause the lower right-hand entry to increase. So the objective function value of the new basic point will not be smaller than that of the old.

To ensure that the new tableau is also primal feasible, we require that all right-hand sides remain nonnegative. So if $\overline{a}_{rs}$ is the pivot entry, we need:

$$\frac{1}{\overline{a}_{rs}}\overline{b}_r \geq 0$$

$$\overline{b}_i - \frac{\overline{a}_{is}}{\overline{a}_{rs}}\overline{b}_r \geq 0, \quad i \neq r$$

There is no problem with the first condition. The second condition is satisfied if $\overline{a}_{is} \leq 0$. For all i such that $\overline{a}_{is} > 0$ we require

$$\frac{\overline{b}_i}{\overline{a}_{is}} \geq \frac{\overline{b}_r}{\overline{a}_{rs}}.$$

This can be ensured by choosing r such that

$$\frac{\overline{b}_r}{\overline{a}_{rs}} = \min_{i:\overline{a}_{is}>0} \left\{ \frac{\overline{b}_i}{\overline{a}_{is}} \right\}.$$

This is called the *ratio test* to determine the pivot row.

Example 9.13 Tableau 5 in Example 9.3 is primal feasible. $\overline{c}_3$ is positive, so the tableau is not dual feasible and x_3 is a candidate for an entering variable. Therefore we wish to pivot on a positive entry of this column. Checking ratios 10/1 and 50/1 we see that we must pivot in the first row. The result is Tableau 3, which is also primal feasible. The objective function value of the corresponding BFS has strictly increased from 300 to 310. □

Here is another way to understand this pivoting process: The equations in $\overline{T}$ are equivalent to the equations in (P). So $\overline{T}$ represents the following equivalent reformulation of (P):

$$\begin{aligned} \max z = \overline{b}_0 + \overline{c}^T x \\ \text{s.t. } \overline{A}x = \overline{b} \\ x \geq O \end{aligned}$$

If all $\overline{c}_j$ are nonpositive, then $\overline{x} = (\overline{x}_B, O)$ is feasible and has objective function value $\overline{b}_0$ since $\overline{c}_B = O$. If $\tilde{x}$ is any other feasible solution, then $z(\tilde{x}) = \overline{b}_0 + \overline{c}^T\tilde{x} \leq \overline{b}_0$ since $\tilde{x} \geq O$ and $\overline{c} \leq O$. Therefore $\overline{x}$ is optimal.

Example 9.14 Consider Tableau 3 in Example 9.3. It represents the set of equations:

$$\begin{aligned} x_3 - 3x_4 + x_5 = 10 \\ x_2 + 2x_4 - x_5 = 40 \\ x_1 - x_4 + x_5 = 30 \\ -3x_4 - x_5 - z = -310 \end{aligned}$$

Setting $x_4 = x_5 = 0$ yields the basic feasible solution $(30, 40, 10, 0, 0)$ with objective function value 310. The last equation implies that for any feasible point, $z = 310 - 3x_4 - x_5 \leq 310$, since both x_4 and x_5 must be nonnegative. Therefore the point $(30, 40, 10, 0, 0)$ is optimal since it attains the objective function value 310. □

Now suppose that there exists an index s such that $\overline{c}_s > 0$. Of course, x_s is a nonbasic variable. The argument in the preceding paragraph suggests that we might be able to do better than $\overline{x}$ by using a positive value of x_s instead of setting it equal to 0. So let's try setting

$\tilde{x}_s = t \geq 0$, keeping $\tilde{x}_j = 0$ for the other nonbasic variables, and finding the appropriate values of $\tilde{x}_B$.

The equations of $\overline{T}$ are

$$\begin{aligned} x_B + \overline{N}x_N &= \overline{b} \\ \overline{c}_N^T x_N - z &= -\overline{b}_0 \end{aligned}$$

or

$$\begin{aligned} x_B &= \overline{b} - \overline{N}x_N \\ z &= \overline{b}_0 + \overline{c}_N^T x_N \end{aligned}$$

Setting $\tilde{x}_s = t$ and $\tilde{x}_j = 0$ for all $j \in N \setminus \{s\}$ yields the point $\tilde{x}(t)$ given by

$$\begin{aligned} \tilde{x}_{N\setminus\{s\}} &= O \\ \tilde{x}_s &= t \\ \tilde{x}_B &= \overline{b} - dt \\ \tilde{z} &= \overline{b}_0 + \overline{c}_s t \end{aligned} \tag{4}$$

where $d = \overline{N}_s$, and the entries of d are indexed by the basis B.

We want to keep all variables nonnegative, but we want to make z large, so choose $t \geq 0$ as large as possible so that $\tilde{x}_B \geq O$. Thus we want

$$\begin{aligned} \overline{b} - dt &\geq O \\ \overline{b} &\geq dt \\ \overline{b}_i &\geq d_i t, \quad i \in B \end{aligned}$$

This last condition is automatically satisfied if $d_i \leq 0$, so we only have to worry about the case when $d_i > 0$. Then we must ensure that

$$\frac{\overline{b}_i}{d_i} \geq t \text{ if } d_i > 0.$$

So choose

$$t = \min_{i:d_i>0} \left\{ \frac{\overline{b}_i}{d_i} \right\}.$$

If this minimum is attained when $i = r$, then with this choice of t the variable x_r takes the value 0. This suggests that we drop r from the basis B and replace it with s, getting the new basis $\tilde{B} = (B \setminus r) \cup \{s\}$, which we write $B - r + s$ for short. x_s is called the *entering variable* and x_r is called the *leaving variable*. To obtain the basic tableau for $\tilde{B}$, pivot on the entry d_r in the tableau $\overline{T}$. This is the entry in column s of $\overline{T}$ in the row associated with variable x_r. The resulting basic feasible tableau $\tilde{T}$ has associated BFS $\tilde{x}$ such that $\tilde{x}_j = 0$ for $j \notin \tilde{B}$. There is a unique such point with this property; hence it must be the one obtained by choosing t according to the method above.

Example 9.15 Let's take the equations associated with Tableau 5 of Example 9.3:

$$\begin{aligned} x_3 - 3x_4 + x_5 &= 10 \\ x_2 + x_3 - x_4 &= 50 \\ x_1 - x_3 + 2x_4 &= 20 \\ x_3 - 6x_4 - z &= -300 \end{aligned}$$

Solving for the basic variables in terms of the nonbasic variables:

$$\begin{aligned} x_5 &= 10 - x_3 + 3x_4 \\ x_2 &= 50 - x_3 + x_4 \\ x_1 &= 20 + x_3 - 2x_4 \\ z &= 300 + x_3 - 6x_4 \end{aligned}$$

Setting the nonbasic variables to zero gives the associated basic feasible solution $(20, 50, 0, 0, 10)$ with objective function value 300. Since the coefficient of x_3 in the expression for z is positive, we try to increase x_3 while maintaining feasibility of the resulting point. Keep $x_4 = 0$ but let $x_3 = t$. Then

$$\begin{aligned} \tilde{x}_4 &= 0 \\ \tilde{x}_3 &= t \\ \begin{bmatrix} \tilde{x}_5 \\ \tilde{x}_2 \\ \tilde{x}_1 \end{bmatrix} &= \begin{bmatrix} 10 \\ 50 \\ 20 \end{bmatrix} - \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} t \\ \tilde{z} &= 300 + t \end{aligned}$$

These correspond to Equations (4). The maximum value of t that maintains the nonnegativity of the basic variables is 10. Setting $t = 10$ yields the new feasible point $(30, 40, 10, 0, 0)$ with objective function value 310. Since x_5 is now zero and x_3 is now positive, x_3 is the entering variable, x_5 is the leaving variable, and our new basis becomes $(3, 2, 1)$. □

Suppose there are several choices of entering variable. Which one should be chosen? One rule is to choose s such that $\bar{c}_s$ is maximum. This is the *largest coefficient* rule and chooses the entering variable with the largest rate of change of objective function value as a function of x_s. Another rule is to choose s that results in the largest total increase in objective function value upon performing the pivot. This is the *largest increase* rule. A third rule is to choose the smallest s such that $\bar{c}_s$ is positive. This is part of the smallest subscript rule, mentioned below. For a discussion of some of the relative merits and disadvantages of these rules, see Chvátal.

What if there are several choices of leaving variable? For now, you can choose to break ties arbitrarily, or perhaps choose the variable with the smallest subscript (see below).

9.3 The (Primal) Simplex Method

The essential idea of the (primal) simplex method is this: from a basic feasible tableau, pivot in the above manner until a basic optimal tableau is reached. But there are some unresolved issues:

1. How can we be certain the algorithm will terminate?

2. How can we find an initial basic feasible tableau?

Let's first consider the possibility that we have a basic feasible tableau such that there exists an s for which $\overline{c}_s > 0$, but $d \leq O$.

$\ominus$	0	$\oplus$
$\vdots$	$\vdots$	$\vdots$
$\ominus$	0	$\oplus$
$* \ \cdots \ * \ + \ * \ \cdots \ *$	1	$-\overline{b}_0$

In this case, we can choose t to be any positive number, and the point given in (4) will be feasible. Further, $\tilde{z} \to \infty$ as $t \to \infty$, so it is clear that (P) is an unbounded LP. Indeed, it is easy to check directly that $\overline{w} = \tilde{x}(1) - \overline{x}$ is a solution to (UP). $\overline{w}$ satisfies:

$$\begin{aligned} \overline{w}_{N \setminus \{s\}} &= O \\ \overline{w}_s &= 1 \\ \overline{w}_B &= -d \end{aligned}$$

You can see that $\overline{A}\overline{w} = O$ (and so $A\overline{w} = O$), $\overline{w} \geq O$, and $\overline{c}^T\overline{w} = \overline{c}_s > 0$. You can verify this directly from the tableau, or by using the fact that $\overline{w} = \tilde{x}(1) - \overline{x}$. Consequently, $c^T\overline{w}$ is also positive, for we have $0 < \overline{c}_s = (c^T - \overline{y}^T A)\overline{w} = c^T\overline{w} - \overline{y}^T B\overline{w}_B - \overline{y}^T N\overline{w}_N = c^T\overline{w} + \overline{y}^T Bd - \overline{y}^T N_s = c^T\overline{w} + \overline{y}^T N_s - \overline{y}^T N_s = c^T\overline{w}$. Geometrically, when the above situation occurs, the BFS $\overline{x}$ corresponds to a corner point of the feasible region, and the vector $\overline{w}$ points in the direction of an unbounded edge of the feasible region.

Example 9.16 Suppose we are solving the linear program

$$\begin{aligned} \max\ & 5x_1 + 4x_2 \\ \text{s.t. } & -x_1 - 2x_2 \leq -120 \\ & -x_1 - x_2 \leq -70 \\ & -2x_1 - x_2 \leq -100 \\ & x_1, x_2 \geq 0 \end{aligned}$$

Inserting slack variables yields the basic tableau:

x_1	x_2	x_3	x_4	x_5	$-z$	
-1	-2	1	0	0	0	-120
-1	-1	0	1	0	0	-70
-2	-1	0	0	1	0	-100
5	4	0	0	0	1	0

The tableau for the basis $(1, 4, 5)$:

x_1	x_2	x_3	x_4	x_5	$-z$	
1	2	-1	0	0	0	120
0	1	-1	1	0	0	50
0	3	-2	0	1	0	140
0	-6	5	0	0	1	-600

has associated basic feasible solution $(120, 0, 0, 50, 140)$ with objective function value 600. We see that $\bar{c}_3$ is positive but there is no positive pivot entry in that column. Writing out the equations, we get:

$$
\begin{gathered}
x_1 + 2x_2 - x_3 = 120 \\
x_2 - x_3 + x_4 = 50 \\
3x_2 - 2x_3 + x_5 = 140 \\
-6x_2 + 5x_3 - z = -600
\end{gathered}
$$

Solving for the basic variables in terms of the nonbasic variables:

$$
\begin{gathered}
x_1 = 120 - 2x_2 + x_3 \\
x_4 = 50 - x_2 + x_3 \\
x_5 = 140 - 3x_2 + 2x_4 \\
z = 600 - 6x_2 + 5x_3
\end{gathered}
$$

Setting the nonbasic variables to zero gives the associated basic feasible solution $(120, 0, 0, 50, 140)$ with objective function value 600. Since the coefficient of x_3 in the expression for z is positive, we try to increase x_3 while maintaining feasibility of the resulting point. Keep $x_2 = 0$ but let $x_3 = t$. Then

$$
\begin{gathered}
\tilde{x}_2 = 0 \\
\tilde{x}_3 = t \\
\begin{bmatrix} \tilde{x}_1 \\ \tilde{x}_4 \\ \tilde{x}_5 \end{bmatrix} = \begin{bmatrix} 120 \\ 50 \\ 140 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} t \\
\tilde{z} = 600 + 5t
\end{gathered}
$$

The number t can be made arbitrarily large without violating nonnegativity, so the linear program is unbounded. We can rewrite the above equations as

$$\begin{bmatrix} \tilde{x}_1 \\ \tilde{x}_2 \\ \tilde{x}_3 \\ \tilde{x}_4 \\ \tilde{x}_5 \end{bmatrix} = \begin{bmatrix} 120 \\ 0 \\ 0 \\ 50 \\ 140 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \\ 2 \end{bmatrix} t$$

The vector $\overline{w}$ is $[1,0,1,1,2]^T$.

We can confirm that $\overline{w}$ is feasible for (UP):

$$\begin{aligned} -\overline{w}_1 - 2\overline{w}_2 + \overline{w}_3 &= 0 \\ -\overline{w}_1 - \overline{w}_2 + \overline{w}_4 &= 0 \\ -2\overline{w}_1 - \overline{w}_2 + \overline{w}_5 &= 0 \\ 5\overline{w}_1 + 4\overline{w}_2 &> 0 \\ \overline{w} &\geq O \end{aligned}$$

You should graph the original LP and confirm the above results geometrically. □

In general, regardless of objective function, if B is any basis and s is any element not in B, there is a unique way of writing column s as a linear combination of the columns of B. That is to say, there is a unique vector $\overline{w}$ such that $A\overline{w} = O$, $\overline{w}_s = 1$ and $\overline{w}_j = 0$ for $j \notin B + s$. When you have such a vector $\overline{w}$ that is also nonnegative, then $\overline{w}$ is called a *basic feasible direction* (BFD). The above discussion shows that if the simplex method halts by discovering that (P) is unbounded, it finds a BFD with positive objective function value.

Now consider the possibility that we never encounter the situation above. Hence a pivot is always possible. There is only a finite number of bases, so as long as the value of the objective function increases with each pivot, we will never repeat tableaux and must therefore terminate with an optimal tableau.

Unfortunately, it may be the case that $\overline{b}_i = 0$ for some i for which $d_i > 0$, forcing $t = 0$. In this case, the new tableau $\tilde{T}$ and the old tableau $\overline{T}$ correspond to different bases, but have the same BFS since none of the values of any of the variables change during the pivot. Such a pivot is called *degenerate*, and the associated BFS is a *degenerate* point.

Example 9.17 Here is an example from Chvátal. Suppose we have the tableau

x_1	x_2	x_3	x_4	x_5	x_6	x_7	$-z$	
0.5	−5.5	−2.5	9	1	0	0	0	0
0.5	−1.5	−0.5	1	0	1	0	0	0
1	0	0	0	0	0	1	0	1
10	−57	−9	−24	0	0	0	1	0

with basis $(5, 6, 7)$, associated basic feasible solution $(0, 0, 0, 0, 0, 0, 1)$ and objective function value 0. We wish to pivot in column 1, but the ratios $0/0.5$, $0/0.5$, and $1/1$ force us to pivot in rows 5 or 6 (we are labeling the rows by the elements of the basis). We will choose to pivot in the row with the smallest label. The resulting tableau is different:

x_1	x_2	x_3	x_4	x_5	x_6	x_7	$-z$	
1	−11	−5	18	2	0	0	0	0
0	4	2	−8	−1	1	0	0	0
0	11	5	−18	−2	0	1	0	1
0	53	41	−204	−20	0	0	1	0

and the associated basis is now $(1, 6, 7)$, but the corresponding basic feasible solution is still $(0, 0, 0, 0, 0, 0, 0, 1)$. □

It is conceivable that a sequence of degenerate pivots might eventually bring you back to a tableau that was seen before. This can in fact occur, and is known as *cycling*.

Example 9.18 Continuing the example from Chvátal, we will pivot in the column with the most positive $\bar{c}_s$. If there is a tie for the leaving variable, we always choose the variable with the smallest index. Starting from the second tableau above, we generate the following sequence of tableaux:

x_1	x_2	x_3	x_4	x_5	x_6	x_7	$-z$	
1	0	0.5	−4	−0.75	2.75	0	0	0
0	1	0.5	−2	−0.25	0.25	0	0	0
0	0	−0.5	4	0.75	−2.75	1	0	1
0	0	14.5	−98	−6.75	−13.25	0	1	0

x_1	x_2	x_3	x_4	x_5	x_6	x_7	$-z$	
2	0	1	−8	−1.5	5.5	0	0	0
−1	1	0	2	0.5	−2.5	0	0	0
1	0	0	0	0	0	1	0	1
−29	0	0	18	15	−93	0	1	0

x_1	x_2	x_3	x_4	x_5	x_6	x_7	$-z$	
−2	4	1	0	0.5	−4.5	0	0	0
−0.5	0.5	0	1	0.25	−1.25	0	0	0
1	0	0	0	0	0	1	0	1
−20	−9	0	0	10.5	−70.5	0	1	0

x_1	x_2	x_3	x_4	x_5	x_6	x_7	$-z$	
-4	8	2	0	1	-9	0	0	0
0.5	-1.5	-0.5	1	0	1	0	0	0
1	0	0	0	0	0	1	0	1
22	-93	-21	0	0	24	0	1	0

x_1	x_2	x_3	x_4	x_5	x_6	x_7	$-z$	
0.5	-5.5	-2.5	9	1	0	0	0	0
0.5	-1.5	-0.5	1	0	1	0	0	0
1	0	0	0	0	0	1	0	1
10	-57	-9	-24	0	0	0	1	0

We have cycled back to the original tableau! □

There are several ways to avoid cycling. One is a very simple rule called *Bland's rule* or the *smallest subscript rule*, which is described and proved in Chvátal. The rule is: If there are several choices of entering variable (i.e., several variables with positive entry in the last row), choose the variable with the smallest subscript. If there are several choices of leaving variable (i.e., several variables for which the minimum ratio of the ratio test is attained), choose the variable with the smallest subscript. If this rule is applied, no tableau will ever be repeated.

Example 9.19 Applying Bland's rule beginning with the tableau of Example 9.17 results in the same sequence of tableaux as before in Examples 9.17 and 9.18, except that when pivoting in the penultimate tableau, x_1 will enter the basis and x_4 will leave the basis. This results in the tableau

x_1	x_2	x_3	x_4	x_5	x_6	x_7	$-z$	
0	-4	-2	8	1	-1	0	0	0
1	-3	-1	2	0	2	0	0	0
0	3	1	-2	0	-2	1	0	1
0	-27	1	-44	0	-20	0	1	0

Now x_3 enters the basis and x_7 leaves, which yields:

x_1	x_2	x_3	x_4	x_5	x_6	x_7	$-z$	
0	2	0	4	1	-5	2	0	2
1	0	0	0	0	0	1	0	1
0	3	1	-2	0	-2	1	0	1
0	-30	0	-42	0	-18	-1	1	-1

This tableau is optimal. □

Cycling can also be avoided by a *perturbation* technique. Adjoin an indeterminate ε to the field $\mathbf{R}$. Consider the field $\mathbf{R}(\varepsilon)$ of rational functions in ε. We can make this an ordered field by defining

$$a_0 + a_1\varepsilon + a_2\varepsilon^2 + \cdots + a_k\varepsilon^k < b_0 + b_1\varepsilon + b_2\varepsilon^2 + \cdots + b_k\varepsilon^k$$

if there exists $0 \le j < k$ such that $a_i = b_i$ for all $0 \le i \le j$ and $a_{j+1} < b_{j+1}$. (Think of ε as being an infinitesimally small positive number if you dare.)

We solve (P) by replacing b with $b + (\varepsilon, \varepsilon^2, \ldots, \varepsilon^m)^T$ and noting that any tableau which is primal (respectively, dual) feasible with respect to $\mathbf{R}(\varepsilon)$ is also primal (respectively, dual) feasible with respect to $\mathbf{R}$ when ε is replaced by 0.

Note that expressions involving ε will only appear in the last column of a basic tableau—pivoting won't cause any ε's to pop up anywhere else.

Example 9.20 Let's try this out on Chvátal's example. Our starting tableau becomes:

x_1	x_2	x_3	x_4	x_5	x_6	x_7	$-z$	
0.5	-5.5	-2.5	9	1	0	0	0	ε
0.5	-1.5	-0.5	1	0	1	0	0	ε^2
1	0	0	0	0	0	1	0	$1+\varepsilon^3$
10	-57	-9	-24	0	0	0	1	0

Pivot in column x_1. Check ratios: $\varepsilon/0.5 = 2\varepsilon$, $\varepsilon^2/0.5 = 2\varepsilon^2$, $(1+\varepsilon^3)/1 = 1+\varepsilon^3$. The second ratio is smaller, so we choose x_6 to leave, getting:

x_1	x_2	x_3	x_4	x_5	x_6	x_7	$-z$	
0	-4	-2	8	1	-1	0	0	$\varepsilon - \varepsilon^2$
1	-3	-1	2	0	2	0	0	$2\varepsilon^2$
0	3	1	-2	0	-2	1	0	$1 - 2\varepsilon^2 + \varepsilon^3$
0	-27	1	-44	0	-20	0	1	$-20\varepsilon^2$

Now pivot in column x_3. You must pivot x_7 out, which yields:

x_1	x_2	x_3	x_4	x_5	x_6	x_7	$-z$	
0	2	0	4	1	-5	2	0	$2 + \varepsilon - 5\varepsilon^2 + 2\varepsilon^3$
1	0	0	0	0	0	1	0	$1 + \varepsilon^3$
0	3	1	-2	0	-2	1	0	$1 - 2\varepsilon^2 + \varepsilon^3$
0	-30	0	-42	0	-18	-1	1	$-1 - 18\varepsilon^2 - \varepsilon^3$

This tableau is optimal. You can set $\varepsilon = 0$ to see the final table for the original, unperturbed problem. The optimal solution is $(1, 0, 1, 0, 2, 0, 0)$ with objective function value 1. □

To show that cycling won't occur under the perturbation method, it suffices to show that $\overline{b} > 0$ in every basic feasible tableau T, for then the objective function value will strictly increase at each pivot.

Assume that B is a basis for which $\overline{b}_k = 0$ for some k. Then $B^{-1}(b+(\varepsilon,\ldots,\varepsilon^m)^T)$ is zero in the kth component. Let p^T be the kth row of B^{-1}. Then $p^T(b+(\varepsilon,\ldots,\varepsilon^m)^T) = 0$ in $\mathbf{R}(\varepsilon)$. So $p^T b + p^T e_1\varepsilon + p^T e_2\varepsilon^2 + \cdots + p^T e_m\varepsilon^m = 0$. (Remember, e_i is the standard unit vector with all components equal to 0 except for a 1 in the ith component.) Therefore $p^T e_i = 0$ for all i, which in turn implies that $p^T = O^T$. But this is impossible since p^T is a row of an invertible matrix.

Geometrically we are moving the constraining hyperplanes of $(\hat{P})$ a "very small" amount parallel to themselves so that we avoid any coincidences of having more than n of them passing through a common point.

Now we know that using this rule will not cause any tableau (i.e., basis) to repeat. Since there is a finite number of bases, we will eventually discover a tableau that is optimal, or else we will discover a tableau that demonstrates that (P) is unbounded.

Now what about getting that initial primal feasible tableau? If $b \geq O$ there is no problem because we can use the initial tableau associated with the equations in (P) itself—our basis consists of the set of slack variables. The GGMC problem provides a good example of this situation.

Example 9.21 Pivoting from the initial tableau to optimality in the GGMC problem:

x_1	x_2	x_3	x_4	x_5	$-z$	
1	2	1	0	0	0	120
1	1	0	1	0	0	70
2	1	0	0	1	0	100
5	4	0	0	0	1	0

x_1	x_2	x_3	x_4	x_5	$-z$	
0	1.5	1	0	−0.5	0	70
0	0.5	0	1	−0.5	0	20
1	0.5	0	0	0.5	0	50
0	1.5	0	0	−2.5	1	−250

x_1	x_2	x_3	x_4	x_5	$-z$	
0	0	1	−3	1	0	10
0	1	0	2	−1	0	40
1	0	0	−1	1	0	30
0	0	0	−3	−1	1	−310

□

What if at least one component of b is negative (as in Example 9.16? The clever idea is to introduce a new nonnegative variable x_0. This variable is called an *artificial* variable. Put it into each of the equations of (P) with a coefficient of -1, getting the system

$$\begin{aligned} Ax - ex_0 &= b \\ x \geq O,\ x_0 &\geq 0 \end{aligned} \tag{5}$$

Now it is obvious (isn't it?) that (P) is feasible if and only if (5) has a feasible solution in which $x_0 = 0$. So let's try to solve the following *Phase I* problem:

$$\begin{aligned} \max\ & -x_0 \\ \text{s.t. } & Ax - ex_0 = b \\ & x \geq O,\ x_0 \geq 0 \end{aligned}$$

Find slack variable x_{n+k}, such that b_k is most negative. Then $\{n+1, n+2, \ldots, n+m\} + 0 - (n+k)$ is a feasible basis. You can see that the corresponding basic point $\overline{x}$ is nonnegative since

$$\begin{aligned} \overline{x}_0 &= -b_k \\ \overline{x}_{n+i} &= b_i - b_k, \quad i \neq k \\ \overline{x}_{n+k} &= 0 \\ \overline{x}_j &= 0, \quad j = 1, \ldots, n \end{aligned}$$

One pivot moves you to this basis from the basis consisting of the set of slack variables (which is not feasible).

Now you have a basic feasible tableau for the Phase I problem, so you can proceed to solve it. While doing so, choose x_0 as a leaving variable if and as soon as you are permitted to do so and then immediately stop since x_0 will have value 0.

Because x_0 is a nonnegative variable, the Phase I problem cannot be unbounded. So there are two possibilities:

1. If the optimal value of the Phase I problem is negative (i.e., x_0 is positive at optimality), then there is no feasible solution to this problem in which x_0 is zero; therefore (P) is infeasible.

