Worksheet 17 - Trigonometric Identities (§7.2 and 7.3)

In Exercises 1 - 6, use the Even / Odd Identities to verify the identity. Assume all quantities are defined.

1. $\sin(3\pi - 2\theta) = -\sin(2\theta - 3\pi)$
2. $\cos\left(-\frac{\pi}{4} - 5t\right) = \cos\left(5t + \frac{\pi}{4}\right)$
3. $\tan(-t^2 + 1) = -\tan(t^2 - 1)$
4. $\csc(-\theta - 5) = -\csc(\theta + 5)$
5. $\sec(-6t) = \sec(6t)$
6. $\cot(9 - 7\theta) = -\cot(7\theta - 9)$

In Exercises 7 - 21, use the Sum and Difference Identities to compute the exact value. You may need other identities as well.

7. $\cos(75^\circ)$
8. $\sec(165^\circ)$
9. $\sin(105^\circ)$
10. $\csc(195^\circ)$
11. $\cot(255^\circ)$
12. $\tan(375^\circ)$
13. $\cos\left(\frac{13\pi}{12}\right)$
14. $\sin\left(\frac{11\pi}{12}\right)$
15. $\tan\left(\frac{13\pi}{12}\right)$
16. $\cos\left(\frac{7\pi}{12}\right)$
17. $\tan\left(\frac{17\pi}{12}\right)$
18. $\sin\left(\frac{\pi}{12}\right)$
19. $\cot\left(\frac{11\pi}{12}\right)$
20. $\csc\left(\frac{5\pi}{12}\right)$
21. $\sec\left(-\frac{\pi}{12}\right)$
22. If α is a Quadrant IV angle with $\cos(\alpha) = \frac{\sqrt{5}}{5}$, and $\sin(\beta) = \frac{\sqrt{10}}{10}$, where $\frac{\pi}{2} < \beta < \pi$, compute:
    (a) $\cos(\alpha + \beta)$
    (b) $\sin(\alpha + \beta)$
    (c) $\tan(\alpha + \beta)$
    (d) $\cos(\alpha - \beta)$
    (e) $\sin(\alpha - \beta)$
    (f) $\tan(\alpha - \beta)$
23. If $\csc(\alpha) = 3$, where $0 < \alpha < \frac{\pi}{2}$, and β is a Quadrant II angle with $\tan(\beta) = -7$, compute:
    (a) $\cos(\alpha + \beta)$
    (b) $\sin(\alpha + \beta)$
    (c) $\tan(\alpha + \beta)$
    (d) $\cos(\alpha - \beta)$
    (e) $\sin(\alpha - \beta)$
    (f) $\tan(\alpha - \beta)$
24. If $\sin(\alpha) = \frac{3}{5}$, where $0 < \alpha < \frac{\pi}{2}$, and $\cos(\beta) = \frac{12}{13}$ where $\frac{3\pi}{2} < \beta < 2\pi$, compute:
    (a) $\sin(\alpha + \beta)$
    (b) $\cos(\alpha - \beta)$
    (c) $\tan(\alpha - \beta)$

25. If $\sec(\alpha) = -\frac{5}{3}$, where $\frac{\pi}{2} < \alpha < \pi$, and $\tan(\beta) = \frac{24}{7}$, where $\pi < \beta < \frac{3\pi}{2}$, compute:

(a) $\csc(\alpha - \beta)$ (b) $\sec(\alpha + \beta)$ (c) $\cot(\alpha + \beta)$

In Exercises 26 - 38, verify the identity.

26. $\cos(\theta - \pi) = -\cos(\theta)$

27. $\sin(\pi - \theta) = \sin(\theta)$

28. $\tan\left(\theta + \frac{\pi}{2}\right) = -\cot(\theta)$

29. $\sin(\alpha + \beta) + \sin(\alpha - \beta) = 2\sin(\alpha)\cos(\beta)$

30. $\sin(\alpha + \beta) - \sin(\alpha - \beta) = 2\cos(\alpha)\sin(\beta)$

31. $\cos(\alpha + \beta) + \cos(\alpha - \beta) = 2\cos(\alpha)\cos(\beta)$

32. $\cos(\alpha + \beta) - \cos(\alpha - \beta) = -2\sin(\alpha)\sin(\beta)$

33. $\dfrac{\sin(\alpha + \beta)}{\sin(\alpha - \beta)} = \dfrac{1 + \cot(\alpha)\tan(\beta)}{1 - \cot(\alpha)\tan(\beta)}$

