RESEARCH STATEMENT

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1. Introduction

I work in the space between algebra and topology. Topology, and more precisely, homotopy theory, furnishes us with many commutative rings associated to finite groups. Some of these include the representation ring, Burnside ring, and the G-equivariant K-theory of a space. Many of these constructions admit extra structure in the form of natural operations on the ring, such as the Adams operations and symmetric powers. I'm especially interested in questions stemming from Adams operations on such rings and their role in stable homotopy theory.

Adams operations arise in a number of ways. The most standard way is through a λ-ring structure. A λ-ring is a ring with a notion of exterior powers. This includes representation rings and more generally the G-equivariant K-theory of a space. These Adams operations are powerful tools when studying K-theory. For instance, Adams and Atiyah gave an elegant proof of the Hopf invariant one problem using Adams operations on K-theory [AA66].

Adams operations also arise in chromatic homotopy theory. In this setting, it's not quite as obvious what makes something an Adams operation. However, Ando constructed Adams operations on Morava E-theory [And95]. Recently, in their disproof of the telescope conjecture, Burklund, Hahn, Levy, and Schlank constructed Adams operations on $BP\langle \mathrm{n} \rangle$.

The Burnside ring of a finite group is a pre-λ-ring. This implies it has Adams operations. These are additive, but not necessarily multiplicative operations. There is another standard way to construct Adams operations, by making use of power operations. Power operations are operations which mimic the n-fold power map. They leverage the fact that the n-fold power has a symmetric group action by permutation. Ando's construction of the Adams operations on Morava E-theory makes use of the power operations supported by Morava E-theory. My work during the last few years has been to reconcile these two ways of constructing Adams operations.

2. Background

We begin by outlining the definition of a λ-ring. A motivational approach with more complete details may be found in Knutson's lecture notes [Knu73], while Gay, Morris and Morris [GMM83] present them as a collection of properties to hold with respect to the operations.

Assume R to be a commutative ring and $\mathbb{N}$ the set of non-negative integers. Then, R is a pre-λ-ring if it is equipped with operations for each n, $\lambda_n : R \to R$ such that λ_0 is the constant map to 1, λ_1 is the identity map, and λ_n is 'exponential' in the following sense for each $x, y \in R$.

$$\lambda_n(x+y) = \sum_{i+j=n} \lambda_i(x)\lambda_j(y)$$

Notably, these operations are not expected to be additive or multiplicative, and in general necessarily fail to be. If the λ operations satisfy certain additional formulae derived from exterior powers regarding multiplication and repeated compositions, then we refer to R as a λ-ring (other literature may reference these additional conditions as necessary for special-λ-ring structure). Both pre-λ-rings and λ-rings contain the data to construct exponential operations $\{\beta_n\}$ analogous to the symmetric powers of a vector space and additive operations $\{\psi_n\}$ known as Adams operations. When R is torsion-free, any one of these three operations is sufficient to recover the other two.

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The complex representation ring of a finite group $R(G)$ is an example of a λ-ring, where the λ operations are precisely the exterior powers and the β operations the symmetric powers. As $R(G)$ is torsion-free, we may obtain Adams operations ψ_n. This will require the calculation of all $\lambda_{\leq n}$ and $\psi_{\leq n-1}$, and thus does pose some additional computational difficulty.

However, we can also obtain the Adams operation on $R(G)$ by another means. We will first need to define the power operation, P_n. This is induced by taking the n^{th} tensor power of the representation, and having G act diagonally while Σ_n, the symmetric group on n elements, permutes the tensor entries.

$$
\begin{aligned}
P_n : \qquad R(G) &\longrightarrow R(G \times \Sigma_n) \\
V &\longmapsto V^{\otimes n}
\end{aligned}
$$

We can then make use of a Kunneth isomorphism, where $R(G \times \Sigma_n) \cong R(G) \otimes R(\Sigma_n)$ due to the decomposition of irreducible representations of a product. We may then observe that the class functions of Σ_n happen to land fully in the integers. Hence, given W, a representation of Σ_n, we can evaluate its character on the conjugacy class of the long cycle, denoted $[\sigma] = [(123....n)]$ to obtain an integer. Then composing appropriately, we will obtain the Adams operation.

$$
\begin{tikzcd}
R(G) \arrow[r, "P_n"] \arrow[rrr, bend right, "\psi_n"'] & R(G \times \Sigma_n) \arrow[r, "\cong"] & R(G) \otimes R(\Sigma_n) \arrow[r, "id\otimes\chi([\sigma])"] & R(G)
\end{tikzcd}
$$

These two alternate constructions give the same Adams operations on the representation ring. Any pre-λ-ring is naturally endowed with the first construction, and the second type is a natural construction within cohomology theories relying on power operations. Morava E-theory is of this second type, and it motivates us to ask if we can create such a factorization for the Burnside ring, since it is both a pre-λ-ring and a cohomology theory.

