Name: ______________________________ Student ID: ________________________

Section: ____________________________

Do not remove this answer page. You will return the whole exam. You will be allowed two hours to complete this test. No books or notes may be used. You may use a graphing calculator during the exam, but NO calculator with a Computer Algebra System (CAS) or communication capabilities is permitted. You may not use a phone or other communication device during the exam.

The exam consists of 12 multiple choice questions that are worth 5 points each and 4 free response questions that are worth 10 points each. You should work the multiple choice questions on the question page. After you have checked your work carefully, record your answers by completely filling in the circle below that corresponds to your answer. If you must change your answer, make a note on the front of the exam. Be sure to check carefully when you transfer your answers to the cover sheet.

Show all work to receive full credit on the free response problems. You do not need to compute a decimal approximation to your answer. For example, the answer 4π is preferred to 12.57.

Multiple Choice Questions

1 (A) (B) (C) (D) (E) **7** (A) (B) (C) (D) (E)

2 (A) (B) (C) (D) (E) **8** (A) (B) (C) (D) (E)

3 (A) (B) (C) (D) (E) **9** (A) (B) (C) (D) (E)

4 (A) (B) (C) (D) (E) **10** (A) (B) (C) (D) (E)

5 (A) (B) (C) (D) (E) **11** (A) (B) (C) (D) (E)

6 (A) (B) (C) (D) (E) **12** (A) (B) (C) (D) (E)

SCORE

Multiple Choice	13	14	15	16	Total Score
60	10	10	10	10	100

Trigonometric functions for special angles

θ	$\cos(\theta)$	$\sin(\theta)$	θ	$\cos(\theta)$	$\sin(\theta)$	θ	$\cos(\theta)$	$\sin(\theta)$	θ	$\cos(\theta)$	$\sin(\theta)$
0	1	0	$\frac{\pi}{2}$	0	1	π	-1	0	$\frac{3\pi}{2}$	0	-1
$\frac{\pi}{6}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\frac{2\pi}{3}$	$-\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{7\pi}{6}$	$-\frac{\sqrt{3}}{2}$	$-\frac{1}{2}$	$\frac{5\pi}{3}$	$\frac{1}{2}$	$-\frac{\sqrt{3}}{2}$
$\frac{\pi}{4}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{3\pi}{4}$	$\frac{-\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{5\pi}{4}$	$\frac{-\sqrt{2}}{2}$	$-\frac{\sqrt{2}}{2}$	$\frac{7\pi}{4}$	$\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{2}}{2}$
$\frac{\pi}{3}$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{5\pi}{6}$	$-\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\frac{4\pi}{3}$	$-\frac{1}{2}$	$-\frac{\sqrt{3}}{2}$	$\frac{11\pi}{6}$	$\frac{\sqrt{3}}{2}$	$-\frac{1}{2}$

Multiple Choice Questions

1. (5 points) Assume that the radius r of a sphere is expanding at a rate of 3 inches/min. The volume of a sphere is $V = \frac{4}{3}\pi r^3$.

   Determine the rate at which the volume is changing with respect to time when $r = 2$ inches.

   A. 4π inches^3/min

   B. 24π inches^3/min

   C. 48π inches^3/min

   D. 3π inches^3/min

   E. 12π inches^3/min

2. (5 points) Suppose a population at time t consists of $P(t) = 2025e^{0.0113t}$ individuals. Find the time it takes for the population to triple.

   A. $\ln(3)/0.0113$

   B. $2025/0.0113$

   C. 6075

   D. 0.0339

   E. $\ln(0.0113)/3$

3. (5 points) Let $f(x) = e^x$. Find the quadratic approximation to f at $x = 0$.

   A. $1 + x + \frac{1}{2}x^2$

   B. $1 - x - \frac{x^2}{2}$

   C. $1 - x - \frac{1}{2}x^2$

   D. $1 + e + \frac{1}{2}e^2$

   E. $x + \frac{1}{2}x^2$

4. (5 points) If possible, find the global maximum and global minimum values of $f(x) = x^3 - 6x^2 + 12x - 8$ on the interval $[0, 3]$.

   A. global maximum 1*; global minimum* -8

   B. global maximum 3; global minimum -8

   C. global maximum 1; global minimum 0

   D. global maximum does not exist; global minimum -8

   E. global maximum 3; global minimum does not exist

5. (5 points) Let $f(x) = 1/x^2$. If possible, find the global maximum and minimum values for f on the interval $[-1, 1]$.

   A. The maximum does not exist, and the minimum is -1.

   B. The maximum is 1, and the minimum does not exist.

   C. The maximum does not exist, and the minimum is 0.

   D. The maximum is 1 and the minimum is 0.

   E. The maximum does not exist, and the minimum is 1.

6. (5 points) Find the intervals where the function

$$f(x) = xe^{-x}$$

is increasing and the ones where it is decreasing.