34. $\dfrac{\cos(\alpha + \beta)}{\cos(\alpha - \beta)} = \dfrac{1 - \tan(\alpha)\tan(\beta)}{1 + \tan(\alpha)\tan(\beta)}$

35. $\dfrac{\tan(\alpha + \beta)}{\tan(\alpha - \beta)} = \dfrac{\sin(\alpha)\cos(\alpha) + \sin(\beta)\cos(\beta)}{\sin(\alpha)\cos(\alpha) - \sin(\beta)\cos(\beta)}$

36. $\dfrac{\sin(t + h) - \sin(t)}{h} = \cos(t)\left(\dfrac{\sin(h)}{h}\right) + \sin(t)\left(\dfrac{\cos(h) - 1}{h}\right)$

37. $\dfrac{\cos(t + h) - \cos(t)}{h} = \cos(t)\left(\dfrac{\cos(h) - 1}{h}\right) - \sin(t)\left(\dfrac{\sin(h)}{h}\right)$

38. $\dfrac{\tan(t + h) - \tan(t)}{h} = \left(\dfrac{\tan(h)}{h}\right)\left(\dfrac{\sec^2(t)}{1 - \tan(t)\tan(h)}\right)$

In Exercises 39 - 48, use the Half Angle Formulas to compute the exact value. You may need other identities as well.

39. $\cos(75°)$ (compare with Exercise 7)

40. $\sin(105°)$ (compare with Exercise 9)

41. $\cos(67.5°)$

42. $\sin(157.5°)$

43. $\tan(112.5°)$

44. $\cos\left(\frac{7\pi}{12}\right)$ (compare with Exercise 16)

45. $\sin\left(\frac{\pi}{12}\right)$ (compare with Exercise 18)

46. $\cos\left(\frac{\pi}{8}\right)$

47. $\sin\left(\frac{5\pi}{8}\right)$

48. $\tan\left(\frac{7\pi}{8}\right)$

In Exercises 49 - 58, use the given information about θ to **compute** the exact values of:

- $\sin(2\theta)$
- $\cos(2\theta)$
- $\tan(2\theta)$
- $\sin\left(\frac{\theta}{2}\right)$
- $\cos\left(\frac{\theta}{2}\right)$
- $\tan\left(\frac{\theta}{2}\right)$

49. $\sin(\theta) = -\frac{7}{25}$ where $\frac{3\pi}{2} < \theta < 2\pi$
50. $\cos(\theta) = \frac{28}{53}$ where $0 < \theta < \frac{\pi}{2}$
51. $\tan(\theta) = \frac{12}{5}$ where $\pi < \theta < \frac{3\pi}{2}$
52. $\csc(\theta) = 4$ where $\frac{\pi}{2} < \theta < \pi$
53. $\cos(\theta) = \frac{3}{5}$ where $0 < \theta < \frac{\pi}{2}$
54. $\sin(\theta) = -\frac{4}{5}$ where $\pi < \theta < \frac{3\pi}{2}$
55. $\cos(\theta) = \frac{12}{13}$ where $\frac{3\pi}{2} < \theta < 2\pi$
56. $\sin(\theta) = \frac{5}{13}$ where $\frac{\pi}{2} < \theta < \pi$
57. $\sec(\theta) = \sqrt{5}$ where $\frac{3\pi}{2} < \theta < 2\pi$
58. $\tan(\theta) = -2$ where $\frac{\pi}{2} < \theta < \pi$

In Exercises 59 - 73, verify the identity. Assume all quantities are defined.

