MA 113 Calculus I — Fall 2013

Exam 3 — Tuesday, 19 November 2013

Name: ______________________________

Section: ______________________________

Last 4 digits of student ID #: ________

This exam has ten multiple choice questions (five points each) and five free response questions (ten points each). Additional blank sheets are available if necessary for scratch work. No books or notes may be used. Turn off your cell phones and do not wear ear-plugs during the exam. You may use a calculator, but not one which has symbolic manipulation capabilities.

On the multiple choice problems:

- Select your answer by placing an X in the appropriate square of the multiple choice answer box on the front page of the exam.
- Carefully check your answers. No credit will be given for answers other than those indicated on the *multiple choice answer box*.

On the free response problems:

- Clearly indicate your answer and the reasoning used to arrive at that answer *(unsupported answers may not receive credit)*,
- Give exact answers, rather than decimal approximations to the answer (unless otherwise stated).

Each free response question is followed by space to write your answer. Please write your solutions neatly in the space below the question.

Multiple Choice Answers

Question					
1	A	B	C	D	E
2	A	B	C	D	E
3	A	B	C	D	E
4	A	B	C	D	E
5	A	B	C	D	E
6	A	B	C	D	E
7	A	B	C	D	E
8	A	B	C	D	E
9	A	B	C	D	E
10	A	B	C	D	E

[B,C,D,A,C, C,C,D,C,C]

Exam Scores

Question	Score	Total
MC		50
11		10
12		10
13		10
14		10
15		10
Total		100

Record the correct answer to the following problems on the front page of this exam.

1. Give the linearization of the function $f(x) = x \sin x$ at the point $x = \pi$:

   (A) $y(x) = \pi - \pi x$
   (B) $y(x) = \pi^2 - \pi x$
   (C) $y(x) = \pi^2 + \pi x$
   (D) $y(x) = x - \pi$
   (E) $y(x) = \pi x$

2. Let f be a differentiable function. Newton's method provides a sequence of successive approximations to a root x^* of the equation $f(x) = 0$. Suppose x_0 is the first guess, and that x_1 and x_2 are computed according to Newton's method. What is the formula for the third approximation x_3 to the root x^*?

   (A) $x_3 = x_0 + f(x_2)/f'(x_2)$
   (B) $x_3 = x_1 - f(x_2)/f'(x_2)$
   (C) $x_3 = x_2 - f(x_2)/f'(x_2)$
   (D) $x_3 = x_2 - f(x_1)/f'(x_1)$
   (E) None of the above

Record the correct answer to the following problems on the front page of this exam.

3. The indefinite integral $\int x^{-1/3}\ dx$ is equal to

   (A) $3x^{1/3} + C$

   (B) $x^{2/3} + C$

   (C) $\frac{2}{3}x^{1/3} + C$

   (D) $\frac{3}{2}x^{2/3} + C$

   (E) None of the above

4. If $\sum_{j=1}^{10} a_j = 20$ and $\sum_{j=1}^{10} b_j = 44$, then the sum

$$\sum_{j=1}^{10}(5a_j - 2b_j)$$

   equals

   (A) 12

   (B) 60

   (C) 88

   (D) 100

   (E) 188

Record the correct answer to the following problems on the front page of this exam.

5. The most general anti-derivative of $f(x) = x^2 + \cos(\pi x)$ is:

   (A) $x^3 + \sin(\pi x) + C$

   (B) $x - \pi \sin(\pi x) + C$

   (C) $\frac{1}{3}x^3 + \frac{1}{\pi}\sin(\pi x) + C$

   (D) $\frac{1}{3}x^3 - \sin(\pi x)$

   (E) $\frac{1}{3}x^3 + \pi \cos(\pi x)$

6. The limit

$$\lim_{x\to\infty}\left(\frac{3x^2 + 7x + 2}{-2x^2 - 4x + 3}\right)$$

   is:

   (A) $3/2$

   (B) $2/3$

   (C) $-3/2$

   (D) $7/4$

   (E) Does not exist

Record the correct answer to the following problems on the front page of this exam.

7. The function $f(x) = x^3 - 6x^2 + 8x$ is:

   (A) concave up on $(-\infty, 0)$ and on $(2, \infty)$
   (B) concave up on $(0, 2)$
   (C) concave down on $(-\infty, 2)$ and concave up on $(2, \infty)$
   (D) concave down on $(2, 4)$
   (E) concave up on $(-\infty, 2)$ and concave down on $(2, \infty)$

8. The limit

$$\lim_{x \to 1} \left(\frac{x^2 - 1}{\sin(\pi x)} \right)$$

   is:

   (A) -2
   (B) $2/\pi$
   (C) -2π
   (D) $-2/\pi$
   (E) Does not exist

Record the correct answer to the following problems on the front page of this exam.