   A. increasing on $(-\infty, 1)$ and decreasing on $(1, \infty)$

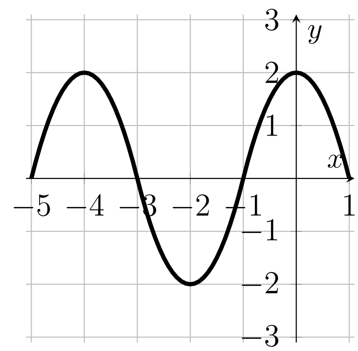
   B. decreasing on $(-\infty, 1)$ and increasing on $(1, \infty)$

   C. is always increasing

   D. is always decreasing

   E. increasing on $(-\infty, 0)$ and decreasing on $(0, \infty)$

7. (5 points) The graph below is the derivative f' of a function f. There are three open intervals on which function f is increasing or decreasing. Give the intervals and describe whether f is increasing or decreasing on each interval.



A. increasing on $(-5,-3)$, decreasing on $(-3,-1)$, and increasing on $(-1,1)$

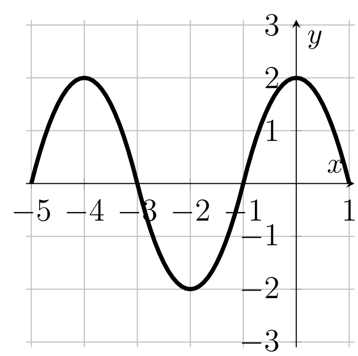
B. is always decreasing

C. is always increasing

D. increasing on $(-5,-4)$, decreasing on $(-2,0)$, and decreasing on $(-4,-2)$

E. increasing on $(-3,-1)$, decreasing on $(-5,-3)$, and decreasing on $(-1,1)$

8. (5 points) The graph below is the derivative f' of a function f. On which interval(s) is f concave up?



A. $(-4,-2)$ and $(0,1)$

B. $(-4,0)$

C. $(-3,-1)$

D. $(-5,-4)$ and $(-2,0)$

E. $(-5,-3)$ and $(-1,1)$

9. (5 points) Find the value of $\lim_{x\to\pi/2} \dfrac{\tan(2x)}{\tan(15x)}$. Hint: rewrite the expression in terms of $\sin(2x)$, $\cos(2x)$, $\sin(15x)$, and $\cos(15x)$.

   A. 0

   B. $\dfrac{2}{15}$

   C. $\dfrac{15}{2}$

   D. 1

   E. Does not exist

10. (5 points) Use L'Hôpital's rule to evaluate the following limit

$$\lim_{x\to\infty} x^2 e^{-x}$$

   A. 1

   B. 2

   C. 0

   D. -1

   E. The limit does not exist

11. (5 points) Let $f(x) = \sin(x) + \cos(x)$. Find F, an anti-derivative of f with $F(0) = 2$.

    A. $F(x) = 3 + \sin(x) - \cos(x)$

    B. $F(x) = 3 - \sin(x) - \cos(x)$

    C. $F(x) = 1 - \sin(x) + \cos(x)$

    D. $F(x) = 1 + \sin(x) + \cos(x)$

    E. $F(x) = 2 - \sin(x) + \cos(x)$

12. (5 points) Find the function f with derivative $f'(x) = e^{113x}$ that passes through the point $(0, 114/113)$.

    A. $f(x) = \dfrac{1}{113}e^{113x} + 3$

    B. $f(x) = \dfrac{1}{113}e^{113x} + 5$

    C. $f(x) = \dfrac{1}{113}e^{113x} - 2$

    D. $f(x) = \dfrac{1}{113}e^{113x} + 1$

    E. $f(x) = \dfrac{1}{113}e^{113x} + 4$

Free response questions: Show work clearly with proper notation.

13. (10 points) Suppose that the growth rate of the population in the Lexington-Fayette metro area is 1.15%. Assuming that we have a population of 350, 000 in 2025 and that this growth rate continues, we can write the following function $P(t) = 350000e^{0.0115t}$ that gives the population after t years.

    (a) Estimate the population in the year 2050. Assume that $t = 0$ corresponds to the year 2025.

    (b) Find the time it takes for the population to double.

Solution:

a) The year 2050 corresponds to $t = 25$, thus $P(25) = 350000e^{0.0115(25)} = 466,582$.

b) We need $350000e^{kT} = 2(350000)$, then $e^{-kT} = 2$, and applying the natural logarithm, we obtain the equation $kT = \ln(2)$. Solving for T, gives $T = \ln(2)/k$. As a decimal $T \approx 60.2737 \approx 60$ years.