We recall the definition of the Burnside ring, denoted $A(G)$. This is the Grothendieck completion of the semiring given by taking coproducts and Cartesian products of G-sets, and it has a canonical basis as an abelian group given by isomorphism classes of transitive G-sets.

$$A(G) \cong_{ab} \bigoplus_{[H] \in C(G)} \mathbb{Z}\{G/H\}$$

The easiest way to see there is a pre-λ-ring structure is to see $A(G)$ supports symmetric powers of G-sets, with the n^{th} symmetric power of a G-set X is given by X^n/Σ_n, i.e the diagonal G action on the set of unordered n-tuples of elements in X. This gives us what would normally be the β operations in $A(G)$. However, as we have discussed before, these operations all determine one another in a torsion-free ring like $A(G)$, and as such, we know there exist λ operations for the Burnside ring. This can be seen concretely by using a series of equalities which hold in pre-λ-rings, and these are discussed by Gay, Morris, and Morris [GMM83].

This gives us a recursive, but calculable, definition for the Adams operation on $A(G)$ of the pre-λ-ring flavor. The question remains if we are able to produce a construction of the second variety on it, as in many ways, the representation ring behaves similarly to the Burnside ring. Notably, we may define the power operation for $A(G)$ in a similar manner as before, taking the cartesian power rather than the vector power.

$$
\begin{aligned}
P_n : \qquad A(G) &\longrightarrow A(G \times \Sigma_n) \\
X &\longmapsto X^{\times n}
\end{aligned}
$$

This suggests we could emulate the construction on the representation ring, however, the Burnside ring does not have a Kunneth isomorphism. Thus, if we want to construct Adams operations from power operations on $A(G)$, we must constructive an additive map $\alpha_{G,n}$ from $A(G\times\Sigma_n)$ to $A(G)$ whose composite with P_n gives ψ_n.

$$\begin{tikzcd} A(G) \arrow[r, "P_n"] \arrow[rr, bend right, "\psi_n"'] & A(G\times\Sigma_n) \arrow[r, dashed, "\alpha_{G,n}"] & A(G) \end{tikzcd}$$

The construction of this map is the focus of my research, in addition to determining for what conditions we expect such an α to occur, as there exist groups for which reasonable conditions on α result in no such factorization existing.

3. Results

In this section we will describe a formula for $\alpha_{G,n}$ in the case that G is abelian, or G is non-abelian and satisfies a certain condition. We begin by considering some restrictions we place on $\alpha_{G,n}$. Firstly, it must be additive and factor through an ideal known as the transfer ideal. Additivity is essentially required to define it on the basis, and as the power operations are not additive, it must factor through the transfer ideal to ensure the composite it additive. Secondly, we require it respects restriction along any group homomorphism. And lastly, we require the following diagram to commute, and the domain of $\alpha_{e,n}$ is taken to be $A(e\times\Sigma_n)\cong A(\Sigma_n)$. In the representation ring case, the map which plays the role of α is an $R(G)$-module map. However, we do not expect it to be an $A(G)$-module map, and thus require this weaker condition.

$$\begin{tikzcd} A(G)\otimes A(\Sigma_n) \arrow[d] \arrow[dr, dashed, "1\otimes\alpha_{e,n}"] & \\ A(G\times\Sigma_n) \arrow[r, "\alpha_{G,n}"] & A(G) \end{tikzcd}$$

Under these conditions, the main result of my thesis describes the settings in which such factorizations occur and gives formulae for them.

We construct $\alpha_{G,n}$ by giving its value on each basis element of $A(G\times\Sigma_n)$. As such, we'll need to understand the structure of subgroups in $G\times\Sigma_n$. We employ Goursat's lemma for this, extracting the following data, where π_G is the projection to the G factor and π_{Σ_n} to the Σ_n factor respectively.

$$\begin{tikzcd} & \ker\pi_G\le\Sigma_n \arrow[d, hook] \arrow[dr] & \\ \ker\pi_{\Sigma_n}\le G \arrow[r, hook] \arrow[dr] & H\le G\times\Sigma_n \arrow[r, "\pi_{\Sigma_n}"] \arrow[d, "\pi_G"] & \operatorname{im}\pi_{\Sigma_n}\le\Sigma_n \\ & \operatorname{im}\pi_G\le G & \end{tikzcd}$$

The data from H further allows us to construct an isomorphism $\operatorname{im}\pi_G/\ker\pi_{\Sigma_n}\cong\operatorname{im}\pi_{\Sigma_n}/\ker\pi_G$. In return, such a selection of normal subgroups and a chosen isomorphism is equivalent data to a subgroup of $G\times\Sigma_n$. This is how we will more easily access these subgroups to define $\alpha_{G,n}$ on the transitive $G\times\Sigma_n$ set associated to them.

For your entertainment, the formula is provided below.

