MA 113 Calculus I, Fall 2023
Written Assignment #2

Instructions: The purpose of this assignment is to develop your ability to formulate and communicate mathematical arguments. *Unreadable work may receive no credit.*

You should provide well-written, complete answers to each of the questions. We will look for correct mathematical arguments, well-written explanations, and correct use of English. Your solution should be formulated in complete sentences. As appropriate, you may want to include diagrams or equations written out on a separate line. You may read your textbook to find examples of how we communicate mathematics. You should also look at the section on "Expectations for student work" in the syllabus web page.

Students may use word processing software, a writing app on a tablet or pencil and paper to prepare their solutions. It may be simpler to draw graphs and mathematical expressions by hand. The final solution *must* be prepared as a single pdf and uploaded to Canvas. For those that write their solutions on paper, a tablet or phone can be used to scan the work into a pdf file. Scanning functionality is built into Google Drive and the Files app on Apple products. Since you are submitting this work to Canvas, there is no need to put your name on your work. We suggest that you not include your name so that the instructors have the option Canvas to grade anonymously.

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1. Consider the semi-circle given by the graph of the function $f(x) = \sqrt{1-x^2}$.

   (a) Find the slope of a radius given by the line segment with endpoints $(0,0)$ and a point (a,b) on the semi-circle. Express the slope in terms of a.

   (b) Form the difference quotient
   $$\frac{f(a+h)-f(a)}{h}$$
   and simplify this difference quotient to obtain an expression which may be evaluated at $h=0$.

   (c) Use the definition of the derivative (see Definition 2.2.1 in CLPI) to find $f'(a)$ for $-1 < a < 1$,
   $$f'(a) = \lim_{h\to 0}\frac{f(a+h)-f(a)}{h}.$$

   (d) Find the slope of the tangent line to the semi-circle at $(a, f(a))$.

   (e) What is the relationship between the slopes of the radius and the tangent line through a point $(a, f(a))$?

Remark: Recall that in geometry, we learned a relation between the tangent line to a circle and a radius of the circle. This assignment shows a way to see this relationship is true using the methods of calculus.

`Solution:` a) The line segment from $(0,0)$ to a point (a,b) will have slope b/a as long as $a \neq 0$. If the point is on the semi-circle, then $b = \sqrt{1-a^2}$ and we may write the slope of the radius in terms of a as
$$\frac{\sqrt{1-a^2}}{a}$$

b) We form the difference quotient and multiply by the conjugate of the numerator to simplify and obtain for $h \neq 0$ that

$$\begin{aligned}
\frac{f(a+h)-f(a)}{h} &= \frac{\sqrt{1-(a+h)^2}-\sqrt{1-a^2}}{h} \\
&= \frac{(\sqrt{1-(a+h)^2}-\sqrt{1-a^2})(\sqrt{1-(a+h)^2}+\sqrt{1-a^2})}{h(\sqrt{1-(a+h)^2}+\sqrt{1-a^2})} \\
&= \frac{1-(a+h)^2-(1-a^2)}{h(\sqrt{1-(a+h)^2}+\sqrt{1-a^2})} \\
&= \frac{1-a^2-2ah-h^2-1+a^2}{h(\sqrt{1-(a+h)^2}+\sqrt{1-a^2})} \\
&= \frac{h(-2a-h)}{h(\sqrt{1-(a+h)^2}+\sqrt{1-a^2})} \\
&= \frac{(-2a-h)}{(\sqrt{1-(a+h)^2}+\sqrt{1-a^2})}
\end{aligned}$$

c) Now that we have simplified the difference quotient to obtain an expression which is a continuous function of h, we may evaluate the limit by substitution,

$$\lim_{h\to 0}\frac{f(a+h)-f(a)}{h} = \lim_{h\to 0}\frac{(-2a-h)}{(\sqrt{1-(a+h)^2}+\sqrt{1-a^2})} = \frac{-a}{\sqrt{1-a^2}}.$$

Thus $f'(a) = -a/\sqrt{1-a^2}$.

d) The slope of the tangent at $(a, f(a))$ is the derivative $f'(a) = -a/\sqrt{1-a^2}$.

e) Slopes of the tangent line through $(a, f(a))$ and the radius through $(a, f(a))$ are negative reciprocals of each other

$$-\frac{a}{\sqrt{1-a^2}} = -\left(\frac{\sqrt{1-a^2}}{a}\right)^{-1}$$

at least when both slopes are defined.

Remark: This relation for the slopes means that the tangent line to a circle is perpendicular to the radius.

Grading: a) Slope in terms of a (1 point) do not deduct if student fails to exclude $a = 0$, b) Multiply by conjugate (1 point), simplify and cancel h (2 points), c) find derivative (1 point), d) restate derivative as slope of tangent line (1 point) e) relation of slopes (1 point), also accept that slopes are perpendicular

(1 point) if majority of solution is expressed in complete sentences.