59. $(\cos(\theta) + \sin(\theta))^2 = 1 + \sin(2\theta)$
60. $(\cos(\theta) - \sin(\theta))^2 = 1 - \sin(2\theta)$
61. $\tan(2\theta) = \dfrac{1}{1 - \tan(\theta)} - \dfrac{1}{1 + \tan(\theta)}$
62. $\csc(2\theta) = \dfrac{\cot(\theta) + \tan(\theta)}{2}$
63. $8\sin^4(\theta) = \cos(4\theta) - 4\cos(2\theta) + 3$
64. $8\cos^4(\theta) = \cos(4\theta) + 4\cos(2\theta) + 3$
65. $\sin(3\theta) = 3\sin(\theta) - 4\sin^3(\theta)$
66. $\sin(4\theta) = 4\sin(\theta)\cos^3(\theta) - 4\sin^3(\theta)\cos(\theta)$
67. $32\sin^2(\theta)\cos^4(\theta) = 2 + \cos(2\theta) - 2\cos(4\theta) - \cos(6\theta)$
68. $32\sin^4(\theta)\cos^2(\theta) = 2 - \cos(2\theta) - 2\cos(4\theta) + \cos(6\theta)$
69. $\cos(4\theta) = 8\cos^4(\theta) - 8\cos^2(\theta) + 1$
70. $\cos(8\theta) = 128\cos^8(\theta) - 256\cos^6(\theta) + 160\cos^4(\theta) - 32\cos^2(\theta) + 1$ (HINT: Use the result to 69.)
71. $\sec(2\theta) = \dfrac{\cos(\theta)}{\cos(\theta) + \sin(\theta)} + \dfrac{\sin(\theta)}{\cos(\theta) - \sin(\theta)}$
72. $\dfrac{1}{\cos(\theta) - \sin(\theta)} + \dfrac{1}{\cos(\theta) + \sin(\theta)} = \dfrac{2\cos(\theta)}{\cos(2\theta)}$
73. $\dfrac{1}{\cos(\theta) - \sin(\theta)} - \dfrac{1}{\cos(\theta) + \sin(\theta)} = \dfrac{2\sin(\theta)}{\cos(2\theta)}$

In Exercises 74 - 79, write the given product as a sum. You may need to use an Even/Odd Identity.

74. $\cos(3\theta)\cos(5\theta)$
75. $\sin(2\theta)\sin(7\theta)$
76. $\sin(9\theta)\cos(\theta)$
77. $\cos(2\theta)\cos(6\theta)$
78. $\sin(3\theta)\sin(2\theta)$
79. $\cos(\theta)\sin(3\theta)$

In Exercises 80 - 85, write the given sum as a product. You may need to use an Even/Odd Identity.

80. $\cos(3\theta)+\cos(5\theta)$
81. $\sin(2\theta)-\sin(7\theta)$
82. $\cos(5\theta)-\cos(6\theta)$
83. $\sin(9\theta)-\sin(-\theta)$
84. $\sin(\theta)+\cos(\theta)$
85. $\cos(\theta)-\sin(\theta)$

86. Suppose θ is a Quadrant I angle with $\sin(\theta)=x$. Verify the following formulas

    (a) $\cos(\theta)=\sqrt{1-x^2}$
    (b) $\sin(2\theta)=2x\sqrt{1-x^2}$
    (c) $\cos(2\theta)=1-2x^2$

87. Do the formulas, if any, in Exercise 86 change if we change assume θ is a Quadrant II, III, or IV angle?

88. Suppose θ is a Quadrant I angle with $\tan(\theta)=x$. Verify the following formulas

    (a) $\cos(\theta)=\dfrac{1}{\sqrt{x^2+1}}$
    (b) $\sin(\theta)=\dfrac{x}{\sqrt{x^2+1}}$
    (c) $\sin(2\theta)=\dfrac{2x}{x^2+1}$
    (d) $\cos(2\theta)=\dfrac{1-x^2}{x^2+1}$

89. Do the formulas, if any, in Exercise 88 change if we change assume θ is a Quadrant II, III, or IV angle?

90. If $\sin(\theta)=\dfrac{x}{2}$ for $-\dfrac{\pi}{2}<\theta<\dfrac{\pi}{2}$, compute an expression for $\cos(2\theta)$ in terms of x.

91. If $\tan(\theta)=\dfrac{x}{7}$ for $-\dfrac{\pi}{2}<\theta<\dfrac{\pi}{2}$, compute an expression for $\sin(2\theta)$ in terms of x.

92. If $\sec(\theta)=\dfrac{x}{4}$ for $0<\theta<\dfrac{\pi}{2}$, compute an expression for $\ln|\sec(\theta)+\tan(\theta)|$ in terms of x.

93. Show that $\cos^2(\theta)-\sin^2(\theta)=2\cos^2(\theta)-1=1-2\sin^2(\theta)$ for all θ.

94. Let θ be a Quadrant III angle with $\cos(\theta)=-\dfrac{1}{5}$. Show that this is not enough information to determine the sign of $\sin\left(\dfrac{\theta}{2}\right)$ by first assuming $3\pi<\theta<\dfrac{7\pi}{2}$ and then assuming $\pi<\theta<\dfrac{3\pi}{2}$ and computing $\sin\left(\dfrac{\theta}{2}\right)$ in both cases.