2. If, on the other hand, the optimal value of the Phase I problem is zero, then it must have been the case that x_0 was removed from the basis at the final pivot; therefore, there is a feasible solution to this problem in which x_0 is zero, and moreover the final basis B is a primal feasible basis for (P). If this happens, drop the x_0 column and

replace the final row of the tableau by the row $(c^T, 1, 0)$. Pivot again on the 1's in the basic columns to make $\bar{c}_B^T = O^T$. (Alternatively, calculate $c^T - c_B^T B^{-1} A$. This isn't hard since $B^{-1}A$ is already sitting in the final tableau.) Now you have a basic feasible tableau for (P). Proceed to solve (P)—this part is called *Phase II.*

Example 9.22 We apply the Phase I procedure to the following linear program:

$$\begin{aligned} \max 5x_1 + 4x_2 \\ \text{s.t. } x_1 + 2x_2 \leq 120 \\ -x_1 - x_2 \leq -70 \\ -2x_1 - x_2 \leq -100 \\ x_1, x_2 \geq 0 \end{aligned}$$

The Phase I problem is:

$$\begin{aligned} \max -x_0 \\ \text{s.t. } x_1 + 2x_2 + x_3 - x_0 = 120 \\ -x_1 - x_2 + x_4 - x_0 = -70 \\ -2x_1 - x_2 + x_5 - x_0 = -100 \\ x_1, x_2, x_3, x_4, x_5, x_0 \geq 0 \end{aligned}$$

The first tableau is:

x_1	x_2	x_3	x_4	x_5	x_0	$-z$	
1	2	1	0	0	−1	0	120
−1	−1	0	1	0	−1	0	−70
−2	−1	0	0	1	−1	0	−100
0	0	0	0	0	−1	1	0

The preliminary pivot takes place in column x_0 and the third row (since the right-hand side −100 is the most negative):

x_1	x_2	x_3	x_4	x_5	x_0	$-z$	
3	3	1	0	−1	0	0	220
1	0	0	1	−1	0	0	30
2	1	0	0	−1	1	0	100
2	1	0	0	−1	0	1	100

Now the tableau is primal feasible, and we proceed to solve the Phase I problem:

x_1	x_2	x_3	x_4	x_5	x_0	$-z$	
0	3	1	−3	2	0	0	130
1	0	0	1	−1	0	0	30
0	1	0	−2	1	1	0	40
0	1	0	−2	1	0	1	40

x_1	x_2	x_3	x_4	x_5	x_0	$-z$	
0	0	1	3	−1	−3	0	10
1	0	0	1	−1	0	0	30
0	1	0	−2	1	1	0	40
0	0	0	0	0	−1	1	0

This tableau is optimal with objective function value 0, so the original LP is feasible. We now delete column x_0 and replace the last row with the original objective function:

x_1	x_2	x_3	x_4	x_5	$-z$	
0	0	1	3	−1	0	10
1	0	0	1	−1	0	30
0	1	0	−2	1	0	40
5	4	0	0	0	1	0

We must perform preliminary pivots in columns x_1 and x_2 to bring the tableau back into basic form:

x_1	x_2	x_3	x_4	x_5	$-z$	
0	0	1	3	−1	0	10
1	0	0	1	−1	0	30
0	1	0	−2	1	0	40
0	4	0	−5	5	1	−150

x_1	x_2	x_3	x_4	x_5	$-z$	
0	0	1	3	−1	0	10
1	0	0	1	−1	0	30
0	1	0	−2	1	0	40
0	0	0	3	1	1	−310

We now have a basic feasible tableau and may begin a sequence of pivots to solve the Phase II problem:

x_1	x_2	x_3	x_4	x_5	$-z$	
0	0	1/3	1	−1/3	0	10/3
1	0	−1/3	0	−2/3	0	80/3
0	1	2/3	0	1/3	0	140/3
0	0	−1	0	2	1	−320

x_1	x_2	x_3	x_4	x_5	$-z$	
0	1	1	1	0	0	50
1	2	1	0	0	0	120
0	3	2	0	1	0	140
0	−6	−5	0	0	1	−600

The optimal solution is $(120, 0, 0, 50, 140)$ with an objective function value of 600. □

Example 9.23 Apply the Phase I procedure to the following problem:

$$\begin{aligned} \max\ & 5x_1 + 4x_2 \\ \text{s.t.}\ & -x_1 - 2x_2 \leq -120 \\ & x_1 + x_2 \leq 70 \\ & -2x_1 - x_2 \leq -100 \\ & x_1, x_2 \geq 0 \end{aligned}$$

The initial tableau is:

x_1	x_2	x_3	x_4	x_5	x_0	$-z$	
−1	−2	1	0	0	−1	0	−120
1	1	0	1	0	−1	0	70
−2	−1	0	0	1	−1	0	−100
0	0	0	0	0	−1	1	0

After the preliminary pivot, the result is:

x_1	x_2	x_3	x_4	x_5	x_0	$-z$	
1	2	−1	0	0	1	0	120
2	3	−1	1	0	0	0	190
−1	1	−1	0	1	0	0	20
1	2	−1	0	0	0	1	120

Two pivots solve the Phase I problem:

x_1	x_2	x_3	x_4	x_5	x_0	$-z$	
0	1/2	−1/2	−1/2	0	1	0	25
1	3/2	−1/2	1/2	0	0	0	95
0	5/2	−3/2	1/2	1	0	0	115
0	1/2	−1/2	−1/2	0	0	1	25

x_1	x_2	x_3	x_4	x_5	x_0	$-z$	
0	0	$-1/5$	$-3/5$	$-1/5$	1	0	2
1	0	$2/5$	$1/5$	$-3/5$	0	0	26
0	1	$-3/5$	$1/5$	$2/5$	0	0	46
0	0	$-1/5$	$-3/5$	$-1/5$	0	1	2

Since the optimal objective function value is nonzero, the original LP is infeasible. □

Some comments: In the form that I have described it, the simplex method behaves poorly numerically and is not really implemented in this manner. Also, it is possible to have an exponential number of pivots in terms of the number of variables or constraints. See Chvátal for more details on resolving the first problem, and a discussion of the second in which a slightly skewed hypercube wreaks havoc with reasonable pivot rules.

The existence of the simplex method, however, gives us a brand new proof of Strong Duality! Just observe that the algorithm (1) terminates by demonstrating that (P), and hence $(\hat{P})$, is infeasible; (2) terminates by demonstrating that (P), and hence $(\hat{P})$, is unbounded; or (3) terminates with a dual pair of optimal solutions to $(\hat{P})$ and $(\hat{D})$ with equal objective function values. In fact, we can conclude something stronger:

Theorem 9.24 *If (P) is feasible, then it has a basic feasible solution. If (P) is unbounded, then it has a basic feasible direction with positive objective function value. If (P) has an optimal solution, then it has an optimal basic feasible solution.*

Suppose we have found optimal $\overline{x}$ and $\overline{y}$ from the optimal final tableau of the simplex method. We already know that they have the same objective function values, so clearly they must also satisfy complementary slackness. This can be seen directly: Suppose $\overline{x}_j > 0$. Then $j \in B$. So $c_j - \overline{y}^T A_j = 0$. Also note that if $\overline{y}_i > 0$, then $0 - \overline{y}^T e_i < 0$, hence $n + i \in N$. Therefore $x_{n+i} = 0$ and the ith constraint in $(\hat{P})$ is satisfied with equality. So the corresponding solutions of $(\hat{P})$ and $(\hat{D})$ also are easily seen to satisfy complementary slackness. The simplex method in effect maintains primal feasibility, enforces complementary slackness, and strives for dual feasibility.

10 Exercises: The Simplex Method

These questions concern the pair of dual linear programs

$$(\hat{P}) \quad \begin{array}{c} \max \hat{c}^T\hat{x} \\ \text{s.t. } \hat{A}\hat{x} \le b \\ \hat{x} \ge O \end{array} \qquad (\hat{D}) \quad \begin{array}{c} \min \hat{y}^T b \\ \text{s.t. } \hat{y}^T\hat{A} \ge \hat{c}^T \\ \hat{y} \ge O \end{array}$$

and the pair of dual linear programs

$$(P) \quad \begin{array}{c} \max c^T x \\ \text{s.t. } Ax = b \\ x \ge O \end{array} \qquad (D) \quad \begin{array}{c} \min y^T b \\ \text{s.t. } y^T A \ge c^T \end{array}$$

where A is $m \times n$ and (P) is obtained from $(\hat{P})$ by introducing m slack variables.

Exercise 10.1 True or false: Every feasible solution of (P) is a BFS of (P). If true, prove it; if false, give a counterexample. □

Exercise 10.2 True or false: Suppose $\overline{T}$ is a basic feasible tableau with associated BFS $\overline{x}$. Then $\overline{x}$ is optimal if and only if $\overline{c}_j \le O$ for all j. (This is Chvátal 3.10.) If true, prove it; if false, give a counterexample. □

Exercise 10.3 Prove that the number of basic feasible solutions for (P) is at most $\binom{m+n}{m}$. Can you construct an example in which this number is achieved? □

Exercise 10.4

1. Geometrically construct a two-variable LP $(\hat{P})$ in which the same BFS $\overline{x}$ is associated with more than one feasible basis.
2. Do the same as in the previous problem, but in such a way that all of the bases associated with $\overline{x}$ are also dual feasible.
3. Do the same as in the previous problem, but in such a way that at least one of the bases associated with $\overline{x}$ is dual feasible, while at least one is not.

□

Exercise 10.5 Geometrically construct a two-variable LP $(\hat{P})$ that has no primal feasible basis and no dual feasible basis. □

Exercise 10.6 Geometrically construct a two-variable LP $(\hat{P})$ such that both the primal and the dual problems have more than one optimal solution. □

Exercise 10.7

1. total unimodularity

2. Suppose that A is a matrix with integer entries and B is a basis such that A_B has determinant -1 or $+1$. Assume that b also has integer entries. Prove that the solution to $A_B x_B = b$ is an integer vector.

3. Suppose $\hat{A}$ is a matrix with integer entries such that every square submatrix (of whatever size) has determinant 0, -1 or $+1$. Assume that b also has integer entries. Prove that if (P) has an optimal solution, then there is an optimal integer solution x^*.

□

Exercise 10.8

1. For variable cost coefficients c, consider the function $z^*(c)$, which is defined to be the optimal objective function value of (P) as a function of c. Take the domain of $z^*(c)$ to be the points c such that (P) has a finite optimal objective function value. Prove that there exists a finite set S such that $z^*(c)$ is of the form

$$z^*(c) = \max_{x \in S} \{c^T x\}$$

on its domain.

2. For variable right hand sides b, consider the function $z^*(b)$, which is defined to be the optimal objective function value of (P) as a function of b. Take the domain of $z^*(b)$ to be the points b such that (P) has a finite optimal objective function value. Prove that there exists a finite set T such that $z^*(b)$ is of the form

$$z^*(b) = \min_{y \in T} \{y^T b\}$$

on its domain.

□

Exercise 10.9 Suppose that $\hat{A}$ and b have integer entries, B is a feasible basis for (P), and $\overline{x}$ is the associated BFS. Let

$$\alpha = \max_{i,j}\{|a_{ij}|\}$$
$$\beta = \max_{i}\{|b_i|\}$$

Prove that the absolute value of each entry of B^{-1} is no more than $(m-1)!\alpha^{m-1}$. Prove that $|\overline{x}_j| \leq m!\alpha^{m-1}\beta$ for all j (Papadimitriou and Steiglitz). □

Exercise 10.10 Suppose $\overline{x}$ is feasible for (P). We say that $\overline{x}$ is an *extreme point* of (P) if there exists no nonzero vector $\overline{w}$ such that $\overline{x}+\overline{w}$ and $\overline{x}-\overline{w}$ are both feasible for (P). Illustrate this definition geometrically. Define a point $\overline{x}$ to be a *convex combination* of points $\overline{x}^1, \overline{x}^2$ if there exists a real number t, $0 \leq t \leq 1$, such that $\overline{x} = t\overline{x}^1 + (1-t)\overline{x}^2$. The *support* of a point $\overline{x}$ is the set of indices $\operatorname{supp}(\overline{x}) = \{i : \overline{x}_i \neq 0\}$.

Prove that the following are equivalent for a feasible point $\overline{x}$ of (P):

1. $\overline{x}$ is a BFS of (P).
2. $\overline{x}$ is an extreme point of (P).
3. $\overline{x}$ cannot be written as a convex combination of any two other feasible points of (P), both different from $\overline{x}$.
4. The set of column vectors $\{A_i : i \in \operatorname{supp}(\overline{x})\}$ is linearly independent.

□

Exercise 10.11 Suppose $\overline{x}$ is feasible for (P). We say that $\overline{x}$ is an *exposed point* of (P) if there exists an objective function $d^T x$ such that $\overline{x}$ is the unique optimal solution to

$$\begin{aligned} \max\, & d^T x \\ \text{s.t. } & Ax = b \\ & x \geq O \end{aligned}$$

Illustrate this definition geometrically. Prove that $\overline{x}$ is an exposed point of (P) if and only if $\overline{x}$ is a BFS of (P). □

Exercise 10.12 Suppose $\overline{z}$ is a nonzero vector in $\mathbf{R}^n$. Define $\overline{z}$ to be a *nonnegative combination* of vectors $\overline{z}^1, \overline{z}^2$ if there exist nonnegative real numbers t_1, t_2 such that $\overline{z} = t_1\overline{z}^1 + t_2\overline{z}^2$. Call $\overline{z}$ an *extreme vector* of (P) if it is a nonzero, nonnegative solution to $Az = O$, and $\overline{z}$ cannot be expressed as a nonnegative combination of any two other nonnegative solutions $\overline{z}^1, \overline{z}^2$ to $Az = O$ unless both $\overline{z}^1, \overline{z}^2$ are themselves nonnegative multiples of $\overline{z}$. The *support* of a vector $\overline{z}$ is the set of indices $\operatorname{supp}(\overline{z}) = \{i : \overline{z}_i \neq 0\}$.

Prove that the following are equivalent for a nonzero, nonnegative solution $\overline{z}$ to $Az = O$:

1. $\overline{z}$ is a positive multiple of a basic feasible direction for (P).

2. $\overline{z}$ is an extreme vector for (P).

3. The set of column vectors $\{A_i : i \in \text{supp}\,(\overline{z})\}$ is linearly dependent, but dropping any one vector from this set results in a linearly independent set.

□

Exercise 10.13 Prove directly (without the simplex method) that if (P) has a feasible solution, then it has a basic feasible solution. Hint: If $\overline{x}$ is feasible and not basic feasible, find an appropriate solution to $A\overline{w} = O$ and consider $\overline{x} \pm \overline{w}$. Similarly, prove directly that if (P) has a optimal solution, then it has an optimal basic feasible solution. □

Exercise 10.14 True or false: If (P) and (D) are both feasible, then there always exist optimal basic feasible solutions $\overline{x}$ and $\overline{y}$ to (P) and (D), respectively, that satisfy strong complementary slackness . If true, prove it; if false, give a counterexample. □

Exercise 10.15 Suppose we start with a linear program

$$\begin{array}{c} \max c'^T x \\ \text{s.t. } A'x' \le b \\ x'_1 \text{ unrestricted} \\ x'_j \ge O, \quad j = 2, \ldots n \end{array}$$

and convert it into a problem in standard form by making the substitution $x'_1 = u_1 - v_1$, where $u_1, v_1 \ge 0$. Prove that the simplex method will not produce an optimal solution in which both u_1 and v_1 are positive. □

Exercise 10.16 Suppose $(\hat{P})$ is infeasible and this is discovered in Phase I of the simplex method. Use the final Phase I tableau to find a solution to

$$\begin{array}{c} y^T \hat{A} \ge O^T \\ y^T b < 0 \\ y \ge O \end{array}$$

□

Exercise 10.17 Chvátal, problems 2.1–2.2, 3.1–3.9, 5.2. Note: You need to read the book in order to do problems 3.3–3.7. □

11 The Simplex Method—Further Considerations

11.1 Uniqueness

Suppose that B is an optimal basis with associated $\overline{T}$, $\overline{x}$, and $\overline{y}$. Assume that $\overline{c}_j < 0$ for all $j \in N$. Then $\overline{x}$ is the unique optimal solution to (P). Here are two ways to see this:

1. We know that any optimal $\tilde{x}$ must satisfy complementary slackness with $\overline{y}$. But $\overline{y}^T A_j > c_j$ for all $j \in N$ since $\overline{c}_j < 0$. So $\tilde{x}_j = 0$ for all $j \in N$. Hence $\tilde{x}_N = \overline{x}_N$. Also $B\tilde{x}_B = b$, so $\tilde{x}_B = B^{-1}b = \overline{x}_B$. So $\tilde{x} = \overline{x}$.

2. Assume that $\tilde{x}$ is optimal. If $\tilde{x}_N \neq O$, then $\tilde{z} = \overline{z} + \overline{c}^T\tilde{x} = \overline{z} + \overline{c}_N^T\tilde{x}_N < \overline{z}$ since $\overline{c}_N < O$. So $\tilde{x}_N = O$, and we again are able to conclude that $\tilde{x}_B = \overline{x}_B$ and so $\tilde{x} = \overline{x}$.

Exercise 11.1 Suppose that B is an optimal basis with associated $\overline{T}$, $\overline{x}$, and $\overline{y}$.

1. Assume that $\overline{x}$ is the unique optimal solution to (P) and that $\overline{b} > O$. Prove that $\overline{c}_j < 0$ for all $j \in N$.

2. Construct an example in which $\overline{x}$ is the unique optimal solution to (P), $\overline{b} \not> O$, and $\overline{c}_N \not< O$.

□

Exercise 11.2 Suppose again that B is an optimal basis with associated $\overline{T}$, $\overline{x}$, and $\overline{y}$. Assume that $\overline{b} > O$. Prove that $\overline{y}$ is the unique optimal solution to (D). □

11.2 Revised Simplex Method

In practice we do not really want or need all of the entries of a tableau $\overline{T}$. Let us assume we have some method of solving nonsingular systems of m equations in m unknowns—for example, we may use matrix factorization techniques. More details can be found in Chvátal.

At some stage in the simplex method we have a feasible basis B, so we know that $\overline{b} \geq O$. Perhaps we also have the associated BFS $\overline{x}$, but if we do not, we can find $\overline{x} = (\overline{x}_B, \overline{x}_N)$ by setting $\overline{x}_N = O$ and solving $Bx_B = b$. To get $\overline{y}^T = c_B^T B^{-1}$, we solve $\overline{y}^T B = c_B^T$. Then we can calculate $\overline{c}_N^T = c_N^T - \overline{y}^T N$. If $\overline{c} \leq O$, then $\overline{x}$ is optimal. If not, then $\overline{c}_s > 0$ for some s. To find $d = \overline{A}_s = B^{-1}A_s = B^{-1}a$, we solve $Bd = a$. If $d \leq O$, then (P) is unbounded. Otherwise we use the ratio test to find the minimum ratio t and the leaving variable x_r. Replace $\overline{x}$ by $\tilde{x}_s = t$, $\tilde{x}_{N-s} = O$, and $\tilde{x}_B = \overline{x}_B - td$. Replace B by $B + s - r$.

During these computations, remember that B is an ordered basis, and that this ordered basis labels the columns of A_B (also denoted B), the rows of B^{-1}, the rows of $\overline{T}$, the elements of d, and the elements of $\overline{b} = \overline{x}_B$.

Example 11.3 Solving GGMC using the revised simplex method. We are given the initial tableau:

x_1	x_2	x_3	x_4	x_5	$-z$	
1	2	1	0	0	0	120
1	1	0	1	0	0	70
2	1	0	0	1	0	100
5	4	0	0	0	1	0

Our starting basis is $(3, 4, 5)$, So

$$B = \begin{array}{c} \begin{matrix} 3 & 4 & 5 \end{matrix} \\ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \end{array}$$

1. Since our "current" tableau is the same as the initial tableau, we can directly see that $\overline{x} = (0, 0, 120, 70, 100)$ and that we can choose x_1 as the entering variable.

   To find the leaving variable, write $\tilde{x}_B = \overline{x}_B - td$:

$$\begin{bmatrix} \tilde{x}_3 \\ \tilde{x}_4 \\ \tilde{x}_5 \end{bmatrix} = \begin{bmatrix} 120 \\ 70 \\ 100 \end{bmatrix} - t \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$$
$$\tilde{x}_1 = t$$
$$\tilde{x}_2 = 0$$

   Therefore $t = 50$, x_5 is the leaving variable, $(3, 4, 1)$ is the new basis, $\overline{x} = (50, 0, 70, 20, 0)$ is the new basic feasible solution, and

$$B = \begin{array}{c} \begin{matrix} 3 & 4 & 1 \end{matrix} \\ \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 2 \end{bmatrix} \end{array}$$

   is the new basis matrix B.

2. Find $\overline{y}$ by solving $y^T B = c_B^T$:

$$\begin{bmatrix} y_1 & y_2 & y_3 \end{bmatrix} \begin{array}{c} \begin{matrix} 3 & 4 & 1 \end{matrix} \\ \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 2 \end{bmatrix} \end{array} = \begin{array}{c} \begin{matrix} 3 & 4 & 1 \end{matrix} \\ \begin{bmatrix} 0 & 0 & 5 \end{bmatrix} \end{array}$$

Thus $\overline{y} = (0, 0, 2.5)^T$.

Calculate $\overline{c}_N^T = c_N - \overline{y}^T N$.

$$\begin{bmatrix} \overline{c}_2 & \overline{c}_5 \end{bmatrix} = \overset{\begin{matrix} 2 & 5 \end{matrix}}{\begin{bmatrix} 4 & 0 \end{bmatrix}} - \begin{bmatrix} 0 & 0 & 2.5 \end{bmatrix} \overset{\begin{matrix} 2 & 5 \end{matrix}}{\begin{bmatrix} 2 & 0 \\ 1 & 0 \\ 1 & 1 \end{bmatrix}} = \overset{\begin{matrix} 2 & \quad 5 \end{matrix}}{\begin{bmatrix} 1.5 & -2.5 \end{bmatrix}}$$

Since $\overline{c}$ has some positive entries, we must pivot. We choose x_2 as the entering variable. Find d by solving $Bd = a$ where $a = A_2$:

$$\overset{\begin{matrix} 3 & 4 & 1 \end{matrix}}{\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 2 \end{bmatrix}} \begin{bmatrix} d_3 \\ d_4 \\ d_1 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$$

We see that

$$\begin{bmatrix} d_3 \\ d_4 \\ d_1 \end{bmatrix} = \begin{bmatrix} 1.5 \\ 0.5 \\ 0.5 \end{bmatrix}$$

To find the leaving variable, write $\tilde{x}_B = \overline{x}_B - td$:

$$\begin{bmatrix} \tilde{x}_3 \\ \tilde{x}_4 \\ \tilde{x}_1 \end{bmatrix} = \begin{bmatrix} 70 \\ 20 \\ 50 \end{bmatrix} - t \begin{bmatrix} 1.5 \\ 0.5 \\ 0.5 \end{bmatrix}$$

$$\tilde{x}_2 = t$$
$$\tilde{x}_5 = 0$$

Therefore $t = 40$, x_4 is the leaving variable, $(3, 2, 1)$ is the new basis, $\overline{x} = (30, 40, 10, 0, 0)$ is the new basic feasible solution, and

$$B = \overset{\begin{matrix} 3 & 2 & 1 \end{matrix}}{\begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 2 \end{bmatrix}}$$

is the new basis matrix B.

3. Find $\overline{y}$ by solving $y^T B = c_B^T$:

$$\begin{bmatrix} y_1 & y_2 & y_3 \end{bmatrix} \begin{array}{c} \begin{array}{ccc} 3 & 2 & 1 \end{array} \\ \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 2 \end{bmatrix} \end{array} = \begin{array}{c} \begin{array}{ccc} 3 & 2 & 1 \end{array} \\ \begin{bmatrix} 0 & 4 & 5 \end{bmatrix} \end{array}$$

Thus $\overline{y} = (0, 3, 1)^T$.
Calculate $\overline{c}_N^T = c_N - \overline{y}^T N$.

$$\begin{bmatrix} \overline{c}_4 & \overline{c}_5 \end{bmatrix} = \begin{array}{c} \begin{array}{cc} 4 & 5 \end{array} \\ \begin{bmatrix} 0 & 0 \end{bmatrix} \end{array} - \begin{bmatrix} 0 & 3 & 1 \end{bmatrix} \begin{array}{c} \begin{array}{cc} 4 & 5 \end{array} \\ \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \end{array} = \begin{array}{c} \begin{array}{cc} 4 & 5 \end{array} \\ \begin{bmatrix} -3 & -1 \end{bmatrix} \end{array}$$

Since $\overline{c}$ is nonpositive, our current solution $\overline{x}$ is optimal.

□

11.3 Dual Simplex Method

What if we have a dual feasible basis B with associated $\overline{T}$, $\overline{x}$, and $\overline{y}$? That is to say, assume $\overline{c} \leq O$. There is a method of pivoting through a sequence of dual feasible bases until either an optimal basis is reached, or else it is demonstrated that (P) is infeasible. A pivot that maintains dual feasibility is called a *dual pivot*, and the process of solving an LP by dual pivots is called the *dual simplex method*.

During this discussion, remember again that B is an ordered basis, and that this ordered basis labels the columns of A_B (also denoted B), the rows of B^{-1}, the rows of $\overline{T}$, and the elements of $\overline{b} = \overline{x}_B$.

If $\overline{b} \geq O$, then B is also primal feasible and $\overline{x}$ is an optimal solution to (P). So suppose

$\overline{b}_r < 0$ for some $r \in B$. Assume that the associated row $\overline{w}$ of $\overline{A}$ is nonnegative:

$$\begin{array}{|ccc|c|c|} \hline & & & 0 & * \\ & & & \vdots & \vdots \\ & & & 0 & * \\ \oplus & \cdots & \oplus & 0 & - \\ & & & 0 & * \\ & & & \vdots & \vdots \\ & & & 0 & * \\ \hline \ominus & \cdots & \ominus & 1 & * \\ \hline \end{array}$$

Then the corresponding equation reads $\overline{w}^T x = \overline{b}_r$, which is clearly infeasible for nonnegative x since $\overline{w} \geq O$ and $\overline{b}_r < 0$. So (P) is infeasible.

Now suppose $\overline{w}$ contains at least one negative entry. We want to pivot on one of these negative entries $\overline{w}_s$, for then the lower right-hand entry of the tableau will not decrease, and so the corresponding objective function value of the dual feasible solution will not increase.

$$\begin{array}{|ccccc|c|c|} \hline & & & & & 0 & * \\ & & & & & \vdots & \vdots \\ & & & & & 0 & * \\ & & -^{\circ} & & & 0 & - \\ & & & & & 0 & * \\ & & & & & \vdots & \vdots \\ & & & & & 0 & * \\ \hline \ominus & \cdots & \ominus & \cdots & \ominus & 1 & * \\ \hline \end{array}$$

We do not want the pivot to destroy dual feasibility, so we require

$$\overline{c}_j - \frac{\overline{c}_s}{\overline{w}_s}\overline{w}_j \leq 0$$

for all j. There is no problem in satisfying this if $\overline{w}_j \geq 0$. For $\overline{w}_j < 0$ we require that

$$\frac{\overline{c}_j}{\overline{w}_j} \geq \frac{\overline{c}_s}{\overline{w}_s}.$$

So choose s such that

$$\frac{\overline{c}_s}{\overline{w}_s} = \min_{j:\overline{w}_j<0} \left\{ \frac{\overline{c}_j}{\overline{w}_j} \right\}.$$

This is the *dual ratio test* to determine the entering variable. Pivoting on $\overline{w}_s$ causes r to leave the basis and s to enter the basis.

Example 11.4 Tableau 4 of Example 9.3 is dual feasible, but not primal feasible. We must pivot in the first row (which corresponds to basic variable x_4). Calculating ratios for the two negative entries in this row, $-1/(-1/3) = 3$, $-2/(-1/3) = 6$, we determine that the pivot column is that for x_3. Pivoting entry results in tableau 3, which is dual feasible. (It also happens to be primal feasible, so this is an optimal tableau.) □

Analogously to the primal simplex method, there are methods to initialize the dual simplex method and to avoid cycling.

11.4 Revised Dual Simplex Method

As in the revised simplex method, let us assume we have some method of solving nonsingular systems of m equations in m unknowns, and we wish to carry out a pivot of the dual simplex method without actually computing all of $\overline{T}$.

Suppose B is a dual feasible basis, so we know that $\overline{c} \leq O$. We can find $\overline{x} = (\overline{x}_B, \overline{x}_N)$ by setting $\overline{x}_N = O$ and solving $Bx_B = b$. If $\overline{x}_B = \overline{b} \geq O$, then B is also primal feasible and $\overline{x}$ is an optimal solution to (P). If not, then $\overline{b}_r < 0$ for some $r \in B$. To get $\overline{y}^T = c_B^T B^{-1}$, we solve $\overline{y}^T B = c_B^T$. Then we can calculate $\overline{c}^T = c^T - \overline{y}^T A$. We need the row $\overline{w}$ of $\overline{A}$ indexed by the basic variable r. Let v^T be the row of B^{-1} indexed by the basic variable r. We can find v^T by solving $v^T B = e_k^T$, where r is the kth ordered element of B. Then $\overline{w} = v^T A$. (Actually, we only need the nonbasic portion of $\overline{w}$,so we compute only $\overline{w}_N = v^T N$.) If $\overline{w} \geq O$, then (P) is infeasible. Otherwise, use the dual ratio test to determine the entering variable s and replace B by $B - r + s$.

Example 11.5 Suppose we are given the initial tableau for the GGMC problem:

x_1	x_2	x_3	x_4	x_5	$-z$	
1	2	1	0	0	0	120
1	1	0	1	0	0	70
2	1	0	0	1	0	100
5	4	0	0	0	1	0

Assume that the current basis is $(4, 2, 1)$.