9. The critical points of the function $f(x) = x^2(x^2 - 4)$ are

   (A) $\{-2, 0, 2\}$
   (B) $\{-\sqrt{2}, \sqrt{2}\}$
   (C) $\{-\sqrt{2}, 0, \sqrt{2}\}$
   (D) $\{-1, 0, 1\}$
   (E) $\{-\sqrt{2}/2, 0, \sqrt{2}/2\}$

10. A baseball is thrown upward and its height in meters from the ground satisfies $h(t) = 20t - 5t^2$, where time is measured in seconds. At what time is the ball's instantaneous velocity equal to its average velocity over the time interval $[0, 2]$?

    (A) 10 seconds
    (B) 2 seconds
    (C) 1 second
    (D) 1/2 second
    (E) 4 seconds

Free Response Questions: Show your work!

11. An adventurer wants to reach a camp 10 km downstream on the opposite side of a perfectly straight 2 km wide river. Starting at the edge of the river, the adventurer swims at 2 km/hr in a straight line to a point on the opposite shore. The adventurer then walks at 4 km/hr the remainder of the way to the camp.

    (a) Sketch the situation labeling all important quantities.

    (b) Find the path that minimizes the time it takes the adventurer to get to the camp 10 km downstream on the opposite side. How many kilometers downstream from his starting point will the adventurer be when he finishes his swim and reaches the opposite shore? Present all your work.

(a) Let x be the distance downstream where the adventurer comes ashore. Diagram should show $10-x$ for walking distance, 10 km total length, 2 km width.

(b)
Swimming: Distance $\sqrt{x^2+4}$, time $\sqrt{x^2+4}/2$
Walking: Distance $10-x$, time $(10-x)/4$

$$T(x) = \frac{1}{2}\sqrt{x^2+4} + \frac{10-x}{4}$$

Domain $0 \leq x \leq 10$
Find critical point:

$$T'(x) = \frac{2x}{4\sqrt{x^2+4}} - \frac{1}{4}$$

so $T'(x) = 0$ when $\dfrac{x}{2\sqrt{x^2+4}} = \dfrac{1}{4}$.

Solving for x we get $x = \dfrac{2\sqrt{3}}{3}$ km
Test endpoints:

$$T(0) = 1 + 5/2 = 7/2 = 3.5 \text{ hours}$$

$$T(4/(2\sqrt{3})) \approx 3.36 \text{ hours}$$

$$T(10) = \frac{1}{2}\sqrt{100+4} \approx 5.1 \text{ hours}$$

Hence, $x = \dfrac{2\sqrt{3}}{3}$ km.

(a) 3 points
1 point for introducing the distance x, 1 point for correctly showing the dimensions of the river (2 km wide, 10 km long), and 1 point for correctly showing the adventurer's path

(b) 7 points
1 point each for swimming and walking times

1 point for correct formula, 1 point for domain

1 point for derivative formula

1 point for correct critical point

1 point for checking endoints

Free Response Questions: Show your work!

12. Find the critical points of the function f on the interval $(-2, 2)$ where

$$f(x) = \begin{cases} 2x & -2 < x < 0 \\ x^2 - 2x & 0 \leq x < 2 \end{cases}$$

Determine if the function f has a local maximum or a local minimum at each critical point. Show all your work and use calculus to justify your answers.

Compute $$f'(x) = \begin{cases} 2 & -2 < x < 0 \\ 2x - 2 & 0 < x < 2 \end{cases}$$	Derivative formula: **2 points** 1 point for $-2 < x < 0$ 1 point for $0 < x < 2$
Critical points: • $x = 1$ since $f'(1) = 0$ • $x = 0$ since $f'(0)$ does not exist	Critical Points: **3 points** 1 point for $x = 1$ 1 point for $x = 0$ with an additional point for observing that $f'(0)$ does not exist
The sign of $f'(x)$ is: • positive in $(-2, 0)$ • negative in $(0, 1)$ • positive in $(1, 2)$	Sign of $f'(x)$: **3 points** 1 point for each interval
By the first derivative test, f has: • a local maximum at $x = 0$ • a local minimum at $x = 1$	Local Extrema: **2 points** 1 point each

Free Response Questions: Show your work!

