MA 113 Calculus I, Fall 2023
Written Assignment #3

Instructions: The purpose of this assignment is to develop your ability to formulate and communicate mathematical arguments. *Unreadable work may receive no credit.*

You should provide well-written, complete answers to each of the questions. We will look for correct mathematical arguments, well-written explanations, and correct use of English. Your solution should be formulated in complete sentences. As appropriate, you may want to include diagrams or equations written out on a separate line. You may read your textbook to find examples of how we communicate mathematics. You should also look at the section on "Expectations for student work" in the syllabus web page.

Students may use word processing software, a writing app on a tablet or pencil and paper to prepare their solutions. It may be simpler to draw graphs and mathematical expressions by hand. The final solution *must* be prepared as a single pdf and uploaded to Canvas. For those that write their solutions on paper, a tablet or phone can be used to scan the work into a pdf file. Scanning functionality is built into Google Drive and the Files app on Apple products. Since you are submitting this work to Canvas, there is no need to put your name on your work. We suggest that you not include your name so that the instructors have the option Canvas to grade anonymously.

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1. Let $f(x) = \dfrac{1-e^x}{1+e^x}$.
   (a) Find the derivative f'. Carefully justify each step using the differentiation rules from the text. (You may identify rules by the number or by a short description such as the quotient rule. For example, you may refer to Theorem 2.4.5 or the quotient rule.)
   (b) Find the interval(s) where $f'(x) > 0$ or $f'(x) < 0$.
   (c) Find $\lim_{x\to\infty} f(x)$ and $\lim_{x\to-\infty} f(x)$.
   (d) Use the information above to sketch the graph of f.

2. (a) Write an equation for the tangent line at a point (a, a^2) on the graph of $y = x^2$.
   (b) Find an equation that a must satisfy if the tangent lines passes through the point $(0, -2)$.
   (c) Solve the equation from part b) and find all tangent lines to the graph of $y = x^2$ that pass through the point $(0, -2)$.
   (d) Graph the curve $y = x^2$ and the two tangent lines to check your answer to part 2b).

`Solution:` 1a) To compute the derivative of $f(x) = \dfrac{1-e^x}{1+e^x}$, we use the rule for differentiating constants (see Example 2.2.2), the rule for differentiating the exponential function (Theorem 2.7.5) and the rule for differentiating a sum (Lemma 2.4.1. or Theorem 2.4.2) to find

$$\frac{d}{dx}(1+e^x) = e^x, \qquad \frac{d}{dx}(1-e^x) = -e^x.$$

Next we use the quotient rule (Theorem 2.4.5) to find

$$\begin{aligned} f'(x) &= \left(\frac{1-e^x}{1+e^x}\right)' \\ &= \frac{(1-e^x)'(1+e^x)-(1-e^x)(1+e^x)'}{(1+e^x)^2} \\ &= \frac{-e^x(1+e^x)-(1-e^x)e^x}{(1+e^x)^2} \\ &= \frac{-2e^x}{(1+e^x)^2}. \end{aligned}$$

b) Since $e^x > 0$ for all x, it follows that $f'(x) = -2e^x/(1+e^x)^2 < 0$ for x in $(-\infty, \infty)$.

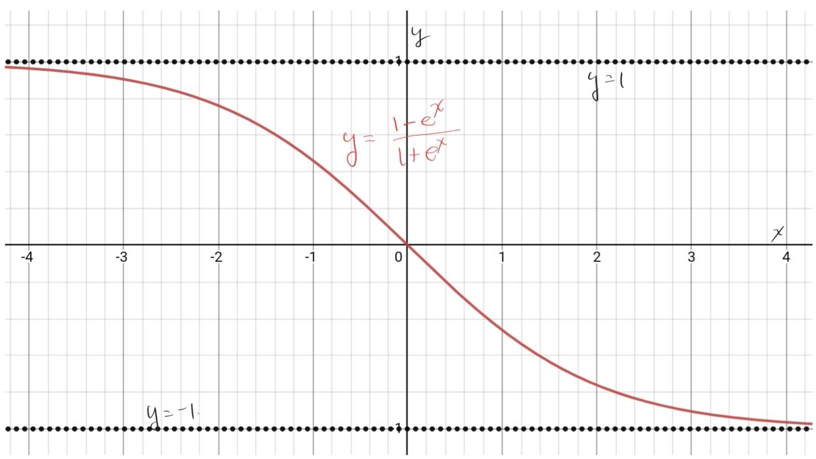
c) Since $\lim_{x\to-\infty} e^x = \lim_{x\to\infty} e^{-x} = 0$, we may use the limit rules for limits at $\pm\infty$ to see

$$\lim_{x\to-\infty} \frac{1-e^x}{1+e^x} = 1$$

and

$$\lim_{x\to+\infty} \frac{1-e^x}{1+e^x} = \lim_{x\to+\infty} \frac{e^x}{e^x}\left(\frac{e^{-x}-1}{e^{-x}+1}\right) = -1.$$

d) Using that the slope of the tangent line at each point is negative and the limits at $\pm\infty$, we may sketch the graph as



2a) The derivative of $f(x) = x^2$ is $f'(x) = 2x$. Thus the tangent line to $y = x^2$ at a is $y - a^2 = 2a(x-a)$ in point-slope form or $y = 2ax - a^2$ in slope-intercept form.

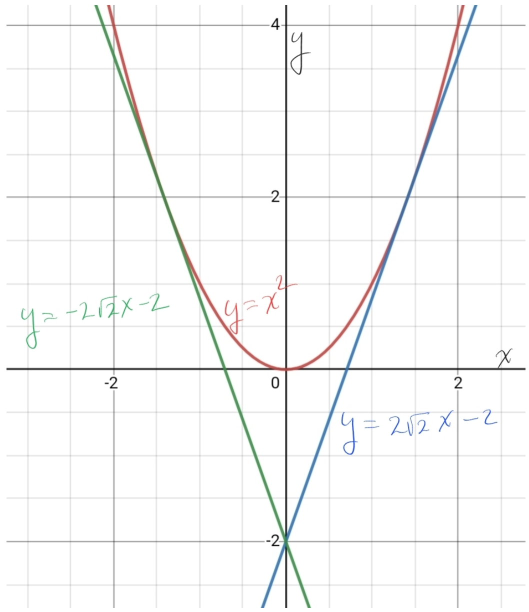
b) This line will pass through $(x, y) = (0, -2)$ if this point satisfies the equation $y = 2ax - a^2$. Substituting $(x, y) = (0, -2)$ gives the equation

$$-2 = 2a \cdot 0 - a^2 \quad \text{or} \quad a^2 = 2.$$

c) Solving the equation $a^2 = 2$ we find two solutions $a = \pm\sqrt{2}$. Using these values of a, we have the equations of the tangent lines are

$$y = 2\sqrt{2}x - 2, \qquad y = -2\sqrt{2}x - 2.$$

d) The graph below gives the graph of $y = x^2$ and the two tangent lines.



Grading: 1a) (1 point) for finding correct derivative, (1 point) for citing at least two differentiation rules (my solution uses 4).

b) Interval (1 point)

c) Values of limits (1 point)

d) Graph. Look for correct monotonicity and limits (1 point). Mark if axes not labelled, but do not deduct.

2a) General tangent line, accept any correct form (1 point)

bc) Values for a (1 point), equations of tangent lines (1 point)

d) Note errors, but no points are assigned to this part. Encourage students to label their graphs.