1. The basis matrix is

$$B = \begin{array}{c} \begin{matrix} 4 & 2 & 1 \end{matrix} \\ \begin{bmatrix} 0 & 2 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 2 \end{bmatrix} \end{array}$$

Find $\overline{x}$ by solving $Bx_B = b$:

$$\begin{array}{c} \begin{matrix} 4 & 2 & 1 \end{matrix} \\ \begin{bmatrix} 0 & 2 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 2 \end{bmatrix} \end{array} \begin{bmatrix} x_4 \\ x_2 \\ x_1 \end{bmatrix} = \begin{bmatrix} 120 \\ 70 \\ 100 \end{bmatrix}$$

and setting $\overline{x}_3 = \overline{x}_5 = 0$. Thus $\overline{x} = (80/3, 140/3, 0, -10/3, 0)$, and we can see that the basis is not primal feasible.

Find $\overline{y}$ by solving $y^T B = c_B^T$:

$$\begin{bmatrix} y_1 & y_2 & y_3 \end{bmatrix} \begin{array}{c} \begin{matrix} 4 & 2 & 1 \end{matrix} \\ \begin{bmatrix} 0 & 2 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 2 \end{bmatrix} \end{array} = \begin{array}{c} \begin{matrix} 4 & 2 & 1 \end{matrix} \\ \begin{bmatrix} 0 & 4 & 5 \end{bmatrix} \end{array}$$

Thus $\overline{y} = (1, 0, 2)^T$.

Calculate $\overline{c}_N^T = c_N - \overline{y}^T N$.

$$\begin{bmatrix} \overline{c}_3 & \overline{c}_5 \end{bmatrix} = \begin{array}{c} \begin{matrix} 3 & 5 \end{matrix} \\ \begin{bmatrix} 0 & 0 \end{bmatrix} \end{array} - \begin{bmatrix} 1 & 0 & 2 \end{bmatrix} \begin{array}{c} \begin{matrix} 3 & 5 \end{matrix} \\ \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} \end{array} = \begin{array}{c} \begin{matrix} 3 & 5 \end{matrix} \\ \begin{bmatrix} -1 & -2 \end{bmatrix} \end{array}$$

Since $\overline{c}$ is nonpositive, the current basis is dual feasible.

Now $\overline{x}_4 < 0$, so we need the row v^T of B^{-1} indexed by x_4. Since the ordered basis is $(4, 2, 1)$, we want the first row of B^{-1}, which we find by solving $v^T B = e_1$:

$$\begin{bmatrix} v_1 & v_2 & v_3 \end{bmatrix} \begin{array}{c} \begin{matrix} 4 & 2 & 1 \end{matrix} \\ \begin{bmatrix} 0 & 2 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 2 \end{bmatrix} \end{array} = \begin{array}{c} \begin{matrix} 4 & 2 & 1 \end{matrix} \\ \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \end{array}$$

The result is $v^T = [-1/3, 1, -1/3]$.

Calculate the nonbasic portion of the first row $\overline{w}$ of $\overline{A}$ (the row of $\overline{A}$ indexed by x_4) by $\overline{w}_N = v^T N$:

$$\begin{bmatrix} \overline{w}_3 & \overline{w}_5 \end{bmatrix} = \begin{bmatrix} -1/3 & 1 & -1/3 \end{bmatrix} \begin{matrix} \begin{matrix} 3 & 5 \end{matrix} \\ \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} \end{matrix} = \begin{matrix} \begin{matrix} 3 & \quad 5 \end{matrix} \\ \begin{bmatrix} -1/3 & -1/3 \end{bmatrix} \end{matrix}$$

Both of these numbers are negative, so both are potential pivot entries. Check the ratios of $\overline{c}_j/\overline{w}_j$ for $j = 3, 5$: $-1/(-1/3) = 3$, $-2/(-1/3) = 6$. The minimum ratio occurs when $j = 3$. So the variable x_3 enters the basis (while x_4 leaves), and the new basis is $(3, 2, 1)$.

2. The basis matrix is now

$$B = \begin{matrix} \begin{matrix} 3 & 2 & 1 \end{matrix} \\ \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 2 \end{bmatrix} \end{matrix}$$

Find $\overline{x}$ by solving $Bx_B = b$:

$$\begin{matrix} \begin{matrix} 3 & 2 & 1 \end{matrix} \\ \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 2 \end{bmatrix} \end{matrix} \begin{bmatrix} x_3 \\ x_2 \\ x_1 \end{bmatrix} = \begin{bmatrix} 120 \\ 70 \\ 100 \end{bmatrix}$$

and setting $\overline{x}_4 = \overline{x}_5 = 0$. Thus $\overline{x} = (30, 40, 10, 0, 0)$, and we can see that the basis is primal feasible, hence optimal.

□

11.5 Sensitivity Analysis

Suppose we have gone to the trouble of solving the linear program (P) and have found an optimal basis B with associated $\overline{T}$, $\overline{x}$, and $\overline{y}$, and then discover that we must solve a new problem in which either b or c has been changed. It turns out that we do not have to start over from scratch.

For example, suppose b is replaced by b'. This affects only the last column of $\overline{T}$, so the basis B is still dual feasible and we can solve the new problem with the dual simplex method starting from the current basis. Or suppose c is replaced by c'. This affects only the last row of $\overline{T}$, so the basis is still primal feasible and we can solve the new problem with the primal simplex method starting from the current basis.

11.5.1 Right Hand Sides

Suppose we replace b by $b+u$. In order for the current basis to remain optimal, we require that $B^{-1}(b+u) \geq O$; i.e., $B^{-1}b + B^{-1}u \geq O$, or $\overline{b} + B^{-1}u \geq O$. Find $v = B^{-1}u$ by solving $Bv = u$.

If the basis is still optimal, then $\overline{y}$ does not change, but the associated BFS becomes $\tilde{x}_N = O$, $\tilde{x}_B = \overline{x}_B + v$. Its objective function value is $c_B^T B^{-1}(b+u) = \overline{y}^T(b+u) = \overline{y}^T b + \overline{y}^T u = \overline{z} + \overline{y}^T u$. So as long as the basis remains optimal, the dual variables $\overline{y}$ give the rate of change of the optimal objective function value as a function of the changes in the right-hand sides. In economic contexts, such as the GGMC problem, the values $\overline{y}_i$ are sometimes known as *shadow prices.*

Exercise 11.6 Prove if $\overline{b} > O$ then there exists $\varepsilon > 0$ such that the basis will remain optimal if $\|u\| < \varepsilon$. □

Sometimes we are interested in parameterized changes in the right-hand sides. We can, for example, replace u by θu where θ is a scalar variable, and determine the optimal value as a function of θ. As we vary θ, if the basis becomes primal infeasible, then we can employ the dual simplex method to obtain a new optimal basis.

In particular, if we want to change only one component of b, use $b + \theta e_k$ for some k. Then v equals the kth column of B^{-1} (which sits in $\overline{T}$), and we require $\overline{b} + \theta v \geq O$.

Example 11.7 Let's vary the second right-hand side in the GGMC problem:

x_1	x_2	x_3	x_4	x_5	$-z$	
1	2	1	0	0	0	120
1	1	0	1	0	0	$70+\theta$
2	1	0	0	1	0	100
5	4	0	0	0	1	0

The final tableau used to be:

x_1	x_2	x_3	x_4	x_5	$-z$	
0	0	1	-3	1	0	10
0	1	0	2	-1	0	40
1	0	0	-1	1	0	30
0	0	0	-3	-1	1	-310

But with the presence of θ it becomes:

x_1	x_2	x_3	x_4	x_5	$-z$	
0	0	1	-3	1	0	$10 - 3\theta$
0	1	0	2	-1	0	$40 + 2\theta$
1	0	0	-1	1	0	$30 - \theta$
0	0	0	-3	-1	1	$-310 - 3\theta$

Note that the coefficients of θ match the entries in column x_4, the slack variable for the second constraint. The basis and tableau remain feasible and optimal if and only if $10 - 3\theta \geq 0$, $40 + 2\theta \geq 0$, and $30 - \theta \geq 0$; i.e., if and only if $-20 \leq \theta \leq 10/3$. In this range,

$$
\begin{aligned}
x^* &= (30 - \theta, 40 + 2\theta, 10 - 3\theta, 0, 0) \\
y^* &= (0, 3, 1) \\
z^* &= 310 + 3\theta
\end{aligned}
$$

Should θ drop below -20, perform a dual simplex pivot in the second row:

x_1	x_2	x_3	x_4	x_5	$-z$	
0	1	1	-1	0	0	$50 - \theta$
0	-1	0	-2	1	0	$-40 - 2\theta$
1	1	0	1	0	0	$70 + \theta$
0	-1	0	-5	0	1	$-350 - 5\theta$

This basis and tableau remain feasible and optimal if and only if $50 - \theta \geq 0$, $-40 - 2\theta \geq 0$, and $70 + \theta \geq 0$; i.e., if and only if $-70 \leq \theta \leq -20$. In this range,

$$
\begin{aligned}
x^* &= (70 + \theta, 0, 50 - \theta, 0, -40 - 2\theta) \\
y^* &= (0, 5, 0) \\
z^* &= 350 + 5\theta
\end{aligned}
$$

Should θ drop below -70, we would want to perform a dual smplex pivot in the third row, but the absence of a negative pivot entry indicates that the problem becomes infeasible.

Should θ rise above 10/3 in the penultimate tableau, perform a dual simplex pivot in the first row:

x_1	x_2	x_3	x_4	x_5	$-z$	
0	0	$-1/3$	1	$-1/3$	0	$-10/3+\theta$
0	1	$2/3$	0	$-1/3$	0	$140/3$
1	0	$-1/3$	0	$2/3$	0	$80/3$
0	0	-1	0	-2	1	-320

This basis and tableau remain feasible and optimal if and only if $-10/3+\theta \geq 0$; i.e., if and only if $\theta \geq 10/3$. In this range,

$$\begin{aligned} x^* &= (80/3, 140/3, 0, -10/3+\theta, 0) \\ y^* &= (1, 0, 2) \\ z^* &= 320 \end{aligned}$$

□

Exercise 11.8 Carry out the above process for the other two right-hand sides of the GGMC problem. Use the graph of the feasible reagion to check your results on the ranges of θ and the values of x^*, y^*, and z^* in these ranges. For each of the three right-hand sides, graph z^* as a function of θ. □

11.5.2 Objective Function Coefficients

Suppose we replace c by $c+u$. In order for the current basis to remain optimal, we calculate $\tilde{y}^T = (c_B+u_B)^T B^{-1} = c_B^T B^{-1} + u_B^T B^{-1} = \overline{y}^T + u_B^T B^{-1}$. We can find $v^T = u_B^T B^{-1}$ by solving $v^T B = u_B^T$. Then we require that $\tilde{c} \leq O$, where

$$\begin{aligned} \tilde{c}^T &= (c+u)^T - \tilde{y}^T A \\ &= c^T + u^T - (\overline{y}^T + v^T)A \\ &= c^T - \overline{y}^T A + u^T - v^T A \\ &= \overline{c}^T + u^T - v^T A. \end{aligned}$$

If the basis is still optimal, then $\overline{x}$ does not change, but the associated dual basic feasible solution becomes $\overline{y}+v$. Its objective function value is $\tilde{y}^T b = (\overline{y}+v)^T b = \overline{y}^T b + u_B^T B^{-1} b = \overline{z} + u^T \overline{x}_B$. So as long as the basis remains feasible, the primal variables $\overline{x}_B$ give the rate of change of the optimal objective function value as a function of the changes in the objective function coefficients.

Exercise 11.9 Prove that if $\overline{c} < 0$ then there exists $\varepsilon > 0$ such that the basis will remain optimal if $\|u\| < \varepsilon$. □

Sometimes we are interested in parameterized changes in the objective function coefficients. We can, for example, replace u by θu where θ is a scalar variable, and determine the optimal value as a function of θ. As we vary θ, if the basis becomes dual infeasible, then we can employ the primal simplex method to obtain a new optimal basis.

In particular, if we want to change only one component of c, use $c + \theta e_k$ for some k. If $k \in N$, then $v = O$ and we simply require that $\overline{c}_k + \theta \le 0$. If $k \in B$, then v equals the ℓth row of B^{-1} (which sits in $\overline{T}$), where k is the ℓth ordered element of the basis. In this case we require $\overline{c}^T + \theta(e_k^T - v^T A) \le O$. Note that $v^T A$ is the ℓth row of $\overline{A}$ and sits within $\overline{T}$.

Example 11.10 Let's vary the first objective function coefficient in the GGMC problem:

x_1	x_2	x_3	x_4	x_5	$-z$	
1	2	1	0	0	0	120
1	1	0	1	0	0	70
2	1	0	0	1	0	100
$5+\theta$	4	0	0	0	1	0

The final tableau used to be:

x_1	x_2	x_3	x_4	x_5	$-z$	
0	0	1	-3	1	0	10
0	1	0	2	-1	0	40
1	0	0	-1	1	0	30
0	0	0	-3	-1	1	-310

But with the presence of θ it becomes:

x_1	x_2	x_3	x_4	x_5	$-z$	
0	0	1	-3	1	0	10
0	1	0	2	-1	0	40
1	0	0	-1	1	0	30
0	0	0	$-3+\theta$	$-1-\theta$	1	$-310-30\theta$

Note that the nonbasic coefficients of θ match the negatives of the entries in the third row, because x_1 is the third element in the ordered basis of the tableau. The basis and tableau remain dual feasible and optimal if and only if $-3+\theta \le 0$ and $-1-\theta \le 0$; i.e., if and only if $-1 \le \theta \le 3$. In this range,

$$\begin{aligned} x^* &= (30, 40, 10, 0, 0) \\ y^* &= (0, 3-\theta, 1+\theta) \\ z^* &= 310 + 30\theta \end{aligned}$$

Should θ rise above 3, peform a simplex pivot in the fourth column:

x_1	x_2	x_3	x_4	x_5	$-z$	
0	$3/2$	1	0	$-1/2$	0	70
0	$1/2$	0	1	$-1/2$	0	20
1	$1/2$	0	0	$1/2$	0	50
0	$3/2-\theta/2$	0	0	$-5/2-\theta/2$	1	$-250-50\theta$

This basis and tableau remain dual feasible and optimal if and only if $3/2-\theta/2 \le 0$ and $-5/2-\theta/2 \le 0$; i.e., if and only if $\theta \ge 3$. In this range,

$$x^* = (50, 0, 70, 20, 0)$$
$$y^* = (0, 0, 5/2 + \theta/2)$$
$$z^* = 250 + 50\theta$$

On the other hand, should θ fall below -1 in the penultimate tableau, perform a simplex pivot in the fifth column:

x_1	x_2	x_3	x_4	x_5	$-z$	
0	0	1	-3	1	0	10
0	1	1	-1	0	0	50
1	0	-1	2	0	0	20
0	0	$1+\theta$	$-6-2\theta$	0	1	$-300-20\theta$

This basis and tableau remain dual feasible and optimal if and only if $1+\theta \le 0$ and $-6-2\theta \le 0$; i.e., if and only if $-3 \le \theta \le -1$. In this range,

$$x^* = (20, 50, 0, 0, 10)$$
$$y^* = (-1-\theta, 6+2\theta, 0)$$
$$z^* = 300 + 20\theta$$

Should θ fall below -3, perform a simplex pivot in the fourth column:

x_1	x_2	x_3	x_4	x_5	$-z$	
$3/2$	0	$-1/2$	0	1	0	40
$1/2$	1	$1/2$	0	0	0	60
$1/2$	0	$-1/2$	1	0	0	10
$3+\theta$	0	-2	0	0	1	-240

This basis and tableau remain dual feasible and optimal if and only if $3+\theta \le 0$; i.e., if and only if $\theta \le -3$. In this range,

$$x^* = (0, 60, 0, 10, 40)$$
$$y^* = (2, 0, 0)$$
$$z^* = 240$$

□

Exercise 11.11 Carry out the above process for the other four objective function coefficients of the GGMC problem. Use the graph of the feasible reagion to check your results on the ranges of θ and the values of x^*, y^*, and z^* in these ranges. For each of the five objective function coefficients, graph z^* as a function of θ. □

11.5.3 New Variable

Suppose we wish to introduce a new variable x_p with associated new column A_p of A, and new objective function coefficient c_p. This will not affect the last column of $\overline{T}$, so the current basis is still primal feasible. Compute $\overline{c}_p = c_p - \overline{y}^T A_p$. If $\overline{c}_p \leq 0$, then the current basis is still optimal. If $\overline{c}_p > 0$, then we can use the primal simplex method to find a new optimal basis. If c_p has not yet been determined, then we can easily find the range of c_p in which the current basis remains optimal by demanding that $c_p \leq \overline{y}^T A_p$. This calculation is sometimes called *pricing out* a new variable or activity.

Example 11.12 Suppose the GGMC proposes producing a new product: bric-a-brac. One kilogram of bric-a-brac requires 1 hours of labor, 2 units of wood, and 2 units of metal. How much profit c_6 must be realized before it becomes advantageous to produce this product? At optimality we saw previously that $\overline{y} = (0, 3, 1)$. Pricing out the bric-a-brac:

$$\overline{y}^T A_6 = [0, 3, 1] \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} = 7.$$

If the profit c_6 from bric-a-brac is no more than 7 dollars then it is not worthwhile to produce it, but if it is more than 7 dollars then a new optimal solution must be found.

Let's suppose $c_6 = 8$. Then $\overline{c}_6 = 8 - 7 = 1$. We do not have to start from scratch, but can amend the optimal tableau by appending a new column $d = B^{-1} A_6$, which can be found by solving $Bd = A_6$:

$$\begin{matrix} 3 & 2 & 1 \end{matrix}$$
$$\begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} d_3 \\ d_2 \\ d_1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$$

The solution is $d = (-4, 3, -1)^T$. Appending this column, and the value of $\overline{c}_6$, to the final tableau, gives:

x_1	x_2	x_3	x_4	x_5	x_6	$-z$	
0	0	1	−3	1	−4	0	10
0	1	0	2	−1	3	0	40
1	0	0	−1	1	−1	0	30
0	0	0	−3	−1	1	1	−310

One simplex pivot brings us to the new optimal tableau:

x_1	x_2	x_3	x_4	x_5	x_6	$-z$	
0	4/3	1	−1/3	−1/3	0	0	190/3
0	1/3	0	2/3	−1/3	1	0	40/3
1	1/3	0	−1/3	2/3	0	0	130/3
0	−1/3	0	−11/3	−2/3	0	1	−970/3

The new optimal strategy is to produce $43\frac{1}{3}$ kilograms of gadgets and $13\frac{1}{3}$ kilograms of bric-a-brac for a profit of $323\frac{1}{3}$ dollars. □

11.5.4 New Constraint

Suppose we wish to add a new constraint $a_p^T x \leq b_p$ to $(\hat{P})$. Introduce a new slack variable x_{n+m+1} with cost zero and add the equation $a_p^T x + x_{n+m+1}$ to (P). Enlarge the current basis by putting x_{n+m+1} into it. This new basis will still be dual feasible for the new system (you should be certain you can verify this), but it won't be primal feasible if the old BFS $\overline{x}$ does not satisfy the new constraint. In this case, use the dual simplex method to find a new optimal basis.

Example 11.13 After solving the GGMC problem, let's add the constraint $x_1 \leq 18$. Using a new slack variable x_6, we enlarge the final tableau:

x_1	x_2	x_3	x_4	x_5	x_6	$-z$	
0	0	1	−3	1	0	0	10
0	1	0	2	−1	0	0	40
1	0	0	−1	1	0	0	30
1	0	0	0	0	1	0	18
0	0	0	−3	−1	0	1	−310

Make the tableau basic again with a preliminary pivot in the first column:

x_1	x_2	x_3	x_4	x_5	x_6	$-z$	
0	0	1	-3	1	0	0	10
0	1	0	2	-1	0	0	40
1	0	0	-1	1	0	0	30
0	0	0	1	-1	1	0	-12
0	0	0	-3	-1	0	1	-310

Pivot to optimality with two dual simplex pivots:

x_1	x_2	x_3	x_4	x_5	x_6	$-z$	
0	0	1	-2	0	1	0	-2
0	1	0	1	0	-1	0	52
1	0	0	0	0	1	0	18
0	0	0	-1	1	-1	0	12
0	0	0	-4	0	-1	1	-298

x_1	x_2	x_3	x_4	x_5	x_6	$-z$	
0	0	$-\frac{1}{2}$	1	0	$-\frac{1}{2}$	0	1
0	1	$\frac{1}{2}$	0	0	$-\frac{1}{2}$	0	51
1	0	0	0	0	1	0	18
0	0	$-\frac{1}{2}$	0	1	$-\frac{3}{2}$	0	13
0	0	-2	0	0	-3	1	-294

So the new optimal solution is to produce 18 kilograms of gadgets and 51 kilograms of gewgaws, with a profit of 294 dollars. □

11.6 LP's With Equations

Suppose our original LP $(\hat{P})$ consists only of equations. We could convert the problem into standard form by converting each equation into two inequalities, but it turns out that the problem can be solved without increasing the number of constraints.

Here is one way; see Chvátal for improvements and a more detailed explanation. First use Gaussian elimination to identify and remove redundant equations. Then multiply some constraints by -1, if necessary, so that all right-hand sides are nonnegative. Then, for each $i = 1, \ldots, m$ introduce a different new artificial variable x_{n+i} with a coefficient of $+1$ into the ith constraint. The Phase I problem minimizes the sum (maximizes the negative of the sum) of the artificial variables. The set of artificial variables is an initial primal feasible basis.

Upon completion of the Phase I problem, either we will have achieved a nonzero objective function value, demonstrating that the original problem is infeasible, or else we will have achieved a zero objective function value with all artificial variables necessarily equalling zero. Suppose in this latter case there is an artificial variable remaining in the basis. In the final tableau, examine the row associated with this variable. It must have at least one nonzero entry in some column corresponding to one of the original variables, otherwise we would discover that the set of equations of the original problem is linearly dependent. Pivot on any such nonzero entry, whether positive or negative, to remove the artificial variable from the basis and replace it with one of the original variables. This "artificial" pivot will not change primal feasibility since the pivot row has a zero in the last column. Repeating this process with each artificial variable in the basis, we obtain a primal feasible basis for the original problem. Now throw away the artificial variables and proceed with the primal simplex method to solve the problem with the original objective function.

If our original problem has a mixture of equations and inequalities, then we can add slack variables to the inequalities to turn them into equations. In this case we may be able to get away with fewer artificial variables by using some of the slack variables in the initial feasible basis.

Example 11.14 To solve the linear program:

$$\begin{gathered} \max 5x_1 + 4x_2 \\ \text{s.t. } x_1 + 2x_2 = 120 \\ x_1 + x_2 \leq 70 \\ 2x_1 + x_2 \leq 100 \\ x_1, x_2 \geq 0 \end{gathered}$$

we insert an artificial variable x_3 into the first equation and slack variables into the next two inequalities. To test feasibility we try to minimize x_3 (or maximize $-x_3$).

$$\begin{gathered} \max -x_3 \\ \text{s.t. } x_1 + 2x_2 + x_3 = 120 \\ x_1 + x_2 + x_4 = 70 \\ 2x_1 + x_2 + x_5 = 100 \\ x_1, x_2, x_3, x_4, x_5 \geq 0 \end{gathered}$$

This is represented by the tableau:

x_1	x_2	x_3	x_4	x_5	$-z$	
1	2	1	0	0	0	120
1	1	0	1	0	0	70
2	1	0	0	1	0	100
0	0	−1	0	0	1	0

Perform a preliminary pivot in the third column to make the tableau basic:

x_1	x_2	x_3	x_4	x_5	$-z$	
1	2	1	0	0	0	120
1	1	0	1	0	0	70
2	1	0	0	1	0	100
1	2	0	0	0	1	120

One pivot results in optimality:

x_1	x_2	x_3	x_4	x_5	$-z$	
1/2	1	1/2	0	0	0	60
1/2	0	−1/2	1	0	0	10
3/2	0	−1/2	0	1	0	40
0	0	−1	0	0	1	0

Since the optimal objective function value is zero, the linear program is found to be feasible. The artificial variable x_3 is nonbasic, so its column can be deleted. Replace the Phase I objective function with the Phase II objective function:

x_1	x_2	x_4	x_5	$-z$	
1/2	1	0	0	0	60
1/2	0	1	0	0	10
3/2	0	0	1	0	40
5	4	0	0	1	0

Perform a preliminary pivot in the second column to make the tableau basic again:

x_1	x_2	x_4	x_5	$-z$	
1/2	1	0	0	0	60
1/2	0	1	0	0	10
3/2	0	0	1	0	40
3	0	0	0	1	−240

Now one more pivot achieves optimality:

x_1	x_2	x_4	x_5	$-z$	
0	1	−1	0	0	50
1	0	2	0	0	20
0	0	−3	1	0	10
0	0	−6	0	1	−300

□

11.7 LP's With Unrestricted Variables

Suppose our original problem has some unrestricted variable x_p. We can replace x_p with the difference $x_p^+ - x_p^-$ of two nonnegative variables, as described in an earlier section. Using the revised simplex method and some simple bookkeeping, we do not increase the size of the problem by this conversion by any significant amount.

11.8 LP's With Bounded Variables

Suppose our original problem has some variables with upper and/or lower bounds, $\ell_j \le x_j \le u_j$, where $\ell_j = -\infty$ if there is no finite lower bound, and $u_j = +\infty$ if there is no finite upper bound. A variable with either a finite upper or lower bound is called *bounded* ; otherwise the variable is *unrestricted* or *free*. We can modify the simplex method easily to handle this case also.

The main change is this: At any stage in the simplex method we will have a basis B, a basic tableau $\overline{T}$, and a selection of values for the nonbasic variables x_N. Instead of getting a BFS by setting $\overline{x}_N = O$, we will instead set each bounded nonbasic variable to one of its finite bounds, and each unrestricted nonbasic variable to zero. Given such a selection, we can then solve for the values of the basic variables by solving $B\overline{x}_B = b - N\overline{x}_N$. If the value of each basic variable falls within its bounds, then we have a *normal basic feasible solution.* The important thing to realize is that we will have more than one normal BFS associated with a fixed tableau $\overline{T}$, depending upon the choices of the values of the nonbasic variables.

Suppose that $\overline{c}_j \le 0$ for every $j \in N$ for which $\overline{x}_j < u_j$, and that $\overline{c}_j \ge 0$ for every $j \in N$ for which $\overline{x}_j > \ell_j$. Then the value of $\overline{z}$ cannot be increased by increasing any nonbasic variable currently at its lower bound, or decreasing any nonbasic variable currently at its upper bound. So the corresponding normal BFS is optimal. (Be certain you are convinced of this and can write this out more formally.)

Suppose that $\overline{c}_s > 0$ for some $s \in N$ for which $\overline{x}_s < u_s$. Then we can try adding $t \ge 0$ to $\overline{x}_s$, keeping the values of the other nonbasic variables constant, and monitoring the changes in the basic variables:

$$\begin{aligned}
\tilde{x}_s &= \overline{x}_s + t \\
\tilde{x}_j &= \overline{x}_j, \quad j \in N - s \\
\tilde{x}_B &= B^{-1}(b - N\tilde{x}_N) \\
&= B^{-1}b - B^{-1}N\overline{x}_N - B^{-1}A_s t \\
&= \overline{x}_B - dt
\end{aligned}$$

where $d = B^{-1}A_s$. Choose t as large as possible so that x_s does not exceed its upper bound and no basic variable drifts outside its upper or lower bound. If t can be made arbitrarily large, then the LP is unbounded, and we stop. If x_s hits its upper bound first, then we do not change the basis B; we just have a new normal BFS with x_s at its upper bound. If one of the basic variables x_r hits its upper or lower bound first, then s enters the basis, r leaves the basis, and x_r is nonbasic at its upper or lower bound.