13. (a) State the Mean Value Theorem.

    (b) Suppose that f is a differentiable function on the real line and $2 \leq f'(x) \leq 4$ for x in the interval $(1, 6)$. If $f(1) = 2$, use the Mean Value Theorem for f on the interval $[1, 6]$ to determine the largest and smallest possible values for $f(6)$.

(a)

Suppose that f is continuous in $[a, b]$ and differentiable in (a, b). Then, there is a point c between a and b so that

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

(a) 4 points

1 point each for:

- f is continuous in $[a, b]$
- f is differentiable in (a, b)
- there eixsts c between a and b
- correct formula for $f'(c)$

(b)

Since f is differentiable on the real line, then f is also continuous, and hence satisfies the hypothesis of the Mean Value Theorem.

By the Mean Value Theorem, there is a point c between 1 and 6 so that

$$\frac{f(6) - f(1)}{5} = f'(c)$$

or

$$\frac{f(6) - 12}{5} = f'(c)$$

so that

$$f(6) = 5f'(c) + 2$$

for some c between 1 and 6. Since $2 \leq f'(c) \leq 4$, we conclude that

$$12 \leq f(6) \leq 22.$$

(b) 6 points

1 point for checking that f is continuous

2 points for stating the MVT as applied to the function f on $[1, 6]$

1 point for substituting $f(1) = 2$

1 points for deducing the equality

$$f(6) = 5f'(c) + 2$$

2 points for concluding that $12 \leq f(6) \leq 22$

14. (a) We know the following sum for $N = 1, 2, 3, \ldots$:

$$\sum_{j=1}^{N} j = \frac{N(N+1)}{2}.$$

Using this, evaluate the sum

$$\frac{1}{N}\sum_{j=1}^{N}\left(5 + \frac{2j}{N}\right)$$

(b) Compute the limit as N tends to infinity of the result obtained in part a).

(a)

Evaluate

$$\frac{1}{N}\sum_{j=1}^{N}\left(5 + \frac{2j}{N}\right)$$
$$= \frac{1}{N}\left(\sum_{j=1}^{N} 5 + \sum_{j=1}^{N}\frac{2j}{N}\right)$$
$$= \frac{1}{N}\left(5N + \frac{N(N+1)}{N}\right)$$
$$= 5 + \frac{N+1}{N}$$

(a) 6 points

2 points for using linearity

2 points for using the identity for $\sum_{j=1}^{N} j$ correctly

2 points for simplification

(b)

$$\lim_{N\to\infty}\left(5 + \frac{N+1}{N}\right)$$
$$= \lim_{N\to\infty} 5 + \lim_{N\to\infty}\left(\frac{N+1}{N}\right)$$
$$= 5 + 1$$
$$= 6$$

(b) 4 points

Remark: Students should receive full credit for evaluating limit in this part, even if answer in part (a) is not correct.

1 point for correct statement of limit

2 points for simplifying using limit laws

1 point for correct evaluation of limit

15. The graph below is the graph of the function $f(x) = x^3 - 5x + 1$.

(a) Taking $x_0 = 2.5$, use Newton's method to find a root of the equation $f(x) = 0$. Show your work below and give the values of x_1 and x_2 correctly rounded to three decimal places in the table to the right.

n	x_n
0	2.500
1	**2.200**
2	**2.132**

(b) Taking $x_0 = -3$, use Newton's method to find a root of the equation $f(x) = 0$. Show your work below and give the values of x_1 and x_2 correctly rounded to three decimal places in the table to the right.

n	x_n
0	-3.000
1	**-2.500**
2	**-2.345**

Since

$$f(x) = x^3 - 5x + 1$$

and

$$f'(x) = 3x^2 - 5$$

Newton's method gives

$$x_{n+1} = x_n - \frac{x_n^3 - 5x_n + 1}{3x_n^2 - 5}.$$

2 points

Correctly compute $f'(x)$ (1 point) and state the iteration formula for finding zeros of $f(x)$ (1 point)

(a)

Numerical answers for x_1, x_2 are shown above.

(a) 4 points

2 points each for correct evaluation of x_1, x_2, to three decimal place accuracy

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(b)

Numerical answers for x_1, x_2 are shown in the table above.

(b) 4 points

2 point each for correct evaluation of x_1, x_2,