Grading. (a) Figuring out $t = 25$ corresponding to 2025 (2 points), applying $P(25)$ (2 points), answer (1 point).

(b) Exponential equation for T, $e^{-kT} = 2$. (3 points). Result of applying ln (1 point). Value for T (1 point). Accept either $\ln(2)/0.0115$ or ≈ 60 years.

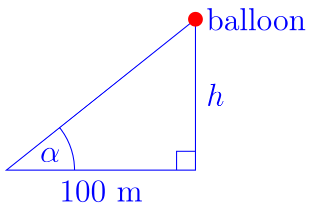
14. (10 points) A helium balloon rising vertically is tracked by an observer located 100 meters from the lift-off point.

   (a) Make a sketch summarizing the information in the problem. Label the quantities you use in your solution.

   (b) At a certain moment, the *angle* between the observer's line-of-sight and the horizontal is $\frac{\pi}{4}$, and it is changing at a constant rate of 0.03 radians per second. How fast is the balloon rising at this moment?

Solution:

a)



b) We have $\tan\alpha = h/100$. Thus, $h = 100\tan\alpha$. Then

$$\frac{dh}{dt} = 100\sec^2\alpha\frac{d\alpha}{dt}$$

but $\frac{d\alpha}{dt} = 0.03$ when $\alpha = \pi/4$ and $\sec(\pi/4) = \sqrt{2}$. Thus,

$$\frac{dh}{dt} = 100\left(\sqrt{2}\right)^2(0.03) = 6$$

Thus, the balloon is rising 6 meters per second.

Grading:

a) Sketch with at least one side or angle labeled (3 points)

b) Equation involving α and h (2 points). Differentiate to find equation involving $\frac{dh}{dt}$ (2 points). Find $\sec(\pi/4)$ (1 point). Find answer for $\frac{dh}{dt}$ (1 point). Units in part a) and b) (1 point).

15. (10 points) Let $f(x) = x^2 e^{-x^2}$. Clearly use calculus to answer the following:

(a) Find the intervals over which $f(x)$ is increasing, and find the intervals over which $f(x)$ is decreasing.

(b) Find each value x where f has a local maximum and each value x where f has a local minimum.

Solution: a) $f'(x) = 2xe^{-x^2}(1 - x^2)$. Thus, critical numbers: $x = 0, -1, 1$.

Signs of $f'(x)$:

		-1		0		1	
$f'(x)$	++		- -		+ +		- -

Thus, f is increasing on $(-\infty, -1)$, decreasing on $(-1, 0)$, increasing on $(0, 1)$, and decreasing on $(1, \infty)$.

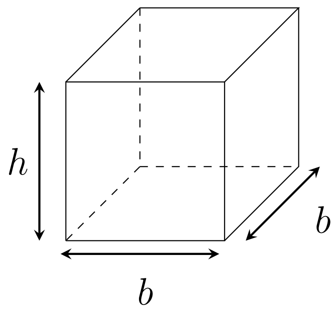
b)

- There is a local maximum at $x = -1$,
- There is a local mininum at $x = 0$, and
- There is a local maximum at $x = 1$.

Grading: a) Find $f'(x)$ (2 points), find critical points (3 points), use sign of f' to decide on increase or decrease (2 points).

c) Use sign of f' to characterize local extrema (3 points)

16. (10 points) A box is constructed out of two different types of metal. The metal for the top and bottom, which are both square, costs \$4 per square foot. The metal for the sides costs \$4 per square foot. Find the dimensions that minimize cost if the box has a volume of 1728 cubic feet.



The volume of the box is

$$V = b^2 h$$

The cost function is

$$C = 4(4bh) + 4b^2 + 4b^2$$

Solution: We want to minimize $C = 4(4bh) + 4b^2 + 4b^2 = 16bh + 8b^2$ with the relation $V = b^2 h = 1728$.

We solve for h in the last equation to find $h = \dfrac{1728}{b^2}$. Substitute into the equation for the cost to find C as $C(b) = \frac{27648}{b} + 8b^2$.

Thus to find the minimum cost, we want to find the minimum value of C for $b > 0$.

We compute $C'(b) = -\frac{27648}{b^2} + 16b = \frac{16b^3 - 27648}{b^2}$ and find that the critical with $b > 0$ is $b = \sqrt[3]{1728} = 12$. For this value of b, $h = 12$.

This will be a minimum since $C'(b) < 0$ for $0 < b < 12$ and $C'(b) > 0$ for $12 < b$.

Thus, the dimensions that minimize the cost function are $b = 12$ ft and $h = 12$ ft.

Grading: Solve area for h (2 points), substitute to express C in terms of b (2 points), find critical point (3 points), give b and h for mininum cost (2 points), units on final answer (1 point).

Problem does not ask for justification that we have found a minimum.