Suppose on the other hand that $\overline{c}_s < 0$ for some $s \in N$ for which $\overline{x}_s > \ell_s$. Then we can try adding $t \leq 0$ to $\overline{x}_s$, keeping the values of the other nonbasic variables constant, and monitoring the changes in the basic variables using the same equations as above. Choose t as negative as possible so that x_s does not exceed its lower bound and no basic variable drifts outside its upper or lower bound. If t can be made arbitrarily negative, then the LP is unbounded, and we stop. If x_s hits its lower bound first, then we do not change the basis B; we just have a new normal BFS with x_s at its lower bound. If one of the basic variables x_r hits its upper or lower bound first, then s enters the basis, r leaves the basis, and x_r is nonbasic at its upper or lower bound.

Example 11.15 Assume that we require $5 \leq x_1 \leq 45$, $0 \leq x_2 \leq 45$, and that the three slack variables have lower bounds of 0 and upper bounds of $+\infty$.

1. We could try to initialize the simplex method with the basis $(3, 4, 5)$, which corresponds to the tableau:

x_1	x_2	x_3	x_4	x_5	$-z$	
1	2	1	0	0	0	120
1	1	0	1	0	0	70
2	1	0	0	1	0	100
5	4	0	0	0	1	0

Let us choose to set both nonbasic variables to their lower bounds. Using the equations from the above tableau to solve for the values of the basic variables, we find:

Nonbasic Variables	Basic Variables
$x_1 = \ell_1 = 5$	$x_3 = 115$
$x_2 = \ell_2 = 0$	$x_4 = 65$
	$x_5 = 90$

Fortunately, the values of the three basic variables fall within their required bounds, so we have an initial normal basic feasible solution.

Since $\overline{c}_1$ and $\overline{c}_2$ are both positive, we wish to increase the value of x_1 or x_2. As they are each presently at their lower bounds, we may increase either one. We will choose to increase x_1 but keep x_2 fixed. Then changes in the basic variables depend upon the entries of the column x_1 of the tableau:

Nonbasic Variables	Basic Variables
$x_1 = \ell_1 = 5 + t$	$x_3 = 115 - t$
$x_2 = \ell_2 = 0$	$x_4 = 65 - t$
	$x_5 = 90 - 2t$

Choose $t \geq 0$ as large as possible, keeping all variable values within the required bounds. Thus $5 + t \leq 45$, $115 - t \geq 0$, $65 - t \geq 0$, and $90 - 2t \geq 0$. This forces $t = 40$, and this value of t is determined by the nonbasic variable x_1. Therefore we do not change the basis, but merely set x_1 to its upper bound of 45. Then we have:

Nonbasic Variables	Basic Variables
$x_1 = u_1 = 45$	$x_3 = 75$
$x_2 = \ell_2 = 0$	$x_4 = 25$
	$x_5 = 10$

2. Now we still have $\bar{c}_1$ and $\bar{c}_2$ both positive, which means we wish to increase x_1 and x_2. But x_1 is at its upper bound; hence cannot be increased. Hence we fix x_1, increase x_2, and use the second column of the tableau to determine the changes of the basic variable values:

Nonbasic Variables	Basic Variables
$x_1 = 45$	$x_3 = 75 - 2t$
$x_2 = 0 + t$	$x_4 = 25 - t$
	$x_5 = 10 - t$

Choose $t \geq 0$ as large as possible, keeping all variable values within the required bounds. Thus $0 + t \leq 45$, $75 - 2t \geq 0$, $25 - t \geq 0$, and $10 - t \geq 0$. This forces $t = 10$, and this value of t is determined by the basic variable x_5. This variable becomes nonbasic at its lower bound value, and x_2 enters the basis. The new basis is $(3, 4, 2)$, and the variable values are:

Nonbasic Variables	Basic Variables
$x_1 = u_1 = 45$	$x_3 = 55$
$x_5 = \ell_5 = 0$	$x_4 = 15$
	$x_2 = 10$

3. The current tableau is now:

x_1	x_2	x_3	x_4	x_5	$-z$	
-3	0	1	0	-2	0	-80
-1	0	0	1	-1	0	-30
2	1	0	0	1	0	100
-3	0	0	0	-4	1	-400

$\overline{c}_1$ and $\overline{c}_5$ are both negative, indicating that we wish to decrease either x_1 or x_5. x_5 is already at its lower bound, and cannot be decreased, but we can decrease x_1. Hence we fix x_5, decrease x_1, and use the first column of the tableau to determine the changes of the basic variable values:

Nonbasic Variables	Basic Variables
$x_1 = 45 + t$	$x_3 = 55 + 3t$
$x_5 = 0$	$x_4 = 15 + t$
	$x_2 = 10 - 2t$

Choose $t \leq 0$ as negative as possible, keeping all variable values within the required bounds. Thus $45 + t \geq 0$, $55 + 3t \geq 0$, $15 + t \geq 0$, and $10 - 2t \leq +\infty$. This forces $t = -15$, and this value of t is determined by the basic variable x_4. This variable becomes nonbasic at its lower bound value, and x_1 enters the basis. The new basis is $(3, 1, 2)$, and the variable values are:

Nonbasic Variables	Basic Variables
$x_4 = \ell_4 = 0$	$x_3 = 10$
$x_5 = \ell_5 = 0$	$x_1 = 30$
	$x_2 = 40$

4. The current tableau is now:

x_1	x_2	x_3	x_4	x_5	$-z$	
0	0	1	-3	1	0	10
1	0	0	-1	1	0	30
0	1	0	2	-1	0	40
0	0	0	-3	-1	1	-310

$\overline{c}_4$ and $\overline{c}_5$ are both negative, indicating that we wish to decrease either x_4 or x_5. Each of these variables is currently at its lower bound, however, so our current solution is optimal. We get the (by now familiar) objective function value by using the original (or the current) equation involving z: $z = 5x_1 + 4x_2 = 310$.

□

What changes do we need to make to the Phase I procedure to find an initial normal basic feasible solution? One method is to assume that we have equations $Ax = b$ and introduce artificial variables $x_{n+1}, \ldots, x_{n+m}$ as before, one for each constraint. Declare each original variable to be nonbasic with Phase I objective function coefficient zero, and set each original variable x_j to a value $\overline{x}_j$, which is either its lower bound or its upper bound (or zero if it is an unrestricted variable). Determine the value $\overline{x}_{n+i}$ of each artificial variable x_{n+i} by

$$\overline{x}_{n+i} = b_i - \sum_{j=1}^{n} a_{ij}\overline{x}_j.$$

If $\overline{x}_{n+i} \geq 0$, give this variable a lower bound of zero, an upper bound of $+\infty$, and a Phase I objective function coefficient of -1. If $\overline{x}_{n+i} < 0$, give this variable a lower bound of $-\infty$, an upper bound of zero, and a Phase I objective function coefficient of $+1$. Then we will have an initial normal basic feasible solution, and we attempt to find a normal basic feasible solution for the original problem by maximizing the Phase I objective function.

11.9 Minimization Problems

We can solve an LP which is a minimization problem by multiplying the objective function by -1 and solving the resulting maximization problem. Alternatively, we can keep the original objective function and make the obvious changes in the criterion for entering variables. For example, if all variables are restricted to be nonnegative, then x_s is a candidate to enter the basis if $\overline{c}_s < 0$ (as opposed to $\overline{c}_s > 0$ in a maximization problem).

Exercise 11.16 For each of the computational exercises in this section in which full tableaux are used, repeat the calculations using the revised simplex or the revised dual simplex methods. □

12 Exercises: More On the Simplex Method

Exercise 12.1 Discuss why it makes economic sense for the shadow prices to be zero for constraints of $(\hat{P})$ that are not satisfied with equality by an optimal basic feasible solution $\overline{x}$. □

Exercise 12.2 Devise a perturbation method to avoid cycling in the dual simplex method, and prove that it works. □

Exercise 12.3 Devise a "Phase I" procedure for the dual simplex method, in case the initial basis consisting of the set of slack variables is not dual feasible. □

Exercise 12.4 If $x^1, \ldots, x^k \in \mathbf{R}^n$, and $t_1, \ldots, t_k$ are nonnegative real numbers that sum to 1, then $t_1 x^1 + \cdots + t_k x^k$ is called a *convex combination* of $x^1, \ldots, x^k$. A *convex set* is a set that is closed under convex combinations. Prove that the set $\{x \in \mathbf{R}^n : Ax \leq b,\ x \geq O\}$ is a convex set. □

Exercise 12.5 Consider the linear programs (P) and $(P(u))$:

$$\begin{array}{rlcrl} & \max c^T x & \qquad & & \max c^T x \\ \text{s.t.} & Ax = b & & \text{s.t.} & Ax = b + u \\ & x \geq O & & & x \geq O \\ & & & & \\ & (P) & & & (P(u)) \end{array}$$

Assume that (P) has an optimal objective function value z^*. Suppose that there exists a vector y^* and a positive real number ε such that the optimal objective function value $z^*(u)$ of $(P(u))$ equals $z^* + u^T y^*$ whenever $\|u\| < \varepsilon$. Prove or disprove: y^* is an optimal solution to the dual of (P). If the statement is false, what additional reasonable assumptions can be made to make it true? Justify your answer. □

Exercise 12.6 Suppose B is an optimal basis for (P). Suppose that $u^1, \ldots, u^k$ are vectors such that B remains an optimal basis if b is replaced by any *one* of $b + u^1, \ldots, b + u^k$. Let $t_1, \ldots, t_k$ be nonnegative real numbers that sum to 1. Prove that B is also an optimal basis for $b + t_1 u^1 + \cdots + t_k u^k$. (This is sometimes called the *100% rule*). □

Exercise 12.7 Give a precise explanation of the following statement: If (P) and (D) are a dual pair of linear programs, performing a dual simplex pivot in a tableau of (P) is "the same" as performing a primal pivot in a tableau of (D). □

Exercise 12.8 Here is another way to turn a system of equations into an equivalent system of inequalities: Show that $(x_1, \ldots, x_n)$ satisfies

$$\sum_{j=1}^{n} a_{ij} x_j = b_i, \quad i = 1, \ldots, m$$

if and only if $(x_1, \ldots, x_n)$ satisfies

$$\sum_{j=1}^{n} a_{ij} x_j \leq b_i, \quad i = 1, \ldots, m$$
$$\sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij} x_j \geq \sum_{i=1}^{m} b_i$$

□

Exercise 12.9 Read Chapter 11 of Chvátal for a good example of using linear programming to model, solve, and report on a "real-life" problem. □

Exercise 12.10 Chvátal, 1.6–1.9, 5.4–5.7, 5.9–5.10, 7.1–7.4, 8.1–8.9, 9.4, 9.6–9.7, 10.1–10.5. You should especially choose some problems to test your understanding of sensitivity analysis. □

13 More On Linear Systems and Geometry

This section focuses on the structural properties of sets described by a system of linear constraints in $\mathbf{R}^n$; e.g., feasible regions of linear programs. Such a set is called a *(convex) polyhedron*. We will usually only consider the case $n \geq 1$.

13.1 Structure Theorems

Theorem 13.1 *If a system of m linear equations has a nonnegative solution, then it has a solution with at most m variables nonzero.*

Proof. Suppose the system in question is

$$\begin{array}{c} Ax = b \\ x \geq O \end{array}$$

Eliminate redundant equations, if necessary. If there is a feasible solution, then Phase I of the simplex method delivers a basic feasible solution. In this solution, the only variables that could be nonzero are the basic ones, and there are at most m basic variables. □

Exercise 13.2 Extend the above theorem, if possible, to mixtures of linear equations and inequalities, with mixtures of free and nonnegative variables. □

Theorem 13.3 *Every infeasible system of linear inequalities in n variables contains an infeasible subsystem of at most $n+1$ inequalities.*

Proof. Suppose the system in question is $Ax \leq b$. If this system is infeasible, then the following system is feasible:

$$\begin{array}{c} A^T y = O \\ b^T y < 0 \\ y \geq O \end{array}$$

By rescaling a feasible solution to the above system by a positive amount, we conclude that the following system (which has $n+1$ equations) is feasible:

$$\begin{array}{c} A^T y = O \\ b^T y = -1 \\ y \geq O \end{array}$$

By the previous theorem, there is a feasible solution $\hat{y}$ in which at most $n+1$ of the variables are positive. Let $S = \{i : \hat{y}_i > 0\}$. Then the system

$$\sum_{j=1}^{n} a_{ij}x_j \leq b_i, \quad i \in S$$

is infeasible since these are the only inequalities used in the contradictory inequality produced by the multipliers $\hat{y}_i$. □

Exercise 13.4 Extend the above theorem, if possible, to mixtures of linear equations and inequalities, with mixtures of free and nonnegative variables. □

Definition 13.5 Assume that A has full row rank. Recall that $\overline{x}$ is a basic feasible solution to the set $S = \{x : Ax = b, \ x \geq O\}$ if there exists a basis B such that $\overline{x} = (\overline{x}_B, \overline{x}_N)$, where $\overline{x}_N = O$ and $\overline{x}_B = B^{-1}b \geq O$. Recall that $\overline{w}$ is a basic feasible direction for S if there exists a basis B and an index $s \in N$ such that $\overline{w}_s = 1$, $\overline{w}_{N-s} = O$, and $\overline{w}_B = -B^{-1}A_s \geq O$. (These are the coefficients of t when it is discovered that an LP is unbounded.) Note that $A\overline{w} = O$.

Exercise 13.6 Assume that A has full row rank. What are the appropriate definitions of *normal basic feasible solution* and *normal basic feasible direction* for the set $S = \{x : Ax = b, \ \ell \leq x \leq u\}$? □

Theorem 13.7 *Assume that A has full row rank. Let $S = \{x : Ax = b, \ x \geq O\}$. Then there exist vectors $v^1, \ldots, v^M$ and vectors $w^1, \ldots, w^N$ such that $S = \{\sum_{i=1}^M r_i v^i + \sum_{i=1}^N s_i w^i : \sum_{i=1}^M r_i = 1, \ r \geq O, \ s \geq O\}$.*

PROOF. Let $v^1, \ldots, v^M$ be the set of basic feasible solutions and $w^1, \ldots, w^M$ be the set of basic feasible directions for S.

First, let $\overline{x} = \sum_{i=1}^M r_i v^i + \sum_{i=1}^N s_i w^i$ where $r, s \geq O$ and $\sum_{i=1}^M r_i = 1$. Then $\overline{x} \geq O$ since all basic feasible solutions and basic feasible directions are nonnegative, and $r, s \geq O$. Also $\overline{x} \in S$, since

$$\begin{aligned}
A\overline{x} &= A(\sum_{i=1}^M r_i v^i + \sum_{i=1}^N s_i w^i) \\
&= \sum_{i=1}^M r_i A v^i + \sum_{i=1}^N s_i A w^i \\
&= \sum_{i=1}^M r_i b + \sum_{i=1}^N s_i O \\
&= (\sum_{i=1}^M r_i) b \\
&= b.
\end{aligned}$$

Now, assume that $\overline{x} \in S$ but $\overline{x}$ cannot be written in the form $\overline{x} = \sum_{i=1}^{M} r_i v^i + \sum_{i=1}^{N} s_i w^i$ where $\sum_{i=1}^{M} r_i = 1$ and $r, s \geq O$. We need to show that $\overline{x} \notin S$. Assume otherwise. Now we are assuming that the following system is infeasible:

$$\begin{bmatrix} v^1 & \cdots & v^M & w^1 & \cdots & w^N \\ 1 & \cdots & 1 & 0 & \cdots & 0 \end{bmatrix} \begin{bmatrix} r \\ s \end{bmatrix} = \begin{bmatrix} \overline{x} \\ 1 \end{bmatrix}$$
$$r, s \geq O$$

But then there exists a vector $[y^T, t]$ such that

$$\begin{bmatrix} y^T & t \end{bmatrix} \begin{bmatrix} v^1 & \cdots & v^M & w^1 & \cdots & w^N \\ 1 & \cdots & 1 & 0 & \cdots & 0 \end{bmatrix} \geq \begin{bmatrix} O^T & O^T \end{bmatrix}$$

and

$$\begin{bmatrix} y^T & t \end{bmatrix} \begin{bmatrix} \overline{x} \\ 1 \end{bmatrix} < 0$$

That is to say,

$$y^T v^i + t \geq 0, \quad i = 1, \ldots, M$$
$$y^T w^i \geq O, \quad i = 1, \ldots, N$$
$$y^T \overline{x} + t < 0$$

Let $c = -y$ and consider the LP

$$\max c^T x$$
$$\text{s.t. } Ax = b$$
$$x \geq O$$

The LP is feasible since $\overline{x} \in S$. The LP is bounded since $c^T w \leq 0$ for all basic feasible directions. Therefore the LP has an optimal basic feasible solution. But the above calculations show that the objective function value of $\overline{x}$ exceeds that of every basic feasible solution, which is a contradiction, since there must be at least one basic feasible solution that is optimal. Therefore $\overline{x}$ can be written in the desired form after all. □

Theorem 13.8 (Finite Basis Theorem) *The same holds for systems of linear equations and inequalities where some variables are free and others nonnegative. In particular, it holds for a set of the form* $S = \{x : Ax \leq b\}$.

PROOF. Convert any given system (I) to another (I') consisting of equations and nonnegative variables by introducing new slack variables to convert inequalities to equations and writing unrestricted variables as the differences of nonnegative variables. There is a linear mapping from the feasible region of (I') onto the feasible region to (I) (that mapping which recovers the values of the originally unrestricted variables and projects away the slack variables). The result now follows from the validity of the theorem for (I'). □

Theorem 13.9 (Minkowski) *Assume that A has full row rank. Let* $S = \{x : Ax = O,\ x \geq O\}$. *Then there exist vectors* $w^1, \ldots, w^N$ *such that* $S = \{\sum_{i=1}^{N} s_i w^i : s \geq O\}$.

PROOF. In this case there is only one basic feasible solution, namely, O (although there may be many basic feasible tableaux). □

Theorem 13.10 (Minkowski) *The same holds for systems of linear equations and inequalities with zero right-hand sides, where some variables are free and others nonnegative. In particular, it holds for a set of the form* $S = \{x : Ax \leq O\}$.

Theorem 13.11 *If* S *is a set of the form* $\{\sum_{i=1}^{M} r_i v^i + \sum_{i=1}^{N} s_i w^i : \sum_{i=1}^{M} r_i = 1,\ r \geq O,\ s \geq O\}$, *then* S *is also a set of the form* $\{x : Ax \leq b\}$.

PROOF. Consider $S' = \{(r,s,x) : \sum_{i=1}^{M} r_i v^i + \sum_{i=1}^{N} s_i w^i - x = O,\ \sum_{i=1}^{M} r_i = 1,\ r \geq O,\ s \geq O\}$. Then $S' = \{(r,s,x) : \sum_{i=1}^{M} r_i v^i + \sum_{i=1}^{N} s_i w^i - x \leq O,\ \sum_{i=1}^{M} r_i v^i + \sum_{i=1}^{N} s_i w^i - x \geq O,\ \sum_{i=1}^{M} r_i \leq 1,\ \sum_{i=1}^{M} r_i \geq 1,\ r \geq O,\ s \geq O\}$. Then a description for S in terms of linear inequalities is obtained from that of S' by using Fourier-Motzkin elimination to eliminate the variables $r_1, \ldots, r_M, s_1, \ldots, s_N$. □

Theorem 13.12 *If* S *is a set of the form* $\{\sum_{i=1}^{N} s_i w^i : s \geq O\}$, *then* S *is also a set of the form* $\{x : Ax \leq O\}$.

Exercise 13.13 Illustrate Theorem 13.11 with the cube, having extreme points $\{(\pm 1, \pm 1, \pm 1)\}$, and also with the octahedron, having extreme points $\{(\pm 1, 0, 0), (0, \pm 1, 0), (0, 0, \pm 1)\}$. □

13.2 Extreme Points and Facets

Definition 13.14 Let $S \subseteq \mathbf{R}^n$. An *extreme point* of S is a point $x \in S$ such that

$$\left\{ \begin{array}{c} x = \sum_{i=1}^{m} \lambda_i x^i \\ \sum_{i=1}^{m} \lambda_i = 1 \\ \lambda_i > 0,\ i = 1, \ldots, m \\ x^i \in S,\ i = 1, \ldots, m \end{array} \right\} \text{ implies } x^i = x,\ i = 1, \ldots, m$$

I.e., x cannot be written as a convex combination of points in S other than copies of x itself.

Theorem 13.15 *Let $P = \{x \in \mathbf{R}^n : a^{iT}x \le b_i,\ i = 1, \ldots, m\}$ be a polyhedron and $v \in P$. Then v is an extreme point of P if and only if* $\dim \operatorname{span}\{a^i : a^{iT}v = b_i\} = n$.

Proof. Let $T = \{a^i : a^{iT}v = b_i\}$. Note that $a^{iT}v < b_i$ for $a^i \notin T$. Assume that $\dim \operatorname{span} T < n$. Then there exists a nonzero $a \in \mathbf{R}^n$ such that $a^T a^i = 0$ for all $a^i \in T$. Consider $v \pm \varepsilon a$ for sufficiently small $\varepsilon > 0$. Then

$$a^{iT}(v \pm \varepsilon a) = a^{iT}v \pm \varepsilon a^{iT}a = \begin{cases} a^{iT}v & = & b_i & \text{if } a^i \in T \\ a^{iT}v \pm \varepsilon a^{iT}a & < & b_i & \text{if } a^i \notin T \end{cases}$$

Thus $v \pm \varepsilon a \in P$. But $v = \frac{1}{2}(v + \varepsilon a) + \frac{1}{2}(v - \varepsilon a)$, so v is not an extreme point of P.

Now suppose that $\dim \operatorname{span} T = n$. Assume that $v = \sum_{i=1}^m \lambda_i x^i$, $\sum_{i=1}^m \lambda_i = 1$, $\lambda_i > 0$, $i = 1, \ldots, m$, and $x^i \in P$, $i = 1, \ldots, m$. Note that if $a^k \in T$, then

$$\begin{aligned} b_k &= a^{kT}v \\ &= \sum_{i=1}^m \lambda_i a^{kT}x^i \\ &\le \sum_{i=1}^m \lambda_i b_k \\ &= b_k \end{aligned}$$

This forces $a^{kT}x^i = b_k$ for all $a^k \in T$, for all $i = 1, \ldots, m$. Hence for any fixed i we have $a^{kT}(v - x^i) = 0$ for all $a^k \in T$. Because $\dim \operatorname{span} T = n$ we conclude $v - x^i = O$ for all i. Therefore v is an extreme point of P. □

Definition 13.16 Let $S = \{x^1, \ldots, x^m\} \subset \mathbf{R}^n$.

1. If $\lambda_1, \ldots, \lambda_m \in \mathbf{R}$ such that $\sum_{i=1}^m \lambda_i = 1$ then $\sum_{i=1}^m \lambda_i x^i$ is called an *affine combination* of $x^1, \ldots, x^m$.

2. If the only solution to

$$\sum_{i=1}^m \lambda_i x^i = O$$
$$\sum_{i=1}^m \lambda_i = 0$$

is $\lambda_i = 0$, $i = 1, \ldots, m$, then the set S is *affinely independent*; otherwise, it is *affinely dependent*.

Exercise 13.17 Let $S = \{x^1, \ldots, x^m\} \subset \mathbf{R}^n$.

1. Prove that S is affinely independent if and only if and only if there is no j such that x^j can be written as an affine combination of $\{x^i \in S : i \neq j\}$.

2. Prove that S is affinely independent if and only if the set $S' = \{(x^1, 1), \ldots, (x^m, 1)\} \subset \mathbf{R}^{n+1}$ is linearly independent.

$\square$

Definition 13.18 Let $S \subseteq \mathbf{R}^n$. The *dimension* of S, $\dim S$, is defined to be one less than the maximum number of affinely independent points in S. (Why does this definition make sense?)

Exercise 13.19 What is $\dim \mathbf{R}^n$? $\square$

Definition 13.20 Let $S \subseteq \mathbf{R}^n$. Then S is *full-dimensional* if $\dim S = n$.

Definition 13.21 Let $P = \{x \in \mathbf{R}^n : Ax \leq b\}$ be a polyhedron. Suppose $a^T x \leq \beta$ is a valid inequality for P. Note that this inequality may not necessarily be one of those used in the above description of P, but it does hold for all points in P. Consider the set $F = \{x \in P : a^T x = \beta\}$. If $F \neq P$ and $F \neq \emptyset$, then F is called a *(proper) face* of P, the inequality $a^T x \leq \beta$ is a *defining inequality* for F, and the points of F are said to be *tight* for that inequality. The empty set and P itself are the two *improper faces* of P.

Definition 13.22 Let $P \subseteq \mathbf{R}^n$ be a polyhedron. Faces of dimension 0, 1, and $\dim P - 1$ are called *vertices*, *edges*, and *facets* of P, respectively. By convention, $\dim \emptyset = -1$.

Definition 13.23 Two inequalities $a^T x \leq \beta$ and $c^T x \leq \gamma$ are *equivalent* if there is a positive number k such that $c = ka$ and $\gamma = k\beta$.

Theorem 13.24 *Assume that $P \subset \mathbf{R}^n$ is a full-dimensional polyhedron and F is a proper face of P. Then F is a facet if and only if all valid inequalities for P defining F are equivalent.*

Proof. Assume that F is a facet and that

$$a^T x \leq \beta \quad (*)$$

is a valid inequality for P defining F. Note that $a \neq O$ since F is a proper face. There exist n affinely independent points $x^1, \ldots, x^n \in F$. Consider the $(n+1) \times n$ matrix

$$\begin{bmatrix} x^1 & \cdots & x^n \\ 1 & \cdots & 1 \end{bmatrix}$$

This matrix has full column rank, so its left nullspace is one dimensional. One element of this nullspace is $(a^T, -\beta)$. Assume that

$$d^T x \leq \gamma \quad (**)$$

is another valid inequality for P defining F. Then $d \neq O$ and $(d^T, -\gamma)$ is another element of the left nullspace. So there exists a nonzero number k such that $(d^T, -\gamma) = k(a^T, -\beta)$. Because P is full-dimensional, there exists a point $w \in P$ such that $d^T w < \gamma$. This implies that $k > 0$, and thus $(**)$ is equivalent to $(*)$.

Now assume that F is not a facet and

$$a^T x \leq \beta \quad (*)$$

is a valid inequality for P defining F. Again, $a \neq O$ since F is proper. Let $x^1, \ldots, x^p$ be a maximum collection of affinely independent points in F. Then $p < n$ and all points in F are affine combinations of these p points. Consider the $(n+1) \times p$ matrix

$$\begin{bmatrix} x^1 & \cdots & x^p \\ 1 & \cdots & 1 \end{bmatrix}$$

This matrix has full column rank, so its left nullspace is at least two dimensional. One member of this nullspace is $(a^T, -\beta)$. Let $(d^T, -\gamma)$ be another, linearly independent, one. Note that $d \neq O$; else γ must also equal zero. Also, $(d^T, -\gamma)(x^i, 1) = 0$, or $d^T x^i = \gamma$, for $i = 1, \ldots, p$. Define $f = a + \varepsilon d$ and $\eta = \alpha + \varepsilon\gamma$ for a sufficiently small nonzero real number ε.

Suppose that $x \in F$, so $a^T x = \beta$. Then x can be written as an affine combination of $x^1, \ldots, x^p$:

$$x = \sum_{i=1}^{p} \lambda_i x^1$$
$$\sum_{i=1}^{p} \lambda_i = 1$$

Thus

$$\begin{aligned} d^T x &= \sum_{i=1}^{p} \lambda_i d^T x^i \\ &= \sum_{i=1}^{p} \lambda_i \gamma \\ &= \gamma \end{aligned}$$

Hence $x \in F$ implies $d^T x = \gamma$. Therefore $x \in F$ implies $f^T x = \eta$.

Now suppose $x \in P \setminus F$. Then $a^T x < \beta$, so $f^T x < \eta$ if ε is sufficiently small.