Theorem 3.1. *For G abelian and $n\in\mathbb{N}$, we may define $\alpha_{G,n}$ inductively as follows for each $H\le G$ and transitive subgroup $T\le\Sigma_n$, such that $\psi_n=\alpha_{G,n}\circ\mathbb{P}_n$. Here, $S\le G$*

$$\alpha_{G,n}(G\times\Sigma_n/(H\times T)=|\Sigma_n/T|G/H$$

$$\alpha_{G,n}((G \times \Sigma_n/((\Sigma_k^{n/k}) \rtimes \Gamma(a_{S/H})) = k(G/S) - \sum_{\substack{\hat{S}<S \\ [\hat{S}:H]=j|n}} \alpha_{G,n}((G \times \Sigma_n/((\Sigma_j^{n/j}) \rtimes \Gamma(a_{\hat{S}/H}))$$

While the construction of the above map gives motivation for how one could extend to other types of groups, there are a number of complications. Particularly, the map was recursively defined so as to apply the principle of inclusion-exclusion to the subgroup lattices of certain quotients. In the general setting, these quotients are not necessarily abelian, or possibly are not groups, resulting in a need to change strategy or require more stringent structure. Hence, a narrowing in scope was made.

Theorem 3.2. *For p prime, $m \in \mathbb{N}$, G such that p is the smallest prime dividing $|G|$ and all quotients of order p^m are abelian, we may define α_{G,p^m} such that $\psi_{p^m} = \alpha_{G,p^m} \circ \mathbb{P}_{p^m}$ and they satisfy the desired properties. The following is a general formula for $\alpha_{G,p}$, and an extension of this can be given for p^m. As before, H is a subgroup of G, T is a transitive subgroup of Σ_p, and additionally, K is a transitive subgroup of $G \times \Sigma_p$.*

$$\alpha_{G,p}(G \times \Sigma_p/(H \times T) = |\Sigma_n/T|G/H$$
$$\alpha_{G,p}(G \times \Sigma_p/K) = p(G/\operatorname{im}_G(K)) - (|\operatorname{im}_{\Sigma_p}(K)|/p)G/\ker_{\Sigma_p}(K)$$

This relationship between G and p is sufficient to find a factorization, however, one may ask if it is strictly required. While it may not be necessary in every case, it is known that a general factorization with the desired properties does not exist, as I have provided work showing no $\alpha_{\Sigma_4,3}$ can exist which factors ψ_3 on Σ_4.

4. Future Projects

4.1. Refining the Factorization Setting. Given that there does not always exist a natural α, there are two clear routes forward. First, I aim to find a larger class of groups than the p-inclusive groups, as Σ_3 shows this condition is not necessary. Second, I also aim to find an infinite class of groups where such a factorization cannot exist for the same reasons that it fails for Σ_4, but this will likely require significant computational assistance.

4.2. Operations on Marks(G). Some of my prior work includes understanding the total power operation on the table of marks. I'd like to conclude that project, and possibly, extend it to develop an α-like map for Marks(G).

4.3. Connections with Feit-Thompson. The Burnside ring turns out to have direct connections to solvability of odd groups. This provides a novel attack vector for a simple proof of the Feit-Thompson theorem, and UKY Bourbon Seminar has been considering various approaches. While this may not bear fruit, it remains an interesting route.

5. Undergraduate Research

While I was an undergraduate, I had an opportunity to work on a research project with Dr. Rachelle Bouchat. It provided me with a route to start asking questions about mathematics which didn't have obvious answers, becoming more interested in how we begin to tackle these problems and what makes for approachable yet interesting research. I hope to provide similar experiences for undergraduates in my future programs.

5.1. The Roots Functor. Hidden in the appendix of his paper Modular isogeny complexes, Charles Rezk describes a novel functor from commutative rings to categories [Rez12]. I previously led a group of undergraduates in a summer research experience, where they investigated the properties of these categories by simplifying the scope to $\mathbb{Z}/n\mathbb{Z}$ and considering the output as a graph. This is a very approachable problem for undergraduates, and while some basic results have been completed, it remains an interesting problem which can use Python or Sage for example-based intuition or verification of possible results.

5.2. **GAP Factorization Search.** In line with my current work, GAP could be used to search for examples of factorizations or counterexamples of such. Students would choose a class of groups or specific group to consider and learn how to use GAP, then apply this knowledge to a computational search.

5.3. **3D Visualization.** Alongside the University of Kentucky Mathlab and David Mehrle, I've worked to produce 3d models which assist in visualizing proofs, knots, and groups. This has included basic concepts like physical Pythagorean theorem re-orientation proofs, as well as engaging with recent mathematical discoveries, such as printing aperiodic monotiles.

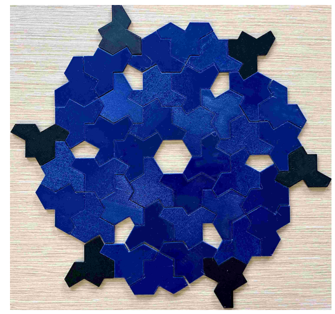

Figure 1: A mosaic made of 3d-printed 'hat' monotiles

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