Therefore $f^T x \leq \eta$ is a valid inequality for P that also defines F, yet is not equivalent to $a^T x \leq \beta$. □

Exercise 13.25 Assume that $P = \{x : a^{iT} x \leq b_i,\ i = 1, \ldots, m\} \subset \mathbf{R}^n$ is a full-dimensional polyhedron and F is a proper face of P. Let $T = \{i : a^{iT} x = b_i \text{ for all } x \in F\}$. Prove that F is the set of points satisfying

$$a^{iT} x \begin{cases} = b_i, & i \in T, \\ \leq b_i, & i \notin T. \end{cases}$$

□

Lemma 13.26 *Assume that $P = \{x : a^{iT} x \leq b_i,\ i = 1, \ldots, m\} \subset \mathbf{R}^n$ is a polyhedron. Then P is full-dimensional if and only if there exists a point $\overline{x} \in P$ satisfying all of the inequalities strictly.*

Proof. Assume P is full-dimensional. Choose $n+1$ affinely independent points $x^1, \ldots, x^{n+1} \in P$. Since these points do not lie on a common hyperplane, for each $i = 1, \ldots, m$ there is at least one x^j for which $a^{iT} x^j < b_i$. Now verify that $\overline{x} = \frac{1}{n+1}(x^1 + \cdots + x^{n+1})$ satisfies all of the m inequalities strictly.

Conversely, assume there exists a point $\overline{x} \in P$ satisfying all of the inequalities strictly. Let $e^1, \ldots, e^n$ be the standard n unit vectors. Verify that for sufficiently small nonzero real number ε, the points $\overline{x}, \overline{x} + \varepsilon e^1, \ldots, \overline{x} + \varepsilon e^n$ satisfy all of the inequalities strictly and are affinely independent. □

Definition 13.27 We say that inequality $a^T x \leq \beta$ is *derivable* from inequalities $a^{iT} x \leq b_i$, $i = 1, \ldots, m$, if there exist real numbers $\lambda_i \geq 0$ such that $\sum_{i=1}^m \lambda_i a^i = a$ and $\sum_{i=1}^m \lambda_i b_i \leq \beta$. Clearly if a point x satisfies all of the inequalities $a^{iT} x \leq b_i$, then it also satisfies any inequality derived from them.

Theorem 13.28 *Assume that $P \subset \mathbf{R}^n$ is a full-dimensional polyhedron, and that $a^T x \leq \beta$ is a facet-defining inequality. Suppose*

$$a^T x \leq \beta \quad (*)$$

is derivable from valid inequalities

$$a^{iT} x \leq b_i,\ i = 1, \ldots, m \quad (**)$$

where $(a^{iT}, b_i) \neq (O^T, 0)$, $i = 1, \ldots, m$, using positive coefficients λ_i, $i = 1, \ldots, m$. Then each inequality in $(**)$ must be equivalent to $(*)$.

Proof. First observe that $a \neq O$ and $a^i \neq O$ for all i. Let F be the set of points of P that are tight for $(*)$. Suppose $v \in F$. Then

$$\begin{aligned}
\beta &= a^T v \\
&= \sum_{i=1}^{m} \lambda_i a^{iT} v \\
&\leq \sum_{i=1}^{m} \lambda_i b_i \\
&\leq \beta
\end{aligned}$$

From this we conclude that v is tight for each of the inequalities in $(**)$.

Since $(*)$ is facet-defining and P is full-dimensional, we can find n affinely independent points $v^1, \ldots, v^n$ that are tight for $(*)$ and hence also for each inequality in $(**)$. The $(n+1) \times n$ matrix

$$\begin{bmatrix} v^1 & \cdots & v^n \\ 1 & \cdots & 1 \end{bmatrix}$$

has full column rank, so the left nullspace is one-dimensional. The vector $(a^T, -\beta)$ is a nonzero member of this nullspace, as is each of $(a^{iT}, -b_i)$, $i = 1, \ldots, m$. Therefore for each i there is a nonzero number k_i such that $a^i = k_i a$ and $b_i = k_i \beta$. Now since P is full-dimensional, there is at least one point $w \in P$ that is not in F. Thus $a^T w < \beta$ and $a^{iT} w < b_i$ for all i. We conclude each k_i must be positive, and therefore that each inequality in $(**)$ is equivalent to $(*)$. □

Definition 13.29 Assume that $P = \{x : a^{iT} x \leq b_i,\ i = 1, \ldots, m\} \subset \mathbf{R}^n$ is a polyhedron. If there is an index k such that $P = \{x : a^{iT} x \leq b_i,\ i \neq k\}$, then the inequality $a^{kT} x \leq b_k$ is said to be *redundant*. I.e., this inequality is redundant if and only if the following system is infeasible:

$$\begin{aligned}
a^{iT} x \leq b_i,\ i \neq k \\
a^{kT} x > b_k
\end{aligned}$$

Theorem 13.30 *Assume that $P = \{x : a^{iT} x \leq b_i,\ i = 1, \ldots, m\} \subset \mathbf{R}^n$ is a nonempty polyhedron. Then the inequality $a^{kT} x \leq b_k$ is redundant if and only if it is derivable from the inequalities $a^{iT} x \leq b_i$, $i \neq k$.*

Proof. It is clear that if one inequality is derivable from the others, then it must be redundant. So assume the inequality $a^{kT} x \leq b_k$ is redundant. Let A be the matrix with rows a^{iT}, $i \neq k$, and b be the vector with entries b_i, $i \neq k$. Consider the dual pair of linear programs:

$$(L) \quad \begin{array}{c} \max t \\ \text{s.t. } Ax \leq b \\ -a^{kT}x + t \leq -b_k \end{array} \qquad (D) \quad \begin{array}{c} \min y^T b - y_0 b_k \\ \text{s.t. } y^T A - y_0 a^{kT} = O^T \\ y_0 = 1 \\ y, y_0 \geq O \end{array}$$

P is nonempty so (L) is feasible (take t to be sufficiently negative).

Then (L) has nonpositive optimal value. Therefore so does (D), and there exists y such that

$$\begin{array}{c} y^T A = a^{kT} \\ y^T b \leq b_k \end{array}$$

Therefore the inequality $a^{kT}x \leq b_k$ is derivable from the others. □

Theorem 13.31 *Assume that $P = \{x : a^{iT}x \leq b_i,\ i = 1, \ldots, m\} \subset \mathbf{R}^n$ is a full-dimensional polyhedron, and that no two of the inequalities $a^{iT}x \leq b_i$ are equivalent. Then the inequality $a^{kT}x \leq b_k$ is not redundant if and only if it is facet-defining.*

Proof. Assume that the inequality is facet-defining. If it were redundant, then by Theorem 13.30 it would be derivable from the other inequalities. But then by Theorem 13.28 it would be equivalent to some of the other inequalities, which is a contradiction.

Now assume that the inequality is not redundant. Then there is a point x^* such that

$$\begin{array}{c} a^{iT}x^* \leq b_i,\ i \neq k \\ a^{kT}x^* > b_k \end{array}$$

Also, since P is full dimensional, by Lemma 13.26 there is a point $\overline{x} \in P$ satisfying all of the inequalities describing P strictly. Consider the (relatively) open line segment joining $\overline{x}$ to x^*. Each point on this segment satisfies all of the inequalities $a^{iT}x < b_i$, $i \neq k$, but one point, v, satisfies the equation $a^{kT}x = b_k$. Choose $n-1$ linearly independent vectors $w^1, \ldots, w^{n-1}$ orthogonal to a^k. Then for $\varepsilon > 0$ sufficiently small, the n points $v, v+\varepsilon w^1, \ldots, v+\varepsilon w^{n-1}$ are affinely independent points in P satisfying the inequality $a^{kT}x \leq b_k$ with equality. Therefore this inequality is facet-defining. □

Theorem 13.32 *Assume that $P = \{x : a^{iT}x \leq b_i,\ i = 1, \ldots, m\} \subset \mathbf{R}^n$ is a full-dimensional polyhedron, and that no two of the inequalities $a^{iT}x \leq b_i$ are equivalent. Then deleting all of the redundant inequalities leaves a system that consists of one facet-defining inequality for each facet of P.*

Proof. By Theorem 13.31, after all redundant inequalities are deleted, only facet-defining inequalities remain. Now suppose

$$a^T x \leq \beta \quad (*)$$

is a facet-defining inequality that is not equivalent to any of the inequalities $a^{iT} x \leq b_i$ in the system describing P. Expand the system by adding in $(*)$. Of course $(*)$ is valid for P, so every point of P must satisfy $(*)$. Therefore $(*)$ is redundant in the expanded system, and hence derivable from the original inequalities by Theorem 13.30. By Theorem 13.28 it must be the case that one of the inequalities $a^{iT} x \leq b_i$ is in fact equivalent to $(*)$. □

Exercise 13.33 Extend the results in this section to polyhedra that are not full-dimensional. □

14 Exercises: Linear Systems and Geometry

Exercise 14.1 Prove that every polyhedron has a finite number of faces. □

Exercise 14.2 If $v^1, \ldots, v^M \in \mathbf{R}^n$ and $\lambda_1, \ldots, \lambda_M$ are nonnegative real numbers that sum to 1, then $\sum_{i=1}^M \lambda_i v^i$ is called a *convex combination* of $v^1, \ldots, v^M$. A set $S \subseteq \mathbf{R}^n$ is called *convex* if any convex combination of any finite collection of elements of S is also in S (i.e., S is closed under convex combinations).

Prove that S is convex if and only if any convex combination of any two elements of S is also in S. □

Exercise 14.3 Prove that the intersection of any collection (whether finite or infinite) of convex sets is also convex. □

Exercise 14.4 For a subset $S \subseteq \mathbf{R}^n$, the *convex hull* of S, denoted $\operatorname{conv} S$, is the intersection of all convex sets containing S.

1. Prove that $\operatorname{conv} S$ equals the collection of all convex combinations of all finite collections of elements of S.

2. Prove that $\operatorname{conv} S$ equals the collection of all convex combinations of all collections of at most $n + 1$ elements of S.

□

Exercise 14.5 Let $S \subseteq \mathbf{R}^n$. Prove that if x is an extreme point of $\operatorname{conv} S$, then $x \in S$. (Remark: This is sometimes very useful in optimization problems. For simplicity, assume that S is a finite subset of $\mathbf{R}^n$ consisting of nonnegative points. If you want to optimize a linear function over the set S, optimize it instead over $\operatorname{conv} S$, which can theoretically be expressed in the form $\{x \in \mathbf{R}^n : Ax \leq b,\ x \geq O\}$. The simplex method will find an optimal basic feasible solution. You can prove that this corresponds to an extreme point of $\operatorname{conv} S$, and hence to an original point of S.) □

Exercise 14.6 Let $v^1, \ldots, v^M \in \mathbf{R}^n$. Let $x \in \mathbf{R}^n$. Prove that either

1. x can be expressed as a convex combination of $v^1, \ldots, v^M$, or else

2. There is a vector a and a scalar α such that $a^T x < \alpha$ but $a^T v^i \geq \alpha$ for all i; i.e., x can be separated from $v^1, \ldots, v^M$ by a hyperplane,

but not both. □

Exercise 14.7 Let $P \subseteq \mathbf{R}^n$ be a polyhedron and $x \in P$. Prove that x is an extreme point of P if and only if x is a vertex of P. Does this result continue to hold if P is replaced by an arbitrary subset of $\mathbf{R}^n$? By a convex subset of $\mathbf{R}^n$? □

Exercise 14.8 A subset K of $\mathbf{R}^n$ is a *finitely generated cone* if there exist P n-vectors $z^1, \ldots, z^P$, for some positive P, such that

$$K = \{\sum_{j=1}^{P} s_j z^j, \ s \geq O\}.$$

Let A be an $m \times n$ matrix, and let b be a *nonnegative* m-vector, such that

$$S = \{x \ : \ Ax \leq b, \ x \geq O\}$$

is *nonempty.* Let

$$\hat{S} = \{\alpha x \ : \ x \in S, \ \alpha \geq 0\}.$$

1. Prove that $\hat{S}$ is a finitely generated cone.
2. Give a simple description of $\hat{S}$ if $b_i > 0$ for all $i = 1, \ldots, m$.
3. Give an example in $\mathbf{R}^2$ to show that $\hat{S}$ need not be a finitely generated cone if b is not required to be nonnegative.

□

15 Exercises: Some Formulations

Exercise 15.1 Prove what I call the *Relaxation Principle*: Suppose $S \subseteq T$ and x^* is an optimal point for $\max_{x \in T} f(x)$. Prove that if $x^* \in S$ then x^* is an optimal point for $\max_{x \in S} f(x)$. I like to call this the *Mount Everest principle*: If you ask what the tallest mountain in Nepal is, and you are told that the tallest mountain in the world is Mt. Everest, and that Mt. Everest is in Nepal, then you can conclude that Mt. Everest is the tallest mountain in Nepal. □

Exercise 15.2 I call this the *Reformulation Principle*: Suppose S and S' are two sets and f and f' are two real-valued functions on S and S', respectively. Assume that $g : S \to S'$ and $g' : S' \to S$ are functions such that $f'(g(x)) \geq f(x)$ and $f(g'(x')) \geq f'(x')$ for all $x \in S$, $x' \in S'$. Let x^* be an optimal point for $\max_{x \in S} f(x)$. Prove that $g(x^*)$ is an optimal point for $\max_{x' \in S'} f'(x')$. (There is an analogous version for minimization problems.) □

Exercise 15.3 Suppose you want to solve the problem

$$(P) \quad \begin{array}{c} \min \|x\|_1 = |x_1| + \cdots + |x_n| \\ \text{s.t. } Ax \leq b \end{array}$$

Use the Reformulation Principle to show that an optimal solution to (P) can be constructed from an optimal solution to the linear program

$$\begin{array}{c} \min u_1 + v_1 + \cdots + u_n + v_n \\ \text{s.t. } Au - Av \leq b \\ u, v \geq O \end{array}$$

Comment on the dual of the above LP. □

Exercise 15.4 Suppose you want to solve the problem

$$(P) \quad \begin{array}{c} \min \|x\|_\infty = \max\{|x_1|, \ldots, |x_n|\} \\ \text{s.t. } Ax \leq b \end{array}$$

Use the Reformulation Principle to show that an optimal solution to (P) can be constructed from an optimal solution to the linear program

$$\begin{array}{c} \min t \\ \text{s.t. } t - u_i - v_i \geq 0, \quad i = 1, \ldots, n \\ Au - Av \leq b \\ u, v \geq O \\ t \text{ unrestricted} \end{array}$$

Comment on the dual of the above LP. □

Exercise 15.5 How would you solve the problem

$$\begin{aligned} \min\, & x_1^2 + \cdots + x_n^2 \\ \text{s.t. } & Ax = b \end{aligned}$$

or the problem

$$\begin{aligned} \min\, & x_1^2 + \cdots + x_n^2 \\ \text{s.t. } & Ax \leq b \end{aligned}$$

□

Exercise 15.6 Suppose you want to solve the problem

$$\begin{aligned} \min\, & f(x_1) \\ \text{s.t. } & Ax \leq b \\ & x \geq O \end{aligned}$$

where $f(x_1)$ is a piecewise linear convex function. For simplicity, assume that $f(x_1)$ has the form

$$f(x_1) = \begin{cases} a_0 + a_1 x_1 & \text{if } 0 \leq x_1 \leq b_1 \\ a_0 + a_1 b_1 + a_2(x_1 - b_1) & \text{if } b_1 \leq x_1 \leq b_2 \end{cases}$$

where $a_1 < a_2$ and $b_1 < b_2$. Prove that an optimal solution to (P) can be constructed from an optimal solution to the linear program

$$\begin{aligned} \min\, & a_1 u_1 + a_2 u_2 \\ \text{s.t. } & Ax \leq b \\ & x_1 = u_1 + u_2 \\ & x \geq O \\ & 0 \leq u_1 \leq b_1 \\ & 0 \leq u_2 \leq b_2 - b_1 \end{aligned}$$

Give an example to show that this method may fail if $a_1 > a_2$. □

Exercise 15.7 In this problem A is an $m \times n$ matrix, x has dimensions $n \times 1$, y has dimensions $m \times 1$, and z is a scalar.

1. Assume $z \geq A_i x$, $i = 1, \ldots, m$, where A_i is the ith row of A. Assume that $y \geq O$ and $y_1 + \cdots + y_m = 1$. Prove that $z \geq y^T A x$.

2. Assume $z \geq y^T A x$ for all y such that $y \geq O$ and $y_1 + \cdots + y_m = 1$. Prove that $z \geq A_i x$, $i = 1, \ldots, m$.

3. Prove that

$$\begin{aligned} z = \max\, & y^T Ax \\ \text{s.t. } & e^T y = 1 \\ & y \geq O \end{aligned}$$

if and only if $z = \max\{A_i x : i = 1, \ldots, m\}$. Here, e is the vector $(1, \ldots, 1)$, of length appropriate for the context.

4. Consider

$$\min_{\substack{x \geq O \\ e^T x = 1}} \max_{\substack{y \geq O \\ e^T y = 1}} y^T Ax$$

Prove that this problem is equivalent to

$$\begin{aligned} \min\, & z \\ \text{s.t. } & z \geq A_i x, \quad i = 1, \ldots, m \\ & e^T x = 1 \\ & x \geq O \end{aligned}$$

5. Prove

$$\min_{\substack{x \geq O \\ e^T x = 1}} \max_{\substack{y \geq O \\ e^T y = 1}} y^T Ax = \max_{\substack{y \geq O \\ e^T y = 1}} \min_{\substack{x \geq O \\ e^T x = 1}} y^T Ax$$

□

16 Networks

This introduction to network problems is very brief and informal; see Chvátal's book for greater precision.

Definition 16.1 A *directed graph (digraph)* $G = (V, E)$ is a finite set V of *nodes* or *vertices* and a finite set E of *arcs* or *edges*. Associated with each arc e is an ordered pair (i, j) of (usually distinct) nodes. i is the *tail* of e, j is the *head* of e, and i and j are called the *endpoints* of e. If i and j are distinct, we may sometimes write $e = ij$ if there is no other arc having the same tail as e and the same head as e. If $i = j$ then e is called a *loop*. We can represent a directed graph by a drawing, using points for nodes and arrows for arcs.

A *subgraph* of a digraph $G = (V, E)$ is a digraph $G' = (V', E')$ where $V' \subseteq V$, $E' \subseteq E$.

A *path* is an alternating sequence $v_0, e_1, v_1, e_2, v_2, \ldots, e_k, v_k$ of nodes and arcs, where the nodes v_i are all distinct and the arcs e_i are all distinct, and v_{i-1} and v_i are the two endpoints of e_i. If you have an alternating sequence in which $v_0 = v_k$ and otherwise the nodes and arcs are distinct, the sequence is called a *cycle*. We also identify paths and cycles with just the sets of arcs in them.

Two nodes are *connected* if there is a path from one to the other. The set of all nodes connected to a given node is a *component* of the digraph. A digraph is *connected* if it has only one component.

A *forest* is a subgraph containing no cycle. A *tree* is a connected forest. A *spanning forest* of a digraph is a forest that contains every node of the digraph. A connected spanning forest is a *spanning tree*.

A *twig* of a digraph is an arc that is not a loop and has an endpoint that is not the endpoint of any other arc in the digraph.

Theorem 16.2 *Spanning forests with at least one arc have twigs.*

PROOF. Assume the contrary. Choose any node in the forest that is the endpoint of some arc in the forest. Begin making a path. If every node encountered is new, at every new node there will be another arc having that node as an endpoint that can be used to continue the path. Since there is only a finite number of nodes, this cannot continue forever. So at some point a node will be encountered for a second time. But then a cycle will be detected, which is also impossible. □

Associated with a digraph having no loops is the *node-arc incidence matrix* A. Rows are indexed by nodes, columns by arcs. The entry in row i column e is -1 if i is the tail of e, $+1$ if i is the head of e, and zero otherwise. (This definition matches that of Chvátal. Note, however, that many other texts interchange -1 and 1 in the definition.)

Theorem 16.3 *For a connected digraph, the left nullspace of A is one dimensional. Thus the rank of A is $|V|-1$.*

Proof. An element of the left nullspace is an assignment of numbers y_i to nodes i such that for every arc $e = ij$, $y_j - y_i = 0$; i.e., $y_i = y_j$. By connectivity, the same number must be assigned to every node. □

For a digraph, an element of the nullspace of A corresponds to an assignment of numbers x_{ij} to arcs $e = ij$ such that for every node j,

$$(*) \qquad \sum_{i:ij\in E} x_{ij} - \sum_{i:ji\in E} x_{ji} = 0.$$

That is to say, for every node j, the sum of the numbers on arcs entering j equals the sum of the numbers on arcs leaving j.

Theorem 16.4 *If a subset of arcs contains a cycle, then the corresponding subset of columns of A is dependent.*

Proof. Trace the cycle in some direction, assigning $+1$ to arcs traversed in the forward direction, and -1 to arcs traversed in the reverse direction. □

Theorem 16.5 *If a subset S of columns of A is dependent, then the corresponding subset of arcs contains a cycle.*

Proof. Let x be a nonzero solution to $Ax = O$. Without loss of generality assume that $x_{ij} \neq 0$ precisely when $ij \in S$. From $(*)$ we can deduce that there can be no twig in the subgraph of G determined by the arcs in S. So the set of arcs cannot be a forest, and hence contains a cycle. □

Theorem 16.6 *Suppose G is connected and H is a subgraph of G. Then H is a spanning tree if and only if it is acyclic and maximal with respect to this property (i.e., introducing any other arc of G creates a cycle).*

Exercise 16.7 Prove the above theorem. □

Therefore if G is connected, bases of the column space of A correspond to spanning trees, and spanning trees have $|V|-1$ arcs (since the row rank of A is $|V|-1$).

Exercise 16.8 If T is a spanning tree and e is an arc not in the tree, then $T+e$ contains a unique cycle. □

Exercise 16.9 Suppose G is connected and H is a subgraph of G. Then H is a spanning tree if and only if it is connected and minimal with respect to this property (i.e., deleting any arc from H results in a disconnected subgraph). □

Twigs can be used to permute rows and columns to get bases into triangular form—see Chvátal for details.

Definition 16.10 Take a digraph $G = (V, E)$. To each arc ij assign a cost c_{ij}. To each node i assign a number b_i. Positive b_i is interpreted as *demand* at node i, negative b_i is interpreted as *supply* at node i, and zero b_i is interpreted to mean that node i has neither supply nor a demand requirement (i is an *intermediate node*). The *transshipment problem* is to find nonnegative flow numbers $x_{ij} \geq 0$ assigned to arcs ij that satisfy the *flow conservation equations*

$$\sum_{i:ij\in E} x_{ij} - \sum_{i:ji\in E} x_{ji} = b_j \text{ for all nodes } j$$

for which the objective function

$$\sum_{ij\in E} c_{ij}x_{ij}$$

is minimized.

Equivalently,

$$\begin{aligned} \min c^T x& \\ \text{s.t. } Ax = b& \\ x \geq O& \end{aligned}$$

where A is the node-arc incidence matrix of G.

Note that the dual LP is

$$\begin{aligned} \max y^T b& \\ \text{s.t. } y^T A \leq c^T& \\ y \text{ unrestricted}& \end{aligned}$$

In particular, for y to be dual feasible, for each arc ij it must satisfy $-y_i + y_j \leq c_{ij}$; i.e., $y_i + c_{ij} \geq y_j$. Note that if y is feasible, then adding the same constant value to each y_i produces another feasible solution.

Without loss of generality assume that G is connected (otherwise we could break the transshipment problem up into several independent problems). We know that A has exactly one redundant row (the dimension of the left nullspace equals one). Note that the sum of the equations is $O^T x = \sum_{i\in V} b_i$, so a necessary condition for feasibility is that the components of b sum to zero (total supply equals total demand).

The bases of A correspond to spanning trees. Given a spanning tree T, it is easy to compute the associated $\overline{x}$ and $\overline{y}$, and to test primal and dual feasibility. For $\overline{x}$, set $\overline{x}_{ij} = 0$ for nontree arcs, and work inward from the twigs to determine values for the tree arcs that satisfy the flow conservation equations. If it turns out that $\overline{x} \geq O$, then $\overline{x}$ and the tree are primal feasible. For $\overline{y}$, note that we have a free choice of one of the components of $\overline{y}$, corresponding to the redundant equation of $Ax = b$. For example, we can decide always to set $\overline{y}_1 = 0$. Then the remaining $\overline{y}_i$ are determined by the condition that $\overline{y}_i + c_{ij} = \overline{y}_j$ for all tree arcs ij. If it turns out that $\overline{y}_i + c_{ij} \geq \overline{y}_j$ for every nontree arc, then $\overline{y}$ and the tree are dual feasible.

If the basis is primal feasible but not dual feasible, then we pivot as follows:

1. Select a nontree arc $e = ij$ for which $\overline{c}_{ij} = c_{ij} + \overline{y}_i - \overline{y}_j < 0$; i.e, $\overline{y}_i + c_{ij} < \overline{y}_j$ (remember, we are minimizing, so we want $\overline{c}_{ij} \geq 0$ at optimality).

2. Find the unique cycle C in $T + e$.

3. Increase x_{ij} to $t \geq 0$ and adjust the flows on the other arcs f in C by adding or subtracting t, depending upon whether f points in the same direction or in the opposite direction around C as e. Let $t \geq 0$ be as large as possible without violating nonnegativity of any basic variable.

4. If t can be made arbitrarily large, then the problem is unbounded. Otherwise, the arc of T that determines the bound on t becomes nonbasic and is dropped from the tree.

Pivot until unboundedness is detected, or else until $\overline{y}_i + c_{ij} \geq \overline{y}_j$ for all nontree arcs, which implies that the final $\overline{x}$ is optimal.

Exercise 16.11 Assume that the problem is feasible. Prove that the problem is unbounded if and only if there is a cycle C with all arcs oriented in the same direction such that the sum of the costs of the arcs in C is negative. □

Exercise 16.12 Can you give an economic interpretation to a dual feasible solution $\overline{y}$? □

If there are ties for the leaving arc, cycling may occur. This can be avoided by such means as Bland's smallest subscript rule, or Cunningham's method (see Chvátal).

If an initial basic feasible spanning tree is not readily available, we can carry out a Phase I procedure as follows: Choose any node w of the network. Construct an artificial arc from w to each demand node. Construct an artificial arc from each supply node and each intermediate node to w. Give each artificial arc a cost of 1 and each original arc a cost of 0. Find a minimum cost transshipment, starting with the initial spanning tree consisting of the set of artificial arcs (clearly primal feasible). Delete all nonbasic artificial arcs. There are three cases to consider:

1. The minimum cost transshipment has positive cost. In this case the original problem is infeasible.

2. The minimum cost transshipment has zero cost and all artificial arcs are nonbasic in the final basis. In this case we drop the artificial arcs to get a basic feasible solution to the original problem.

3. The minimum cost transshipment has zero cost but not all artificial arcs are nonbasic in the final basis. In this case every basic artificial arc must have a flow of zero on it. Choose such an artificial arc uv. Partition the nodes of the network into $R = \{i : \overline{y}_i \leq \overline{y}_u\}$ and $S = \{i : \overline{y}_i > \overline{y}_u\}$. Note that $v \in S$. Summing the constraints for nodes $j \in R$, we get

$$\sum_{j \in R} \sum_{i: ij \in E} x_{ij} - \sum_{j \in R} \sum_{i: ji \in E} x_{ji} = \sum_{j \in R} b_j,$$

   which we abbreviate

$$x(V, R) - x(R, V) = b(R),$$

   which implies

$$\sum_{ij \in E: i \in S, j \in R} x_{ij} - \sum_{ji \in E: i \in S, j \in R} x_{ji} = \sum_{j \in R} b_j,$$

   which we abbreviate

$$x(S, R) - x(R, S) = b(R).$$

   By the optimality criterion that $\overline{y}_i + c_{ij} \geq \overline{y}_j$ for all arcs in the digraph, there can be no original arc from R to S (these have cost zero). So $x(S, R) = b(R)$ holds for all x feasible for the original problem. There can be no tree arc from S to R (since these cannot satisfy the optimality criterion with equality), so for the feasible solution x^* to the original problem found in Phase I, $0 = x^*(S, R) = b(R)$. Therefore $x(S, R) = 0$ for all feasible x. This implies that no arcs from S to R are used in any feasible solution, so they can be dropped from the digraph. Drop uv and any artificial arcs in (R, S) also, and you have two independent subproblems on two separate components, one on R and the other on S. Repeat this procedure if there are additional nonbasic artificial arcs; each such arc will break a problem down into two subproblems.

A corollary of the algorithm is the following theorem.

Theorem 16.13 (Integrality Theorem) *Assume that a transshipment problem has integer supplies and demands. If the problem is feasible, then it has an integer feasible solution. If it has an optimal solution, then it has an integer optimal solution.*

The algorithm can easily be modified to handle transshipment problems with upper and lower bounds on the arcs. In this case a nontree variable in a basic solution will be set to either a finite upper bound, a finite lower bound, or zero if neither bound is finite.

Exercise 16.14 Think about pivoting and Phase I for transshipment problems with upper and lower bounds. Then check your understanding by looking at Chvátal. □

17 Exercises: Networks

Exercise 17.1 Let $G = (V, E)$ be a connected digraph, and let B be the matrix whose rows and columns are indexed by nodes, such that the ij entry of B equals the negative of the number of arcs joining i and j in either direction if $i \neq j$, and the number of arcs with endpoint i if $i = j$. Prove $B = AA^T$, where A is the node-arc incidence matrix of G. □

Exercise 17.2 Suppose you are given a feasible solution to a transshipment problem for some commodity. The values of the variables x_{ij} don't actually explicitly give the specific route through the network that each unit of commodity takes. Describe a method for obtaining a routing plan from the given feasible solution. Your method should provide you with a list of directed paths from supply nodes to demand nodes, specifying how much of the commodity should be sent along each path. (In a directed path, all of the arcs point in the same (forward) direction along the path.) □

Exercise 17.3 Let $G = (V, E)$ be a digraph and $c_{ij} \geq 0$ be an assignment of nonnegative costs to arcs ij of the digraph. Let s and t be two distinct nodes of G. An (s,t)-*dipath* is an path from s to t in which all arcs point forward as the path is traversed from s to t. It is desired to find an (s,t)-dipath of minimum total cost. Explain how to formulate and solve this problem as a transshipment problem. Be explicit on how to construct an optimal dipath. Where do you use the assumption that $c_{ij} \geq 0$ for all arcs? □

Exercise 17.4 (PERT Problem) Suppose you have a project which consists of a set $\{1, \ldots, n\}$ of tasks to perform. Associated with each task is a completion time a_i and a set S_i of tasks that must first be completed before task i is begun. Assume that task 1 is an artificial "starting task" with $a_1 = 0$, and that task n is an artificial "ending task" with $a_n = 0$. The problem is to complete in the shortest possible time. Let t_i be a variable that represents the starting time of task i. Prove that this problem can be formulated and solved as a dual of a transshipment problem. In particular, comment on why there is no difficulty in solving the transshipment problem by the network simplex method. □

Definition 17.5 Given digraph $G = (V, E)$, $X, Y \subseteq V$, and any function $g : E \to \mathbf{R} \cup \{\infty\}$. We will use the notation (X, Y) to denote the set of all arcs with tail in X and head in Y, and $g(X, Y)$ to denote $\sum_{(u,v)\in(X,Y)} g(u, v)$. Also, if we have any function $h : V \to \mathbf{R} \cup \{\infty\}$, and any subset of nodes $X \subset V$, define $h(X)$ to be $\sum_{v\in X} h(v)$.

Exercise 17.6 By analyzing the outcome of Phase I, prove Gale's Theorem: A transshipment problem is feasible if and only if it is never the case that there exists a subset W of V such that $b(\overline{W}) > 0$ and $(W, \overline{W}) = \emptyset$ ($\overline{W}$ is the complement of W). I.e., it is never the case that the net demand within a subset $\overline{W}$ is positive when there are no arcs from W into $\overline{W}$. □

Exercise 17.7 Consider a transshipment problem with a nonnegative upper bound (capacity) u_{ij} (possibly infinite) on each arc ij. Assume each arc is given a lower bound of zero. Extend the Phase I and Phase II procedures to such transshipment problems. □

Exercise 17.8 Use the Phase I procedure that you developed in the preceding exercise to extend Gale's theorem: A transshipment problem is feasible if and only if $b(\overline{W}) \leq u(W, \overline{W})$ for all subsets W ($\overline{W}$ is the complement of W). I.e., the net demand within $\overline{W}$ never exceeds the total capacity of all arcs entering $\overline{W}$. □

Definition 17.9 An *undirected graph* $G = (V, E)$ is a finite set V of *nodes* or *vertices* and a finite set E of *arcs* or *edges*. Associated with each edge e is an unordered pair $\{i, j\}$ of (usually distinct) nodes. i and j are called the *endpoints* of e. We can represent a directed graph by a drawing, using points for nodes and curves for edges. The *degree* of a node in a graph is the number of edges for which it is an endpoint. A graph is *simple* if no two edges share exactly the same pair of endpoints. A *bipartite graph* is an undirected graph $G = (V, E)$ such that the nodes V can be partitioned into two sets X and Y with the property that every edge of G has one endpoint in X and the other in Y. In particular, bipartite graphs cannot have loops.

Exercise 17.10 [Bipartite Graph Realizability] Let $a_1, \ldots, a_m$, $b_1, \ldots, b_n$ be nonnegative integers. Does there exist a simple, bipartite graph $G = (S \cup T, E)$ such that $S = \{1, \ldots, m\}$, $T = \{1, \ldots, n\}$, $\deg(u_i) = a_i$, $i = 1, \ldots, m$, and $\deg(v_j) = b_j$, $j = 1, \ldots, n$? Assume that $a(S) = b(T)$, and prove that such a G exists if and only if for all $X \subseteq S$, $Y \subseteq T$, we have $b(Y) - a(X) \leq |S \setminus X||Y|$. Hint: Transform this problem into a transshipment problem with upper bounds on the arcs. □

Exercise 17.11 [Matrix Scaling] Suppose we are given an $m \times n$ matrix A. We wish to find row multipliers $r_i > 0$ and column multipliers $c_j > 0$ that will be used to scale the matrix A, obtaining matrix $B = (b_{ij})$, where $b_{ij} = r_i a_{ij} c_j$. The goal is to select r and c to minimize the product of the absolute values of the nonzero b_{ij}, subject to the condition that every nonzero b_{ij} has absolute value ≥ 1. Show how to reduce the above problem to that of solving the dual of a transshipment problem. Hint: formulate the problem in terms of the logarithms of r_i, c_j, and $|a_{ij}|$. □

Definition 17.12 Given a digraph $G = (V, E)$ with arc capacities $u : E \to \mathbf{R}_+ \cup \{\infty\}$, nonnegative arc lower bounds $\ell : E \to \mathbf{R}_+$, and arc costs $c : E \to \mathbf{R}$. A *circulation* is a flow x satisfying $0 \le \ell \le x \le u$, and also flow conservation

$$\sum_{i:ij \in E} x_{ij} - \sum_{i:ji \in E} x_{ji} = 0$$

at every node j. The *cost* of x is $\sum_{ij \in E} c_{ij} x_{ij}$.

Exercise 17.13 Show how to convert a minimum cost circulation problem to a minimum cost transshipment problem by making the change of variable $x = x' + \ell$. Then prove Hoffman's theorem: A circulation network admits a feasible circulation if and only if $u(W, \overline{W}) \ge \ell(\overline{W}, W)$ for all subsets W. □

Definition 17.14 A *mixed graph* is a graph $G = (V, E \cup A)$ in which some arcs (those in A) are directed and the others (those in E) are undirected. The mixed graph is said to be *Eulerian* if there is a closed walk using each arc of G exactly once (though nodes can be repeated), traversing all of the arcs of A in the correct direction. Such a closed walk is called an *Euler tour.*

Exercise 17.15 Let $G = (V, E \cup A)$ be a mixed graph such that

1. G is connected.
2. Every node of G is incident with an even number of arcs, and
3. The number of inwardly directed arcs equals the number of outwardly directed arcs at each node.

Show that G is Eulerian. □

Exercise 17.16 Given a mixed graph $G = (V, E \cup A)$. Is G Eulerian? We will attempt to direct some of the undirected arcs of G in order to transform G to the type of mixed graph considered in the above exercise.

If G does not satisfy conditions (1) or (2) of the exercise, then G cannot be Eulerian. Suppose both (1) and (2) are satisfied. Construct a digraph N by replacing each undirected edge $e \in E$ by a pair of oppositely directed arcs, each given lower bound 0 and upper bound 1. Call this set of newly created arcs A'. Give each remaining arc $e \in A$ lower bound 1 and upper bound 1. Determine whether or not N admits a circulation.

Suppose N has a feasible circulation. Then N has an integral circulation, one for which x_e equals 0 or 1 for all $e \in A' \cup A$. Now orient some of the originally undirected arcs e of G

as follows: If $uv \in E$ and if $x_{uv} = 1$, $x_{vu} = 0$ in G', direct e from u to v. This yields a new mixed graph G' satisfying the conditions of the exercise. An Euler tour for G' provides an Euler tour for G.

Suppose N has no feasible circulation. Then Hoffman's Theorem implies there is some $X \subseteq V$ such that in the original mixed graph G the difference between the number of directed arcs from X to $\overline{X}$ and the number of directed arcs from $\overline{X}$ to X exceeds the number of undirected arcs joining X and $\overline{X}$. Hence G cannot be Eulerian.

Complete this discussion by filling in the details of the above paragraph, thus proving the following theorem:

The mixed graph $G = (V, E \cup A)$ is Eulerian if and only if

1. The underlying undirected graph of G is connected.
2. Every node of G is incident with an even number of arcs, and
3. For every $X \subseteq V$, the difference between the number of directed arcs from X to $\overline{X}$ and the number of directed arcs from $\overline{X}$ to X does not exceed the number of undirected arcs joining X and $\overline{X}$.

□

Exercise 17.17 Describe a pivot of the dual simplex method as applied to a transshipment problem. □

Exercise 17.18 Let's consider a two-commodity transshipment problem. Suppose you have a directed graph $G = (V, E)$ and four special nodes $s_1, s_2, t_1, t_2 \in V$. Assume that each arc $ij \in E$ has a capacity u_{ij}. The goal is to ship commodity 1 from s_1 to t_1 and commodity 2 from s_2 to t_2. Let x_{ij} denote the amount of commodity 1 shipped on arc ij and y_{ij} denote the amount of commodity 2 shipped on arc ij. We want x and y to satisfy

$$\begin{array}{c} x(V,i) - x(i,V) = 0 \text{ for all } i \neq s_1, t_1 \\ y(V,i) - y(i,V) = 0 \text{ for all } i \neq s_2, t_2 \\ x_{ij} + y_{ij} \leq u_{ij} \text{ for all } ij \\ x, y \geq O \end{array}$$

For such a feasible x and y, define $z_1 = x(V, t_1) - x(t_1, V)$, the net flow of commodity 1 into t_1; and $z_2 = y(V, t_2) - y(t_2, V)$, the net flow of commodity 2 into t_2. Suppose that we wish to maximize $z_1 + z_2$. Construct a simple example of such a problem to show that the maximum value of $z_1 + z_2$ need not be integer even if all arc capacities u_{ij} are integer. Note: your example should be so small that feasible and optimal solutions can be found by inspection; there is no need to try to use the simplex method. □

Exercise 17.19 Chvátal, 19.1–19.2, 19.4–19.8, 19.9–19.11. Read Chapters 19–21. □

18 Total Unimodularity

Some of the material for this section was drawn directly from the book *Combinatorial Optimization: Networks and Matroids*, by Lawler.

We are interested in instances when we can solve an integer linear program by ignoring the integer requirements, solving the LP relaxation (requiring only that the variables be nonnegative) and observing that an integer optimal solution exists.

The feasible set of an integer linear program (ILP) is the set of integral points lying in some polyhedral set, while the feasible set of the LP relaxation is the polyhedral set itself. If all of the extreme points of the latter set are integer, then the LP relaxation can be used to solve the ILP, no matter what the objective function. On the other hand, if some extreme point of the polyhedral set is not an integer vector, then we can find an objective function for which that noninteger point is the unique optimal solution of the LP relaxation, and therefore may encounter difficulties when trying to solve the ILP in this manner.

For a given set of integer points, there may be several ways of describing the set using linear inequalities. For example, consider:

$$\begin{array}{rcl} 2x+4y & \leq & 7 \\ 4x+2y & \leq & 7 \\ x,y & \geq & 0 \\ & x,y \text{ integer} & \end{array} \tag{6}$$

and

$$\begin{array}{rcl} x & \leq & 1 \\ y & \leq & 1 \\ x,y & \geq & 0 \\ & x,y \text{ integer} & \end{array} \tag{7}$$

Both (6) and (7) describe the same set of points, namely, $\{(0,0),(0,1),(1,0),(1,1)\}$. But their respective LP relaxations describe different regions. The constraints in (7) are to be preferred to (6), because they are precisely the constraints of the convex hull of the desired set of points. This is an extremely desirable situation. The following describes one way in which we can guarantee that we have a polyhedron with all extreme points being integer, and is taken directly from Lawler's book.

Theorem 18.1 (Hoffman and Kruskal) *Let a linear program have constraints $Ax = b$, $x \geq 0$, where A is an integer matrix with linearly independent rows and b is an integer vector. The following three conditions are equivalent:*

1. *The determinant of every basis (maximal nonsingular submatrix of A) is ± 1.*

2. *The extreme points of the convex polyhedron C defined by $Ax = b$, $x \geq 0$ are integral for all integer vectors b.*

3. *The inverse B^{-1} of every basis B is integer.*

Proof. This proof is due to Vienott and Dantzig.

(1) implies (2): Let $x = (x_B, x_N)$ be an extreme point of the convex polyhedron C and B be the associated basis. By Cramer's rule, $\det B = \pm 1$ implies that the solution to $Bx_B = b$ is integral if b is integral.

(2) implies (3): Let B be a basis and y be any integer vector such that $y + B^{-1}e_i \geq 0$, where e_i is the ith unit column. Let $z = y + B^{-1}e_i \geq 0$. Then $Bz = By + e_i$ is an integer vector since B, y and e_i are all integral. Because b can be any integer vector, we shall let $b = Bz$. Now $Bz = b$ and $z \geq 0$, which shows that z, when padded with zeroes, is an extreme point of the convex polyhedron C defined by $Ax = b$, $x \geq 0$. By (2), z is an integer vector. But $z - y = B^{-1}e_i$, from which it follows that $B^{-1}e_i$ is integral. The vector $B^{-1}e_i$ is the ith column vector of B^{-1}, and the argument can be repeated for $i = 1, 2, \ldots, m$ to show that B^{-1} is an integer matrix.

(3) implies (1): Let B be a basis. By assumption B is an integer matrix and $\det B$ is an integer. By condition (3), B^{-1} is an integer matrix so $\det B^{-1}$ is also an integer. But $(\det B)(\det B^{-1}) = 1$, which implies that $\det B = \det B^{-1} = \pm 1$. □

Definition 18.2 A matrix A is *totally unimodular* (TU) if every subdeterminant of A is either 1, −1, or 0.

Corollary 18.3 *Let C' be the convex polyhedron defined by the inequality constraints $A'x \leq b$, $x \geq 0$, where A' is an integer matrix. The following three conditions are equivalent:*

1. *A' is totally unimodular.*

2. *The extreme points of C' are all integral for any integer vector b.*

3. *Every nonsingular submatrix of A' has an integer inverse.*

Proof. Let $A = [A', I]$. It is not hard to establish the equivalence of (1), (2) and (3) of the theorem with (1), (2) and (3) of the corollary, respectively. For example, if M is any nonsingular submatrix of A', then a basis of A can be found, after permuting rows, of the form:

$$B = \begin{bmatrix} M & 0 \\ N & I_k \end{bmatrix}$$

where I_k is a $k \times k$ identity matrix. Then $\det B = \det M$, so that $\det B = \pm 1$. Similar transformations suffice to establish other equivalences. □

Exercise 18.4 Fill in the details of the above equivalences. □

Exercise 18.5 Prove that a matrix A is totally unimodular if and only if any one of the matrices A^T, $-A$, $[A, A]$, $[A, -A]$, $[A, I]$ is totally unimodular. □

Exercise 18.6 Prove that if A is totally unimodular, the extreme points of the polyhedron defined by $Ax = b$, $x \geq 0$ are integral for all integral b. □

There is an easily tested set of sufficient (but not necessary) conditions for total unimodularity:

Theorem 18.7 *A* $(0, \pm 1)$ *matrix* A *is totally unimodular if both of the following conditions are satisfied:*

1. *Each column contains at most two nonzero elements.*
2. *The rows of* A *can be partitioned into two sets* A_1 *and* A_2 *such that two nonzero entries in a column are in the same set of rows if they have different signs and in different sets of rows if they have the same sign.*

Proof. A submatrix of a $(0, \pm 1)$ matrix satisfying the conditions of the theorem must also satisfy the same conditions. Hence it is sufficient to prove that $\det A' = 0, \pm 1$ for all square matrices satisfying the conditions. For any 1×1 matrix A', clearly $\det A' = 0, \pm 1$. Now suppose, by inductive hypothesis, that $\det A' = 0, \pm 1$ for all $(n-1) \times (n-1)$ matrices A' satisfying the conditions. Let A' be $n \times n$. If A' contains a zero column, $\det A' = 0$. If some column of A' contains exactly one nonzero entry, then $\det A' = \pm 1 \cdot \det A'' = 0, \pm 1$, where A'' is the cofactor of that entry. If every column of A' contains exactly two nonzero entries, then $\sum_{i \in A'_1} a_{ij} = \sum_{i \in A'_2} a_{ij}$ for $j = 1, 2, \ldots, n$. This implies that $\det A' = 0$ and the proof is complete. □

Exercise 18.8 Prove that the converse of the above theorem is false. □

Corollary 18.9 *A* $(0, \pm 1)$ *matrix* A *is totally unimodular if it contains no more than one 1 and no more than one* -1 *in each column.*

A characterization of totally-unimodular matrices reminiscent of the above results is given by the following theorem:

Theorem 18.10 (Ghouila-Houri) *An $m \times n$ matrix A is totally unimodular if and only if each subset $J \subseteq \{1, \ldots, n\}$ of the columns can be partitioned into two classes J_1 and J_2 such that for each row $i \in \{1, \ldots, m\}$ we have*

$$\left| \sum_{j \in J_1} a_{ij} - \sum_{j \in J_2} a_{ij} \right| \leq 1.$$

Results of Seymour yield an efficient algorithm to determine whether or not any given matrix is totally unimodular (the details of which we omit).

Corollary 18.11 *The node-arc incidence of any directed graph is totally unimodular.*

PROOF. An immediate corollary of the previous theorem. □

Definition 18.12 Let $G = (V, E)$ be an undirected graph. Define the node-edge incidence matrix B of G, with rows indexed by nodes and columns indexed by arcs, as follows: In the case that e is not a loop, let the entry in row i column e be $+1$ if i is an endpoint of e and zero otherwise. In the case that e is a loop, let the entry in row i column e be $+2$ if i is the endpoint of e and zero otherwise.

Corollary 18.13 *The node-edge incidence matrix of any bipartite undirected graph is totally unimodular. (See Definition 17.9.)*

PROOF. An immediate corollary of the previous theorem. □

19 Exercises: Total Unimodularity

Exercise 19.1 Prove that an undirected graph is bipartite if and only if its node-edge incidence matrix is totally unimodular. □

Exercise 19.2 A matrix M is a *consecutive ones matrix* if every entry is either 0 or 1, and if for every column ℓ there are rows j and k such that $m_{i\ell} = 1$ if $j \le i \le k$ and $m_{i\ell} = 0$ otherwise. Prove that such a matrix is totally unimodular. Hint: Permute the rows and columns into a convenient form, apply some elementary row operations, and expand the determinant along a particular column. □

Exercise 19.3 Let M be a totally unimodular matrix, and let M' be the matrix obtained from M by pivoting on some element m_{rs} of M. Prove that M' is totally unimodular. □

Exercise 19.4 Let $G = (V, E)$ be a bipartite graph. A *matching* M in G is a subset of edges such that no two edges share a common endpoint. The *maximum matching problem* is the problem of finding a matching of maximum cardinality. A *cover* C in G is a subset of nodes such that at least one endpoint of every edge is in C. The *minimum cover problem* is the problem of finding a cover of minimum cardinality.

1. Formulate each of these problems as as integer linear program. For the matching problem, use one variable for each edge of G, and one constraint for each node of G (apart from nonnegativity requirements).

2. Prove that the corresponding linear programming relaxations have optimal integer solutions.

3. Prove that the corresponding linear programming relaxations are dual to each other.

4. Prove that the cardinality of a maximum cardinality matching equals the cardinality of a minimum cardinality cover.

□

Exercise 19.5 [Birkhoff-von Neumann Theorem] A *permutation* matrix is a square matrix with exactly one 1 in each row and column, and zeroes elsewhere. A *doubly stochastic* matrix is a square matrix with nonnegative entries such that the entries in each row and in each column sum to 1. Prove that every doubly-stochastic matrix is a convex combination of permutation matrices. □

Exercise 19.6 [Matrix Tree Theorem] Let $G = (V, E)$ be a connected digraph, and let B be the matrix in Exercise 17.1. Let B' be the matrix obtained from B by crossing off any one row and any one column of B. Prove that the number of spanning trees of G equals $|\det B'|$. Hint: The Cauchy-Binet Theorem states that if X is an $n \times m$ matrix, Y is an $m \times n$ matrix, and $Z = XY$ is an $n \times n$ matrix, then

$$\det Z = \sum_S \det X_S \det Y_S,$$

where the sum is taken over all subsets S of the index set $\{1, \ldots, m\}$ of cardinality n, X_S is the $n \times n$ submatrix of X consisting of all the rows of X but only the columns of X indexed by S, and Y_S is the $n \times n$ submatrix of Y consisting of all the columns of Y but only the rows of Y indexed by S. □

20 Knapsack and Cutting Stock Problems

This material is summarized directly from Chvátal, Chapter 13.

The *knapsack problem* is

$$\begin{aligned} \max z = \sum_{i=1}^{m} c_i x_i \\ \text{s.t. } \sum_{i=1}^{m} a_i x_i \leq b \\ x_i \geq 0 \text{ and integer, } i = 1, \ldots, m \end{aligned}$$

An example is

$$\begin{aligned} \max 4x_1 + 5x_2 + 5x_3 + 2x_4 \\ \text{s.t. } 33x_1 + 49x_2 + 51x_3 + 22x_4 \leq 120 \\ x_1, x_2, x_3, x_4 \geq 0 \text{ and integer} \end{aligned}$$

We assume that the numbers a_i are positive integers and that b is a nonnegative integer. Without loss of generality we may also assume that the numbers c_i are nonnegative; otherwise, those variables x_i for which $c_i \leq 0$ need not be used in an optimal solution and can be dropped from the problem.

Order the variables $x_1, \ldots, x_m$ so that

$$\frac{c_1}{a_1} \geq \frac{c_2}{a_2} \geq \cdots \geq \frac{c_m}{a_m}.$$

Note that every optimal solution x^* satisfies

$$b - \sum_{i=1}^{m} a_i x_i < a_m,$$

otherwise x_m^* could be increased for a better solution. Feasible solutions satisfying the above inequality are called *sensible*.

An *initial segment* is an index k, $0 \leq k \leq m$, and nonnegative integer values $\overline{x}_1, \ldots, \overline{x}_k$ such that

$$\sum_{i=1}^{k} a_i \overline{x}_i \leq b.$$

The *completion* of an initial segment, completion$(\overline{x}_1, \ldots, \overline{x}_k)$, is the feasible point x with $x_j = \overline{x}_j$, $1 \leq j \leq k$, and

$$x_j = \left\lfloor (b - \sum_{i=1}^{j-1} a_i x_i)/a_j \right\rfloor$$

for $k+1 \leq j \leq m$.

We could theoretically solve the knapsack problem by enumerating all feasible solutions. One way to do this is to let the first solution be the completion of the empty segment. For a given solution x, the succeeding solution is obtained by finding $\max\{k : x_k > 0\}$ and then computing completion$(x_1, \ldots, x_{k-1}, x_k - 1)$. Stop when $x = O$.

For the example, we would get the list in Figure 1, which could be made shorter if solutions which were not sensible are eliminated. The values of each of these solutions can be determined, and the solution with the highest value is optimal.

But we can be smarter than this and avoid looking at all feasible solutions. Suppose we are given an initial segment $(\overline{x}_1, \ldots, \overline{x}_k)$. We can obtain an upper bound for the objective function value of any feasible point x such that $x_i \leq \overline{x}_i$, $1 \leq i \leq k$ by

$$\begin{aligned}
z(x) &= \sum_{i=1}^{k} c_i x_i + \sum_{i=k+1}^{m} c_i x_i \\
&= \sum_{i=1}^{k} c_i \overline{x}_i + \sum_{i=1}^{k} c_i (x_i - \overline{x}_i) + \sum_{i=k+1}^{m} c_i x_i \\
&= \sum_{i=1}^{k} c_i \overline{x}_i + \sum_{i=1}^{k} \frac{c_i}{a_i} a_i (x_i - \overline{x}_i) + \sum_{i=k+1}^{m} \frac{c_i}{a_i} a_i x_i \\
&\leq \sum_{i=1}^{k} c_i \overline{x}_i + \sum_{i=1}^{k} \frac{c_{k+1}}{a_{k+1}} a_i (x_i - \overline{x}_i) + \sum_{i=k+1}^{m} \frac{c_{k+1}}{a_{k+1}} a_i x_i \\
&= \sum_{i=1}^{k} c_i \overline{x}_i + \frac{c_{k+1}}{a_{k+1}} \left(\sum_{i=1}^{k} a_i (x_i - \overline{x}_i) + \sum_{i=k+1}^{m} a_i x_i \right) \\
&\leq \sum_{i=1}^{k} c_i \overline{x}_i + \frac{c_{k+1}}{a_{k+1}} \left(\sum_{i=1}^{k} a_i (x_i - \overline{x}_i) + b - \sum_{i=1}^{k} a_i x_i \right) \\
&= \sum_{i=i}^{k} c_i \overline{x}_i + \frac{c_{k+1}}{a_{k+1}} \left(b - \sum_{i=1}^{k} a_i \overline{x}_i \right).
\end{aligned}$$

We now have established the following proposition:

Proposition 20.1 *Assume $(\overline{x}_1, \ldots, \overline{x}_k)$ is an initial segment and x is any feasible point such that $x_i \leq \overline{x}_i$, $i = 1, \ldots, k$. Then*

$$z(x) \leq \sum_{i=1}^{k} c_i \overline{x}_i + \frac{c_{k+1}}{a_{k+1}} \left(b - \sum_{i=1}^{k} a_i \overline{x}_i \right). \tag{8}$$

If all c_i are integer, this last inequality can be strengthened to

$$z(x) \leq \left\lfloor \sum_{i=1}^{k} c_i \overline{x}_i + \frac{c_{k+1}}{a_{k+1}} \left(b - \sum_{i=1}^{k} a_i \overline{x}_i \right) \right\rfloor. \tag{9}$$

Figure 1: Feasible Solutions for the Example

x_1	x_2	x_3	x_4	
3	0	0	0	
2	1	0	0	
2	0	1	0	
2	0	0	2	
2	0	0	1	(not sensible)
2	0	0	0	(not sensible)
1	1	0	1	
1	1	0	0	(not sensible)
1	0	1	1	
1	0	1	0	(not sensible)
1	0	0	3	
1	0	0	2	(not sensible)
1	0	0	1	(not sensible)
1	0	0	0	(not sensible)
0	2	0	1	
0	2	0	0	(not sensible)
0	1	1	0	
0	1	0	3	
0	1	0	2	(not sensible)
0	1	0	1	(not sensible)
0	1	0	0	(not sensible)
0	0	2	0	
0	0	1	3	
0	0	1	2	(not sensible)
0	0	1	1	(not sensible)
0	0	1	0	(not sensible)
0	0	0	5	
0	0	0	4	(not sensible)
0	0	0	3	(not sensible)
0	0	0	2	(not sensible)
0	0	0	1	(not sensible)
0	0	0	0	(not sensible)

Use the notation bound$(\overline{x}_1, \ldots, \overline{x}_k)$ for the bound in (8) (or the bound in (9) if the c_i are all integer).

We can exploit this bound to find the optimal solution x^* in a more efficient manner with the following *branch and bound* algorithm.

1. [Initialize.] Set $M = 0$, $k = 0$.
2. [Find a promising solution.] Let $\overline{x}$ =completion$(\overline{x}_1, \ldots, \overline{x}_k)$. Let $k = m$.
3. [Is this solution better?] If $z(\overline{x}) > M$, replace M by $z(\overline{x})$ and x^* by $\overline{x}$.
4. [Go to the next solution.]
   (a) If $k = 1$, stop, else replace k by $k - 1$.
   (b) If $\overline{x}_k = 0$, return to (4a), else replace $\overline{x}_k$ by $\overline{x}_k - 1$.
5. [Is this initial segment worth exploring?] If bound$(\overline{x}_1, \ldots, \overline{x}_k) > M$, return to (2), else return to (4).

Applying the algorithm to the example:

- $\overline{x}$ =completion$(\emptyset) = (3, 0, 0, 0)$. $M = z(\overline{x}) = 12$. $x^* = \overline{x}$.
- bound$(2) = 13 > 12$. $\overline{x}$ =completion$(2) = (2, 1, 0, 0)$. $z(\overline{x}) = 13$, so $M = 13$, $x^* = \overline{x}$.
- bound$(2, 0) = 13 \not> 13$.
- bound$(1) = 12 \not> 13$; stop.

We now turn to the *cutting stock problem.* In this problem we have a set of rolls of raw material, each of width (length) r, and we require b_i pieces of width w_i, $1 \leq i \leq m$. These pieces are obtained by cutting the raw rolls. A raw roll can be cut into a_i pieces of width w_i if

$$\sum_{i=1}^{m} w_i a_i \leq r.$$

Such a vector $[a_1, \ldots, a_m]^T$ of nonnegative integers is called a *feasible cutting pattern.* We can list all such patterns as the columns of a matrix A. We wish to use the fewest number of raw rolls to meet our demands, so the problem can be formulated:

$$\begin{aligned} \min e^T x \\ \text{s.t. } Ax = b \\ x \geq 0 \text{ and integer} \end{aligned}$$

We will solve this problem by omitting the integer requirements and rounding up the optimal solution of the resulting LP (P). When solving (P), we will not enumerate all of the columns of A in advance but generate them as needed on the fly as we carry out the revised simplex method. A contains an identity matrix (can you see why?) and we could use this as an initial feasible basis.

Suppose we have a given feasible basis B with associated BFS $\overline{x}$. We find $\overline{y}$ by solving $\overline{y}^T B = c_B^T$. We need to pivot if there exists a column A_s of A for which $c_s - \overline{y}^T A_s < 0$ (remember, we are minimizing). Since $c_s = 1$, we seek a column a of A (a cutting pattern) such that $1 - \overline{y}^T a < 0$; i.e., $\overline{y}^T a > 1$. Thus we wish to know if the optimal value of the following problem exceeds 1:

$$\begin{aligned} \max \sum_{i=1}^{m} \overline{y}_i a_i \\ \text{s.t. } \sum_{i=1}^{m} w_i a_i \leq r \\ a_i \geq 0 \text{ and integer, } i = 1, \ldots, m \end{aligned}$$

This is a knapsack problem. If the optimal value does not exceed 1, our current basis and solution for (P) are optimal. If the optimal value exceeds 1, we have found a new basis column for (P), and can use the ratio test to determine which old basis column is to be discarded, and what the values of the new basic variables will be. See Chvátal for an example of this procedure.

Exercise 20.2 Consider the knapsack problem

$$(P_b) \qquad \begin{aligned} \max z = \sum_{i=1}^{m} c_i x_i \\ \text{s.t. } \sum_{i=1}^{m} a_i x_i \leq b \\ x_i \geq 0 \text{ and integer, } i = 1, \ldots, m \end{aligned}$$

where b is a nonnegative integer, $a > O$, and $c \geq O$. Let $z^*(b)$ denote the optimal objective function value and $x^*(b)$ denote an optimal solution of (P_b). Explain how to find $z^*(b)$ and $x^*(b)$ if you already know $z^*(i)$ and $x^*(i)$, $i = 0, \ldots, b-1$. This leads to a procedure for solving P_b by solving the sequence of problems $P_0, P_1, P_2, \ldots, P_b$. □

21 Dantzig-Wolfe Decomposition

Suppose we wish to solve a linear program of the form

$$(P) \qquad \begin{aligned} &\max c^T x \\ &\text{s.t. } Ax = b \\ &x \in S \end{aligned}$$

where the set S is described by "easy" linear constraints, but the constraints in $Ax = b$ are "harder" or "complicating" constraints. For example, S might be described by a totally unimodular or node-arc incidence matrix.

Recall that $S = \{\sum_{i=1}^{M} r_i v^i + \sum_{i=1}^{N} s_i w^i : \sum_{i=1}^{M} r_i = 1,\ r \geq O,\ s \geq O\}$, where the v^i are the normal basic feasible solutions for S, and the w^i are the normal basic feasible directions for S. We create a new linear program, called the *master* program, by substituting

$$x = \sum_{i=1}^{M} r_i v^i + \sum_{i=1}^{N} s_i w^i \tag{10}$$

into the constraints $Ax = b$ and the objective function $c^T x$:

$$\begin{aligned} &\max c^T\left(\sum_{i=1}^{M} r_i v^i + \sum_{i=1}^{N} s_i w^i\right) \\ &\text{s.t. } A\left(\sum_{i=1}^{M} r_i v^i + \sum_{i=1}^{N} s_i w^i\right) = b \\ &\sum_{i=1}^{M} r_i = 1 \\ &r, s \geq O \end{aligned}$$

or equivalently,

$$(M) \qquad \begin{aligned} &\max \sum_{i=1}^{M} (c^T v^i) r_i + \sum_{i=1}^{N} (c^T w^i) s_i \\ &\sum_{i=1}^{M} (A v^i) r_i + \sum_{i=1}^{N} (A w^i) s_i = b \\ &\sum_{i=1}^{M} r_i = 1 \\ &r, s \geq O \end{aligned}$$

Note that the columns of the constraint matrix of M are of the form

$$\begin{bmatrix} Av^i \\ 1 \end{bmatrix}$$

with cost $c^T v^i$, and

$$\begin{bmatrix} Aw^i \\ 0 \end{bmatrix}$$

with cost $c^T w^i$. Let c' denote the cost vector for (M).

We can solve (P) by solving (M), and then recovering the optimal solution for (P) by using (10). The difficulty is that (M) may be too large to write out explicitly. We get around this problem by generating the columns of (M) as needed (this is similar in spirit to our approach to the cutting stock problem).

Suppose at some stage we have a primal feasible basis for (M), with associated basis matrix B. Designate the vector of dual variables as (y, y_0), distinguishing the last component because of the special nature of the last constraint of (M). Find the current dual variable values by solving $(\overline{y}^T, \overline{y}_0)B = c_B'^T$. We now wish to know if there is any candidate for an entering column. The reduced cost for the variable r_i is

$$c^T v^i - (\overline{y}^T, \overline{y}_0) \begin{bmatrix} Av^i \\ 1 \end{bmatrix} = (c^T - \overline{y}^T A)v^i - \overline{y}_0.$$

If this is positive, then the variable r_i can enter the basis. The reduced cost for the variable s_i is

$$c^T w^i - (\overline{y}^T, \overline{y}_0) \begin{bmatrix} Aw^i \\ 0 \end{bmatrix} = (c^T - \overline{y}^T A)w^i.$$

If this is positive, then the variable s_i can enter the basis.

How can we determine if there is some v^i or some w^i for which the above holds, without checking all of them explicitly? We solve the following *subproblem*:

$$(Q) \qquad \begin{array}{c} \max (c^T - \overline{y}^T A)x \\ \text{s.t. } x \in S. \end{array}$$

Presumably this linear program will not be difficult to solve because the constraints describing S are "easy."

There are four possible outcomes:

1. If (Q) is infeasible, then so is (P), and we are done.

2. If (Q) is unbounded, then a normal basic feasible direction w^i is found for which $(c^T - \overline{y}^T A)w^i > 0$. In this case s_i can enter the basis of (M).

3. If (Q) has an optimal solution, and the optimal value is greater than $\overline{y}_0$, then a normal basic feasible solution v^i is found for which $(c^T - \overline{y}^T A)v^i - \overline{y}_0 > 0$. In this case r_i can enter the basis of (M).

4. If (Q) has an optimal solution, and the optimal value is not greater than $\overline{y}_0$, then there are no normal basic feasible solutions v^i for which $(c^T - \overline{y}^T A)v^i > \overline{y}_0$ and no normal basic feasible directions w^i for which $(c^T - \overline{y}^T A)w^i > 0$. Therefore there are no entering variables for (M) and the current solution for (M) is optimal.

See Chvátal for an example. We can initialize (M), if necessary, by introducing artificial variables into (M), and carrying out a Phase I procedure. You should think carefully about how this would be done.

Exercise 21.1 Consider the following two-commodity transshipment problem: You are given a directed graph $G = (V, E)$. There are two demand vectors, b^1 and b^2 (one for each commodity); two cost vectors c^1 and c^2 (one for each commodity), and a nonnegative upper bound vector u (one upper bound for each arc). The transshipment problem is:

$$\begin{aligned} \min c^{1T}x^1 + c^{2T}x^2 \\ \text{s.t. } Ax^1 = b^1 \\ Ax^2 = b^2 \\ O \le x^1 + x^2 \le u \end{aligned}$$

Here, A is the node-arc incidence matrix of G. Discuss the application of the Dantzig-Wolfe procedure to this problem. In particular, what are the "easy" constraints, what are the "complicating" constraints, and what is a reasonable Phase I procedure that takes advantage of the network structure? □

22 Subgradient Optimization

This material is modified from notes taken by Joy Williams on lectures given by Jon Lee. See Nemhauser-Wolsey, I.2.4.

22.1 Subgradients

Definition 22.1 Let f be a function from $\mathbf{R}^n$ to $\mathbf{R}$. The function f is *convex* if $f(\lambda x^1 + (1-\lambda)x^2) \leq \lambda f(x^1) + (1-\lambda)f(x^2)$ for all $x^1, x^2 \in \mathbf{R}^n$, and for all $0 \leq \lambda \leq 1$.

Exercise 22.2 Let $a_i \in \mathbf{R}^n$, $i = 1, \ldots, m$, $\alpha_i \in \mathbf{R}$, $i = 1, \ldots, m$ and define $f(x) = \max_i \{a_i^T x + \alpha_i\}$. Prove that f is convex. □

Definition 22.3 Let f be a convex function from $\mathbf{R}^n$ to $\mathbf{R}$. A vector γ is a *subgradient* of f at $\overline{x}$ if $f(x) \geq f(\overline{x}) + (x - \overline{x})^T \gamma$ for all x; i.e., linearly extrapolating f at $\overline{x}$ using the linear function $f(\overline{x}) + (x - \overline{x})^T \gamma$ never overestimates the value of $f(x)$.

Let epi$(f) \subset \mathbf{R}^{n+1}$ denote the region on or above the graph of the convex function f. Then γ is a subgradient of f at $\overline{x}$ if and only if $(\gamma, -1)$ is the outer normal to a supporting hyperplane to epi(f) at the point $(\overline{x}, f(\overline{x}))$. In the case that f is differentiable at $\overline{x}$, then γ is in fact the gradient of f at $\overline{x}$. (Recall that the gradient gives the direction of greatest local increase of a function.)

Exercise 22.4 Let f be a convex function. Prove that $\overline{x}$ is a minimizer of f if and only if O is a subgradient to f at $\overline{x}$. □

Definition 22.5 The *Subgradient Method* of minimizing a convex function f is to start with any point x^0 and generate a sequence of points by

$$x^{k+1} = x^k - s_k \gamma^k$$

where $s_k > 0$ is a scalar stepsize and γ^k is a subgradient of f at x^k. We want the points x^k to converge to some x^* that minimizes $f(x)$. It can be proved that this will occur if the stepsizes have the properties:

1. $s_k \to 0$.
2. $\sum_{k=0}^{M} s_k \to \infty$.

We could, for example, choose $s_k = 1/k$, but the convergence is too slow for this to be satisfactory in practice. Another possibility is

$$s_k = \rho^k \frac{f(x^k) - \hat{f}}{\|\gamma^k\|^2}$$

where $\hat{f}$ is an estimate of $\min f(x)$ and $0 < \rho < 1$. Convergence is more rapid, but the limit point may not be optimal.

22.2 Linear Programs with Two Types of Constraints

Suppose we have a linear program

$$(P) \qquad \begin{array}{c} z^* = \max c^T x \\ \text{s.t. } Ax = b \\ Ex = d \\ x \geq O \end{array}$$

where the constraints $Ax = b$ are "easy" and the constraints $Ex = d$ are "complicating." Let $\pi \in \mathbf{R}^n$ and consider the problem

$$(L_\pi) \qquad \begin{array}{c} w(\pi) = \max c^T x - \pi^T(Ex - d) \\ \text{s.t. } Ax = b \\ x \geq O \end{array}$$

Proposition 22.6 *Assume that (P) has an optimal solution. Then for any π, $w(\pi) \geq z^*$; i.e., the optimal value of (L_π) provides an upper bound for the optimal value of (P).*

PROOF. Suppose x^* is optimal for (P). Then $Ex^* = d$ so $Ex^* - d = O$. Thus x^*, which is feasible for (L_π), has an objective function value of $c^T x^* = z^*$. But x^* may not be optimal for (L_π), so $w(\pi) \geq z^*$. □

As a consequence, the best upper bound that we could get this way is $\min_\pi w(\pi)$.

Theorem 22.7 *Assume that (P) has an optimal solution. Then $min_\pi w(\pi) = z^*$.*

PROOF. Noting that $\pi^T d$ is a constant for fixed π,

$$\begin{aligned}
\min_\pi w(\pi) &= \min_\pi \left(\begin{array}{c} \max c^T x - \pi^T(Ex - d) \\ \text{s.t. } Ax = b \\ x \geq O \end{array} \right) \\
&= \min_\pi \left(\begin{array}{c} \max (c^T - \pi^T E)x + \pi^T d \\ \text{s.t. } Ax = b \\ x \geq O \end{array} \right) \\
&= \min_\pi \left(\begin{array}{c} \min y^T b + \pi^T d \\ \text{s.t. } y^T A \geq c^T - \pi^T E \end{array} \right) \\
&= \begin{array}{c} \min y^T b + \pi^T d \\ \text{s.t. } y^T A + \pi^T E \geq c^T \end{array} \\
&= \begin{array}{c} \max c^T x \\ \text{s.t. } Ax = b \\ Ex = d \\ x \geq O \end{array} \\
&= z^*
\end{aligned}$$

□

Now the goal is to determine $\min_\pi(w(\pi))$. Observe that

$$w(\pi) = \begin{array}{c} \max c^T x - \pi^T(Ex - d) \\ \text{s.t. } Ax = b \\ x \geq O \end{array}$$

is a piecewise linear, continuous, convex function of π. To see this, let S be the (finite) collection of basic feasible solutions to $\{x : Ax = b,\ x \geq O\}$. Then when $w(\pi)$ is finite, $w(\pi) = \max\{c^T x - \pi^T(Ex - d) : x \in S\}$. Now apply Exercise 22.2.

Proposition 22.8 *Fix π and suppose x is an optimal solution of (L_π). Then $d - Ex$ is a subgradient for w at π.*

PROOF. We need to show $w(\hat{\pi}) \geq w(\pi) + (\hat{\pi} - \pi)^T(d - Ex)$; i.e.,

$$\begin{array}{c} \max c^T \hat{x} - \hat{\pi}^T(E\hat{x} - d) \\ \text{s.t. } A\hat{x} = b \\ \hat{x} \geq O \end{array} \quad \geq \quad c^T x - \pi^T(Ex - d) + (\hat{\pi} - \pi)^T(d - Ex)$$

or

$$\begin{array}{l} \max c^T\hat{x} - \hat{\pi}^T E\hat{x} + \hat{\pi}^T d \\ \quad \text{s.t. } A\hat{x} = b \\ \qquad \hat{x} \geq O \end{array} \quad \geq \quad c^Tx - \pi^T Ex + \pi^T d + \hat{\pi}^T d - \pi^T d - \hat{\pi}^T Ex + \pi^T Ex$$

or

$$\begin{array}{l} \max c^T\hat{x} - \hat{\pi}^T E\hat{x} + \hat{\pi}^T d \\ \quad \text{s.t. } A\hat{x} = b \\ \qquad \hat{x} \geq O \end{array} \quad \geq \quad c^Tx - \hat{\pi}^T Ex + \hat{\pi}^T d.$$

This latter inequality holds because the point x on the right is feasible for the program $(L_{\hat{\pi}})$ on the left (since x is feasible for (L_π)), but isn't necessarily optimal for $(L_{\hat{\pi}})$. □

So we can minimize $w(\pi)$ using the subgradient optimization method:

1. Set $k = 0$ and start with any $\pi^0 \in \mathbf{R}^n$ (e.g., $\pi^0 = O$).

2. Solve (L_{π^k}) to get x^k.

3. Calculate the subgradient $\gamma^k = d - Ex^k$. If $\gamma^k = O$, stop.

4. Let $\pi^{k+1} = \pi^k - s_k\gamma^k$.

5. Set $k = k + 1$ and return to Step 2.

In the above, we could choose

$$s_k = \frac{w(\pi^k) - \hat{w}}{\|\gamma^k\|^2},$$

where $\hat{w}$ is an estimate for the minimum value of $w(\pi)$.

22.3 Finding Feasible Points

Exercise 22.9 Prove that If $\overline{x} \in \mathbf{R}^n$, $O \neq a \in \mathbf{R}^n$, and $\alpha \in \mathbf{R}$, then the distance of $\overline{x}$ from the hyperplane $H = \{x : a^Tx = \alpha\}$ is

$$\frac{|a^T\overline{x} - \alpha|}{\|a\|}.$$

Also, prove that if x is the closest point in H to $\overline{x}$, then

$$x = \overline{x} - \frac{a^T\overline{x} - \alpha}{\|a\|^2}a.$$

□

Finding an optimal solution of the linear program

$$\begin{aligned} \max c^T x \\ \text{s.t. } Ax \le b \\ x \ge O \end{aligned}$$

is equivalent to finding a feasible solution to the system

$$\begin{aligned} Ax \le b \\ x \ge O \\ y^T A \ge c^T \\ y \ge O \\ y^T b \le c^T x \end{aligned}$$

So we are interested in ways of finding a feasible solution to a system of linear inequalities.

Suppose we want to find a feasible point for the system $Ax \le b$. Define

$$f(x) = \max\{0, \max_i\{A_i x - b_i\}\},$$

where A_i denotes the ith row of A. Thus $f(x)$ gives the largest constraint violation, and $f(x) = 0$ if and only if x satisfies $Ax \le b$. Note that $f(x)$ is a convex function.

Proposition 22.10 *If x^k does not satisfy $Ax \le b$ and $A_\ell x^k - b_\ell = f(x^k)$ (i.e., inequality $A_\ell x - b_\ell$ is the most violated), then A_ℓ is a subgradient of f at x^k.*

PROOF. We must verify

$$f(x) \ge f(x^k) + A_\ell(x - x^k)$$

for all x. But the right hand side equals $A_\ell x^k - b_\ell + A_\ell x - A_\ell x^k = A_\ell x - b_\ell$. Since $f(x)$ is defined as the maximum constraint violation, it follows that $f(x) \ge A_\ell x - b_\ell$. □

The following application of subgradient optimization is called the *Ball Method*.

1. Take 0 for the estimate $\hat{f}$ of f, because we want to converge to a point x for which $f(x) = 0$, indicating no violated constraints. Set $x^0 = O$.

2. $$x^{k+1} = x^k - \left[\frac{A_\ell x^k - b_\ell}{\|A_\ell\|^2}\right] A_\ell^T,$$

   where $A_\ell x - b_\ell$ is the constraint most violated by x^k.

The x^k will converge to a feasible solution of $Ax \leq b$ if there is one.

From Exercise 22.9, we can see that at each stage of the Ball Method, x^{k+1} is the orthogonal projection of x^k onto the hyperplane with equation $A_\ell x = b_\ell$.

The name "Ball Method" comes from the following observation. Let S be the feasible region for $Ax \leq b$. Suppose in addition to points x^k we also have positive real numbers r_k so that

1. If $S \neq \emptyset$ then the ball centered at x^0 with radius r_0 intersects S in a set X of positive volume; and
2. $$r_{k+1} = \sqrt{r_k^2 - \frac{(A_\ell x^k - b_\ell)^2}{\|A_\ell\|^2}}$$

Then it can be shown that for each k, the ball centered at x^k with radius r_k contains X (draw a picture and use the Pythagorean theorem!).

Exercise 22.11 Prove the above assertion. □

So if we can estimate the rate at which the volumes of the balls shrink, we can place an upper bound on the number of iterations required before either x^k is found to be feasible or else S is provably empty.

It turns out that we get better convergence results by using ellipsoids instead of balls, and this leads to the Ellipsoid Method.

Exercise 22.12 In what ways can the various techniques of this chapter be applied to the two-commodity flow problem of Exercise 21.1? □

23 The Ellipsoid Method

Much of this material comes from Chvátal, Appendix, and Nemhauser and Wolsey, I.6.1–2.

Definition 23.1 For nonnegative real number r and $\overline{x} \in \mathbf{R}^n$, define $B(\overline{x}, r) = \{x \in \mathbf{R}^n : \|x - \overline{x}\| \leq r\}$, the ball of radius r centered at $\overline{x}$.

Exercise 23.2 Prove if $\overline{x} \in \mathbf{R}^n$, $O \neq a \in \mathbf{R}^n$, $\alpha \in \mathbf{R}$, $0 < \epsilon \in \mathbf{R}$, $r = \epsilon / \|a\|$, and $a^T \overline{x} \leq \alpha$, then $B(\overline{x}, r) \subset \{x : a^T x \leq \alpha + \epsilon\}$. □

The *size* of a nonzero integer N is the number of binary digits required to encode it, including its sign: $\lfloor \log_2 |N| \rfloor + 2$. (The size of 0 is 2.) The size of a collection of matrices and vectors is the sum of the sizes of the individual entries.

More generally, the size of an instance of a problem is the total number of binary digits required to encode it. An algorithm to solve a problem is *polynomial* if the number of steps required by the algorithm to solve any given instance of a problem is bounded by a polynomial in the size of that instance. The Ellipsoid Method provides a polynomial algorithm to solve linear programs.

Proposition 23.3 *Given an integer matrix A and an integer vector b. There exists a polynomial algorithm that finds a solution to $Ax = b$ or else demonstrates that the system is infeasible.*

Proof. See Chvátal, pp. 443–4, for a discussion on implementing Gaussian elimination efficiently. □

Proposition 23.4 *The problem of finding an optimal solution to a linear program can be reduced to the problem of finding a feasible solution to a system of linear inequalities.*

Proof. You can find an optimal solution to

$$\begin{array}{c} \max c^T x \\ \text{s.t. } Ax \leq b \\ x \geq O \end{array}$$

by finding a feasible solution to

$$\begin{array}{c} Ax \leq b \\ x \geq O \\ y^T A \geq c^T \\ y \geq O \\ c^T x \geq y^T b \end{array}$$

□

Proposition 23.5 *The problem of finding a feasible solution to a system of linear inequalities can be reduced to the problem of testing the feasibility of systems of linear inqualities (without actually finding a solution), together with the problem of finding a solution to a system of linear equations.*

Proof. See Chvátal, pp. 445–6. Suppose you have a system $Ax \leq b$. First confirm that it is feasible. Then consider the variables $x_1, \ldots, x_n$ sequentially. If x_j can be set to zero without making the system infeasible, do so. Now consider the inequalities sequentially, $i = 1, \ldots m$. If inequality i can be changed to an equation without making the system infeasible, do so. In the end you will have a system of the form:

$$\sum_{j \in J} a_{ij} x_j = b_i, \quad i \in I$$
$$\sum_{j \in J} a_{ij} x_j \leq b_i, \quad i \notin I$$

(The variables x_j for $j \notin J$ are those that have been set to zero and hence no longer appear explicitly in the system.) Claim: The system of equations

$$\sum_{j \in J} a_{ij} x_j = b_i, \quad i \in I$$

has a unique solution, and that solution is in fact feasible for the original system $Ax \leq b$. □

Exercise 23.6 Verify the claim. □

For A an integer $m \times n$ matrix and b an integer vector of length m, define

$$\gamma = \max\{\max_{i,j} |a_{ij}|, \max_i |b_i|\}.$$

Proposition 23.7 *Let A be an integer $m \times n$ matrix and b be an integer vector of length m. If the system $Ax \leq b$ is feasible, then there exists a feasible point $\overline{x}$ such that $|\overline{x}_j| \leq m!\gamma^m$, $j = 1, \ldots, n$, and $\|\overline{x}\| \leq (\sqrt{m})m!\gamma^m$.*

Proof. Introduce slack variables and let $\hat{x}$ be a normal basic feasible solution, with associated basis B. Nonbasic variables have value zero. The values of the basic variables are found by solving $B\hat{x}_B = b$. That $|\hat{x}_j| \leq m!\gamma^m$ is an immediate consequence of Cramer's rule.

$$\|\hat{x}\| = \sqrt{\sum_{j \in B} \hat{x}_j^2} \leq \sqrt{\sum_{j \in B} (m!\gamma^m)^2} = (\sqrt{m})m!\gamma^m.$$

Drop the slack variables from $\hat{x}$ to get a feasible solution $\overline{x}$ to $Ax \leq b$, and note that $\|\overline{x}\| \leq \|\hat{x}\|$. □

Note that occurrences of m in the above proposition can be replaced by n if $n < m$.

Proposition 23.8 *Let A be an integer $m \times n$ matrix and b be an integer vector of length m. Let*

$$\epsilon = \frac{1}{2mn!\gamma^n}$$

and e be the vector of all 1's. Then the system

$$(I) \qquad Ax \le b$$

is feasible if and only if the system

$$(II) \qquad Ax \le b + \epsilon e$$

is feasible.

PROOF. If (I) is feasible, then clearly (II) is feasible. Now assume (I) is not feasible. By a theorem of the alternatives, the following system is feasible:

$$\begin{aligned} y^T A &= O^T \\ y^T b &= -1 \\ y &\ge O \end{aligned}$$

This system has $n+1$ equations. Let $\overline{y}$ be a basic feasible solution to this system. Then Cramer's Rule (taking into account the special nature of the right-hand side) implies $|\overline{y}_i| \le n!\gamma^n$. Therefore

$$\begin{aligned} \overline{y}^T(b+\epsilon e) &= \overline{y}^T b + \epsilon \overline{y}^T e \\ &\le -1 + \epsilon m n! \gamma^n \\ &= -1 + \tfrac{1}{2} \\ &= -\tfrac{1}{2} \\ &< 0. \end{aligned}$$

Therefore $\overline{y}$ is feasible for the system

$$\begin{aligned} y^T A &= O^T \\ y^T(b+\epsilon e) &< 0 \\ y &\ge O \end{aligned}$$

By a theorem of the alternatives, (II) is infeasible. □

Note that we can use $\epsilon = (2(n+1)n!\gamma^n)^{-1}$ if $n+1 \le m$ and $\epsilon = (2mm!\gamma^m)^{-1}$ if $m < n+1$.

Corollary 23.9 *Let A, b and ϵ be as above. Set $b' = b + \epsilon e$. Then the system $Ax \le b$ is feasible if and only if the system $Ax < b'$ is feasible.*

Proposition 23.10 *Let A be an integer $m \times n$ matrix and b be an integer vector of length m. Let ϵ be as above, $r = \min_i\{\epsilon/\|A_i\|\}$ where A_i is the ith row of A, and $R = (\sqrt{m})m!\gamma^m + r$. Let S be the set of points satisfying system (II) above. Then S is nonempty if and only if there exists a point $\overline{x} \in S$ such that $B(\overline{x}, r) \subset B(O, R) \cap S$.*

PROOF. Assume $S \neq \emptyset$. By Proposition 23.7 there exists $\overline{x}$ feasible for $Ax \leq b$ such that $\|\overline{x}\| \leq (\sqrt{m})m!\gamma^m$. By Exercise 23.2, $B(\overline{x}, r) \subset S$. Also, if $x \in B(\overline{x}, r)$, then $\|x\| = \|x - \overline{x} + \overline{x}\| \leq \|x - \overline{x}\| + \|\overline{x}\| \leq r + (\sqrt{m})m!\gamma^m$. □

Definition 23.11 An $n \times n$ matrix D is *positive definite* if $x^T Dx > 0$ for all $x \in \mathbf{R}^n \setminus \{O\}$. An *ellipsoid* with center y is a set $E = E(D, y) = \{x \in \mathbf{R}^n : (x-y)^T D^{-1}(x-y) \leq 1\}$, where D is $n \times n$ positive definite symmetric, and $y \in \mathbf{R}^n$. So a ball is an ellipsoid with $D = r^2 I_n$. The *standard unit ball* is $B^n = E(I, 0) = B(O, 1)$.

The Ellipsoid Method relies on the following important property of ellipsoids—this should remind you of the situation when we were examining the Ball Method.

Theorem 23.12 *Given an ellipsoid $E = E(D, y)$ and nonzero vector d, the half-ellipsoid $H = E(D, y) \cap \{x \in \mathbf{R}^n : d^T x \leq d^T y\}$ is contained in an ellipsoid E' such that* $\mathrm{vol}\,(E')/\mathrm{vol}\,(E) \leq e^{-1/2(n+1)}$.

PROOF. Postponed. □

Now we can describe the Ellipsoid Method. From our discussion we have demonstrated that we can solve a linear program if we can solve the following problem: Given a set $S = \{x : Ax < b\}$ where A is $m \times n$, a ball $B(O, R)$, and numbers v and V such that $V = \mathrm{vol}\,(B(O, R))$ and the volume of $X = B(O, R) \cap S$ exceeds v if S is nonempty, determine whether S is empty or not. The Ellipsoid Method solves this problem in the following way:

1. Let $t^* = \lceil 2(n+1)(\ln V - \ln v) \rceil$.
2. Let $D_0 = R^2 I$, $x_0 = O$, and $E_0 = E(D_0, x_0) = B(O, R)$. Set $t = 0$.
3. If $x_t \in S$, stop; a feasible solution has been found.
4. Otherwise, if $t \geq t^*$, stop; $S = \emptyset$.
5. Otherwise, since $x_t \notin S$, suppose $a^{i(t)} x_t \geq b_{i(t)}$. Find an ellipsoid E_{t+1} containing the half ellipsoid $H_t = E_t \cap \{x \in \mathbf{R}^n : a^{i(t)} x \leq a^{i(t)} x_t\}$. Let $E_{t+1} = E(D_{t+1}, x_{t+1})$ and $t \leftarrow t + 1$. Return to Step 3.

(To ease notational strain, we have omitted transposes.)

Theorem 23.13 *The Ellipsoid Method for S terminates correctly after no more than t^* iterations.*

Proof. We need to show that if $x_t \notin X$ for $t = 0, \ldots, t^*$, then $X = \emptyset$, and hence $S = \emptyset$. Use induction to show that $X \subseteq E_t$ for all $t \leq t^*$. This is true for $t = 0$ by the construction of E_0. Assume that $X \subseteq E_t$. Since $a^{i(t)} x_t \geq b_{i(t)}$,

$$X \subseteq \{x \in \mathbf{R}^n : a^{i(t)} x \leq b_{i(t)}\} \subseteq \{x \in \mathbf{R}^n : a^{i(t)} x \leq a^{i(t)} x_t\}.$$

Hence

$$X \subseteq E_t \cap \{x \in \mathbf{R}^n : a^{i(t)} x \leq a^{i(t)} x_t\} = H_t \subset E_{t+1}.$$

Now consider the volume of E_{t^*}. Since $\mathrm{vol}\,(E_{t+1})/\mathrm{vol}\,(E_t) \leq e^{-1/2(n+1)}$, it follows that $\mathrm{vol}\,(E_{t^*})/\mathrm{vol}\,(E_0) \leq e^{-t^*/2(n+1)}$. Hence

$$\begin{array}{rcl} \mathrm{vol}\,(E_{t^*}) & \leq & V e^{-t^*/2(n+1)} \\ & = & V e^{-\lceil 2(n+1)(\ln V - \ln v) \rceil / 2(n+1)} \\ & \leq & V e^{-(\ln V - \ln v)} \\ & \leq & V e^{-\ln(V/v)} \\ & = & v. \end{array}$$

But if $X \neq \emptyset$ then we would have $\mathrm{vol}\,(X) > v$, $\mathrm{vol}\,(E_{t^*}) \leq v$, and $X \subseteq E_{t^*}$, which is impossible. Therefore $X = \emptyset$. □

We still need to explain how to construct E_{t+1} and in the process prove Theorem 23.12. Let $d = a^{i(t)}$ and $D = D_t$. Define

$$x_{t+1} = x_t - \frac{1}{n+1} \frac{Dd}{\sqrt{d^T D d}}, \tag{11}$$

and

$$D_{t+1} = \frac{n^2}{n^2 - 1} \left(D - \frac{2}{n+1} \frac{(Dd)(Dd)^T}{d^T D d} \right). \tag{12}$$

We will show that $E_{t+1} = E(D_{t+1}, x_{t+1})$ is the desired ellipsoid by working through a sequence of arguments.

Proposition 23.14 *Every symmetric positive definite $n \times n$ matrix D has a decomposition $D = Q_1^T Q_1$, where Q_1 is an $n \times n$ nonsingular matrix.*

Exercise 23.15 Prove the above proposition. □

Definition 23.16 If A is an $n \times n$ nonsingular matrix, $b \in \mathbf{R}^n$, and $T_A : \mathbf{R}^n \to \mathbf{R}^n$ is defined by $T_A(x) = Ax + b$, then T_A is called an *affine transformation* .

Proposition 23.17 *If $L \subseteq \mathbf{R}^n$ is full-dimensional and convex and T_A is an affine transformation, then* $\mathrm{vol}\,(T_A(L)) = |\det A|\mathrm{vol}\,(L)$*.*

Exercise 23.18 Prove this proposition. You may assume without proof that L and $T_A(L)$ are measurable sets and that the volume of a set equals the integral of the constant function 1 over the set. □

Proposition 23.19 *Let $E = E(D, y)$ be an ellipsoid with $D = Q_1^T Q_1$ and d be a nonzero vector in $\mathbf{R}^n$. Let T_1 be the affine transformation given by $T_1(x) = (Q_1^T)^{-1}x - (Q_1^T)^{-1}y$. Then $T_1(E) = B^n$ and $T_1(\{x : d^T x \le d^T y\}) = \{z : (Q_1 d)^T z \le 0\}$.*

Proof.

$$\begin{aligned} E &= \{x : (x-y)^T D^{-1}(x-y) \le 1\} \\ T_1(E) &= \{(Q_1^T)^{-1}(x-y) : (x-y)^T D^{-1}(x-y) \le 1\} \\ &= \{(Q_1^T)^{-1}(x-y) : (x-y)^T Q_1^{-1}(Q_1^T)^{-1}(x-y) \le 1\}. \end{aligned}$$

Let $z = (Q_1^T)^{-1}(x-y)$. Then $T_1(E) = \{z : z^T z \le 1\}$. Also, noting that $x = Q_1^T z + y$,

$$\begin{aligned} T_1(\{x : d^T x \le d^T y\}) &= \{(Q_1^T)^{-1}(x-y) : d^T x \le d^T y\} \\ &= \{z : d^T(Q_1^T z + y) \le d^T y\} \\ &= \{z : d^T Q_1^T z \le 0\} \\ &= \{z : (Q_1 d)^T z \le 0\}. \ \square \end{aligned}$$

Proposition 23.20 *Under the assumptions of the previous proposition,* $\mathrm{vol}\,(E) = |\det Q_1|\mathrm{vol}\,(B^n)$*.*

Proof. This follows immediate from Proposition 23.17. □

Definition 23.21 An $n \times n$ matrix Q is called *orthogonal* if $Q^T Q = I$.

Exercise 23.22 Prove that if Q is an orthogonal matrix then $\|Q(x-y)\| = \|x-y\|$ for all $x \in \mathbf{R}^n$; i.e., multiplication by Q is distance preserving. □

Proposition 23.23 *Given an arbitrary nonzero vector $d \in \mathbf{R}^n$, there exists an $n \times n$ orthogonal matrix Q_2 such that $Q_2 d = -\|d\|e_1$, where $e_1 = (1, 0, \ldots, 0)^T$.*

Exercise 23.24 Prove the above proposition. □

Proposition 23.25 *Given nonzero vector* $\overline{d}$ *and orthogonal matrix* Q_2 *such that* $Q_2\overline{d} = -\|\overline{d}\|e_1$*, then* $Q_2(B^n) = B^n$ *and* $Q_2(\{x : \overline{d}^T x \leq 0\}) = \{z : z_1 \geq 0\}$*.*

Proof. Taking $z = Q_2 x$, so that $x = Q_2^T z$,

$$\begin{aligned} Q_2(B^n) &= \{Q_2 x : x^T x \leq 1\} \\ &= \{z : z^T Q_2 Q_2^T z \leq 1\} \\ &= \{z : z^T z \leq 1\}. \end{aligned}$$

$$\begin{aligned} Q_2(\{x : \overline{d}^T x \leq 0\}) &= \{Q_2 x : \overline{d}^T x \leq 0\} \\ &= \{z : \overline{d}^T Q_2^T z \leq 0\} \\ &= \{z : (Q_2\overline{d})^T z \leq 0\} \\ &= \{z : -\|\overline{d}\| e_1^T z \leq 0\} \\ &= \{z : z_1 \geq 0\}. \ \square \end{aligned}$$

We now apply these propositions to our ellipsoid $E_t = (D_t, x_t)$ and half-ellipsoid $H_t = E_t \cap \{x : d^T x \leq d^T x_t\}$ in the following way: Find nonsingular Q_1 such that $Q_1^T Q_1 = D = D_t$. Let $\overline{d} = Q_1 d$. Find orthogonal Q_2 such that $Q_2\overline{d} = -\|\overline{d}\|e_1$. Define the map $T(x)$ to be the result of applying the map $T_1(x) = (Q_1^T)^{-1}(x - x_t)$, followed by multiplication by Q_2. From the propositions we conclude that $T(E_t) = B^n$ and $T(H_t) = B^n \cap \{z : z_1 \geq 0\}$. Note that

$$\begin{aligned} T(x) &= Q_2(Q_1^T)^{-1}(x - x_t) \\ &= (Q_2^T)^{-1}(Q_1^T)^{-1}(x - x_t) \\ &= (Q_1^T Q_2^T)^{-1}(x - x_t) \\ &= ((Q_2 Q_1)^T)^{-1}(x - x_t) \\ &= (Q^T)^{-1}(x - x_t), \end{aligned}$$

where $Q = Q_2 Q_1$. Note also that $T^{-1}(z) = Q^T z + x_t$.

Our plan of attack is to find the appropriate ellipsoid $\overline{E}_{t+1}$ containing $B^n \cap \{z : z_1 \geq 0\}$ and then recover E_{t+1} as $T^{-1}(\overline{E}_{t+1})$. During this process we will assume that $n \geq 2$.

Proposition 23.26 *Let*

$$z_{t+1} = \frac{1}{n+1} e_1$$

and

$$\Delta = \frac{n^2}{n^2 - 1}\left(I - \frac{2}{n+1} e_1 e_1^T\right).$$

Then the ellipsoid $\overline{E}_{t+1} = E(\Delta, z_{t+1})$ *contains* $\overline{H}_t = B^n \cap \{z : z_1 \geq 0\}$*.*

Proof. Note that Δ is symmetric, positive definite:

$$\Delta = \frac{n^2}{n^2-1}\begin{bmatrix} 1-\frac{2}{n+1} & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{bmatrix} = \frac{n^2}{n^2-1}\begin{bmatrix} \frac{n-1}{n+1} & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{bmatrix}.$$

Note also that $z - z_{t+1} = z - \frac{e_1}{n+1} = (z_1 - \frac{1}{n+1}, z_2, \ldots, z_n)$. Now

$$\begin{aligned}
\overline{E}_{t+1} &= \{z : (z - z_{t+1})^T \Delta^{-1} (z - z_{t+1}) \leq 1\} \\
&= \{z : \frac{n^2-1}{n^2}\frac{n+1}{n-1}(z_1 - \frac{1}{n+1})^2 + \frac{n^2-1}{n^2}\sum_{i=2}^{n} z_i^2 \leq 1\} \\
&= \{z : (\frac{n+1}{n})^2(z_1 - \frac{1}{n+1})^2 + \frac{n^2-1}{n^2}\sum_{i=1}^{n} z_i^2 - \frac{n^2-1}{n^2}z_1^2 \leq 1\} \\
&= \{z : \frac{n^2+2n+1}{n^2}z_1^2 - \frac{2(n+1)}{n^2}z_1 + \frac{1}{n^2} - \frac{n^2-1}{n^2}z_1^2 + \frac{n^2-1}{n^2}\sum_{i=1}^{n} z_i^2 \leq 1\} \\
&= \{z : \frac{2(n+1)}{n^2}z_1^2 - \frac{2(n+1)}{n^2}z_1 + \frac{1}{n^2} + \frac{n^2-1}{n^2}\sum_{i=1}^{n} z_i^2 \leq 1\} \\
&= \{z : \frac{2(n+1)}{n^2}z_1(z_1 - 1) + \frac{1}{n^2} + \frac{n^2-1}{n^2}\sum_{i=1}^{n} z_i^2 \leq 1\}.
\end{aligned}$$

Assume $z \in \overline{H}_t$. Then $\|z\|^2 \leq 1$, $z_1 \geq 0$, and $z_1 - 1 \leq 0$. So

$$\begin{aligned}
\frac{2(n+1)}{n^2}z_1(z_1 - 1) + \frac{1}{n^2} + \frac{n^2-1}{n^2}\sum_{i=1}^{n} z_i^2 &\leq \frac{1}{n^2} + \frac{n^2-1}{n^2} \\
&= 1.
\end{aligned}$$

Hence $z \in \overline{E}_{t+1}$. Therefore $\overline{H}_t \subseteq \overline{E}_{t+1}$. $\square$

Proposition 23.27 $\mathrm{vol}\,(\overline{E}_{t+1})/\mathrm{vol}\,(B^n) \leq e^{\frac{-1}{2(n+1)}}$.

Proof. By Proposition 23.20,

$$\begin{aligned}
\mathrm{vol}\,(\overline{E}_{t+1})/\mathrm{vol}\,(B^n) &= |\det\sqrt{\Delta}| \\
&= \sqrt{|\det\Delta|} \\
&= \sqrt{\left(\frac{n^2}{n^2-1}\right)^n \frac{n-1}{n+1}} \\
&= \sqrt{\left(\frac{n^2}{n^2-1}\right)^{n-1} \frac{n^2}{(n+1)^2}} \\
&= \frac{n}{n+1}\left(\frac{n^2}{n^2-1}\right)^{\frac{n-1}{2}}.
\end{aligned}$$

Using the fact that $1+x \le e^x$ for all x, we have

$$\frac{n}{n+1} = 1 + \frac{-1}{n+1} \le e^{\frac{-1}{n+1}}$$

and

$$\frac{n^2}{n^2-1} = 1 + \frac{1}{n^2-1} \le e^{\frac{1}{n^2-1}}.$$

So

$$\begin{aligned}
\frac{n}{n+1}\left(\frac{n^2}{n^2-1}\right)^{\frac{n-1}{2}} &\le e^{\frac{-1}{n+1}}\left(e^{\frac{1}{n^2-1}}\right)^{\frac{n-1}{2}} \\
&= e^{\frac{-1}{n+1}+\frac{1}{2(n+1)}} \\
&= e^{\frac{-1}{2(n+1)}}. \quad \square
\end{aligned}$$

We are now ready to verify formulas (11) and (12).

Proposition 23.28 $T^{-1}(z_{t+1}) = x_{t+1}$.

PROOF.

$$
\begin{aligned}
T^{-1}(z_{t+1}) &= Q^T(\frac{1}{n+1}e_1) + x_t \\
&= \frac{1}{n+1}(Q_2Q_1)^T e_1 + x_t \\
&= \frac{1}{n+1}Q_1^T Q_2^T e_1 + x_t \\
&= \frac{1}{n+1}Q_1^T \left(Q_2^T e_1\right) + x_t \\
&= \frac{1}{n+1}Q_1^T \left(-\frac{\overline{d}}{\|\overline{d}\|}\right) + x_t \\
&= -\frac{1}{n+1}Q_1^T \left(\frac{Q_1 d}{\|Q_1 d\|}\right) + x_t \\
&= x_t - \frac{1}{n+1}\frac{Q_1^T Q_1 d}{\|Q_1 d\|} \\
&= x_t - \frac{1}{n+1}\frac{Dd}{\|Q_1 d\|} \\
&= x_t - \frac{1}{n+1}\frac{Dd}{\sqrt{(Q_1 d)^T (Q_1 d)}} \\
&= x_t - \frac{1}{n+1}\frac{Dd}{\sqrt{d^T Q_1^T Q_1 d}} \\
&= x_t - \frac{1}{n+1}\frac{Dd}{\sqrt{d^T D d}} \\
&= x_{t+1}. \ \square
\end{aligned}
$$

Proposition 23.29 $T^{-1}(\overline{E}_{t+1}) = E_{t+1} = E(D_{t+1}, x_{t+1})$.

Proof. Using the change of variables $z = T(x)$,

$$\begin{aligned} T^{-1}(\overline{E}_{t+1}) &= \{T^{-1}(z) : (z - z_{t+1})^T \Delta^{-1} (z - z_{t+1}) \leq 1\} \\ &= \{x : (T(x) - T(x_{t+1}))^T \Delta^{-1} (T(x) - T(x_{t+1})) \leq 1\} \\ &= \{x : ((Q^T)^{-1}(x - x_t) - (Q^T)^{-1}(x_{t+1} - x_t))^T \Delta^{-1} ((Q^T)^{-1}(x- \\ &\quad x_t) - (Q^T)^{-1}(x_{t+1} - x_t)) \leq 1\} \\ &= \{x : (x - x_{t+1})^T Q^{-1} \Delta^{-1} (Q^T)^{-1} (x - x_{t+1}) \leq 1\}. \end{aligned}$$

So we must show that $D_{t+1} = Q^T \Delta Q$.

$$\begin{aligned} Q^T \Delta Q &= Q_1^T Q_2^T \Delta Q_2 Q_1 \\ &= Q_1^T Q_2^T \frac{n^2}{n^2-1} \left(I - \frac{2}{n+1} e_1 e_1^T \right) Q_2 Q_1 \\ &= \frac{n^2}{n^2-1} \left(Q_1^T Q_2^T Q_2 Q_1 - \frac{2}{n+1} Q_1^T Q_2^T e_1 e_1^T Q_2 Q_1 \right) \\ &= \frac{n^2}{n^2-1} \left(Q_1^T Q_1 - \frac{2}{n+1} Q_1^T \left(\frac{-\overline{d}}{\|\overline{d}\|} \right) \left(\frac{-\overline{d}}{\|\overline{d}\|} \right)^T Q_1 \right) \\ &= \frac{n^2}{n^2-1} \left(D - \frac{2}{n+1} Q_1^T \left(\frac{Q_1 d}{\|Q_1 d\|} \right) \left(\frac{Q_1 d}{\|Q_1 d\|} \right)^T Q_1 \right) \\ &= \frac{n^2}{n^2-1} \left(D - \frac{2}{n+1} \frac{(Dd)(Dd)^T}{d^T D d} \right) \\ &= D_{t+1}. \end{aligned}$$

Note also that $Q^T \Delta Q$ is symmetric, positive definite, since Δ is, and since $x^T(Q^T \Delta Q)x = (Qx)^T \Delta (Qx) > 0$ when $x \neq O$ (using the fact that Q is nonsingular). □

We finish our analysis by verifying that the volume of E_{t+1} is appropriately small.

Proposition 23.30 $\operatorname{vol}(E_{t+1})/\operatorname{vol}(E_t) \leq e^{\frac{-1}{2(n+1)}}$.

Proof. Using Proposition 23.17,

$$\begin{aligned}
\frac{\mathrm{vol}\,(E_{t+1})}{\mathrm{vol}\,(E_t)} &= \frac{\mathrm{vol}\,(T^{-1}(\overline{E}_{t+1}))}{\mathrm{vol}\,(T^{-1}(B^n))} \\
&= \frac{|\det Q|\mathrm{vol}\,(\overline{E}_{t+1})}{|\det Q|\mathrm{vol}\,(B^n)} \\
&= \frac{\mathrm{vol}\,(\overline{E}_{t+1})}{\mathrm{vol}\,(B^n)} \\
&\le e^{\frac{-1}{2(n+1)}}. \ \square
\end{aligned}$$

See Nemhauser-Wolsey for a numerical example. See Nemhauser-Wolsey and Chvátal for an explanation of why the number of steps needed to carry out the Ellipsoid Method is polynomial in the size of the problem, including how to overcome the difficulties that the presence of the square root introduces.

At the beginning of this section we went through a sequence of transformations to convert a linear program into a form suitable for application of the Ellipsoid Method. You should verify that each transformation increases the size of the problem by at most a polynomial amount, and so a polynomial algorithm for the Ellipsoid Method yields ultimately a polynomial algorithm for solving LP's.

The above method moves the hyperplane of the violated constraint to the center of the ellipsoid E_t and finds an ellipsoid E_{t+1} containing the half-ellipsoid $E_t \cap \{x : a^{i(t)}x \le a^{i(t)}x_{t+1}\}$. This is called the *shallow cut* method. Another variation does not move the violated constraint but finds an ellipsoid containing the set $E_t \cap \{x : a^{i(t)}x \le b_{i(t)}\}$. This is called the *deep cut* method. The formulas for E_{t+1} are, not surprisingly, more complicated. See Chvátal and Nemhauser-Wolsey for further discussion.

24 Appendix: Some More Linear Algebra

Definition 24.1 If M is an $n \times n$ matrix, $O \neq v \in \mathbf{R}^n$, $\lambda \in \mathbf{R}$, and $Mv = \lambda v$, then v is an *eigenvector* of M with *eigenvalue* λ.

Proposition 24.2 *If M is an $n \times n$ symmetric matrix, v^1 and v^2 are eigenvectors of M with eigenvalues λ_1, λ_2, respectively, and $\lambda_1 \neq \lambda_2$, then $v^{1T} v_2 = 0$.*

Proof.

$$\begin{aligned}
\lambda_1 v^{1T} v^2 &= (Mv^1)^T v^2 \\
&= v^{1T} M^T v^2 \\
&= v^{1T} M v^2 \\
&= v^{1T} \lambda_2 v^2 \\
&= \lambda_2 v^{1T} v^2
\end{aligned}$$

So $(\lambda_1 - \lambda_2) v^{1T} v^2 = 0$. Hence $v^{1T} v^2 = 0$. $\square$

Theorem 24.3 *Let M be a symmetric $n \times n$ matrix. Then there is an orthonormal basis of $\mathbf{R}^n$ consisting of eigenvectors of M with real eigenvalues.*

Proof. **Claim 1:** M has at least one eigenvector v with $\|v\| = 1$ and real eigenvalue λ. Proof: Consider the problem

$$\max_{x:\|x\|=1} x^T M x.$$

We are maximizing a continuous function over a closed, bounded set, so it is attained at some vector v. Let y be any vector such that $y^T v = 0$ and $\|y\| = 1$. Let $z = z(t) = (\cos t)v + (\sin t)y$. It is easy to check that $\|z\| = 1$. Let $f(t) = z^T M z$. f is differentiable and $f'(0) = 0$. Computing,

$$\begin{aligned}
f(t) &= z^T M z \\
&= ((\cos t)v^T + (\sin t)y^T) M ((\cos t)v + (\sin t)y) \\
&= (\cos^2 t) v^T M v + (\cos t \sin t) v^T M y + (\sin t \cos t) y^T M v + (\sin^2 t) y^T M y \\
&= (\cos^2 t) v^T M v + (2 \sin t \cos t) y^T M v + (\sin^2 t) y^T M y.
\end{aligned}$$

(We used the fact that since M is symmetric, $v^T M y = v^T M^T y = (v^T M^T y)^T = y^T M v$.) Then

$$f'(t) = (-2 \cos t \sin t) v^T M v + (-2 \sin^2 t + 2 \cos^2 t) y^T M v + (2 \sin t \cos t) y^T M y,$$

So $f'(0) = 2 y^T M v$, and we conclude $y^T M v = 0$. Hence Mv is orthogonal to every vector orthogonal to v, whence Mv is in the span of v. Therefore $Mv = \lambda v$ for some real number λ, and v is an eigenvector of M. This proves Claim 1.

Claim 2: If v is an eigenvector of M and $v^Ty = 0$, then $v^TMy = 0$. Proof: Let λ be the eigenvalue of v. Then $v^TMy = v^TM^Ty = (Mv)^Ty = \lambda v^Ty = 0$. This proves Claim 2.

Claim 3: If $v^1, \ldots, v^k$ are eigenvectors of M for some $0 \leq k < n$, then there exists another eigenvector v such that $\|v\| = 1$ and $v^Tv^j = 0$ for $0 \leq j \leq k$. Proof: This is true for $k = 0$ by Claim 1, so assume $k > 0$. Let $L = \text{span}\,\{v^1, \ldots, v^k\}$. Consider the problem

$$\max_{x \in L^\perp : \|x\|=1} x^TMx.$$

We are maximizing a continuous function over a closed, bounded set, so it is attained at some vector v. If $\dim L^\perp = 1$ then $Mv \in L^\perp$ by Claim 2, and so $Mv = \lambda v$ for some real number λ. If, on the other hand, $\dim L^\perp > 1$, let $y \in L^\perp$ be any vector such that $y^Tv = 0$ and $\|y\| = 1$. By an argument identical to that used in the proof of Claim 1 we can conclude that $y^TMv = 0$. By Claim 2, $v^{jT}Mv = 0$ for $1 \leq j \leq k$. So Mv is orthogonal to every vector orthogonal to v. Therefore $Mv = \lambda v$ for some real number λ. In either case, v is an eigenvalue of M. This proves Claim 3.

The proof of the theorem now follows by repeated application of Claim 3 (or by recasting the proof as a proof by induction). □

Definition 24.4 An $n \times n$ matrix M is *orthogonal* (or *orthonormal*) if $M^TM = I$. Thus, if M is orthogonal, then $M^{-1} = M^T$.

Theorem 24.5 *If M is a symmetric matrix, then there exists an orthogonal matrix Q and a diagonal matrix Λ such that $M = Q\Lambda Q^T$.*

Proof. Let $v^1, \ldots, v^n$ be an orthonormal basis of $\mathbf{R}^n$ consisting of eigenvectors of M. Let $\lambda_1, \ldots, \lambda_n$ be the associated eigenvalues, respectively. Let Q be the matrix whose columns are the vectors v^i, and Λ be the diagonal matrix whose diagonal entries are the eigenvalues λ_i. Note that Q is orthogonal. Then $MQ = Q\Lambda$, so $M = Q\Lambda Q^T$. □

Corollary 24.6 *If M is an $n \times n$ symmetric, positive definite matrix, then there exists a nonsingular $n \times n$ matrix Q_1 such that $M = Q_1^TQ_1$.*

Proof. If v is an eigenvector of M with eigenvalue λ, then $v^TMv = v^T\lambda v = \lambda\|v\|^2$. But M is positive definite, so $v^TMv > 0$, whence $\lambda > 0$. By the proof of the above theorem, $M = Q\Lambda Q^T$ where Λ is the diagonal matrix of eigenvalues of M. Every diagonal entry of Λ is positive. Let $\sqrt{\Lambda}$ be the diagonal matrix with diagonal entries $\sqrt{\lambda_i}$. Then $M = Q\sqrt{\Lambda}\sqrt{\Lambda}^TQ^T$. Take $Q_1 = \sqrt{\Lambda}^TQ^T$. □

Theorem 24.7 *Suppose $\{v^1, \ldots, v^n\}$ is a basis of $\mathbf{R}^n$. Then there exists an orthonormal basis $\{w^1, \ldots, w^n\}$ of $\mathbf{R}^n$ such that $w^1 = v^1/\|v^1\|$.*

Proof. Apply the Gram-Schmidt algorithm to $\{v^1, \ldots, v^n\}$. □.

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