MA 113 Calculus I, Fall 2023
Written Assignment #4

Instructions: The purpose of this assignment is to develop your ability to formulate and communicate mathematical arguments. *Unreadable work may receive no credit.*

You should provide well-written, complete answers to each of the questions. We will look for correct mathematical arguments, well-written explanations, and correct use of English. Your solution should be formulated in complete sentences. As appropriate, you may want to include diagrams or equations written out on a separate line. You may read your textbook to find examples of how we communicate mathematics. You should also look at the section on "Expectations for student work" in the syllabus web page.

Students may use word processing software, a writing app on a tablet or pencil and paper to prepare their solutions. It may be simpler to draw graphs and mathematical expressions by hand. The final solution *must* be prepared as a single pdf and uploaded to Canvas. For those that write their solutions on paper, a tablet or phone can be used to scan the work into a pdf file. Scanning functionality is built into Google Drive and the Files app on Apple products. Since you are submitting this work to Canvas, there is no need to put your name on your work. We suggest that you not include your name so that the instructors have the option Canvas to grade anonymously.

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1. Consider the ellipse given by the equation $x^2 + 2y^2 = 6$.
   (a) Find all points on the ellipse where the slope is equal to 1.
   (b) Find all tangent lines to the ellipse with slope equal to 1.
2. (a) Let $f(x) = 2x - \cos(x)$ and explain why f' is never zero.
   (b) If $f(x) = 0$ has two distinct solutions $f(a) = f(b) = 0$, use the mean value theorem to conclude that there must be a point c between a and b with $f'(c) = 0$.
   (c) Explain why the equation $2x - \cos(x) = 0$ has at most one solution. Hint: You may want to study Example 2.13.3 in our textbook CLP1.
   (d) Find an interval which contains a solution of the $2x - \cos(x) = 0$ and use the intermediate value theorem to show that your answer is right.

`Solution:` 1. a) We differentiate the equation of the ellipse implicitly to find $2x + 4yy' = 0$ and solve for $y' = -2x/(4y) = -x/(2y)$. If we require $y' = -x/(2y) = 1$, we have $x = -2y$. Using this relation in the equation of the ellipse we conclude $(-2y)^2 + 2y^2 = 4y^2 + 2y^2 = 6$ or $y = \pm 1$. If $y = \pm 1$, then $x = -2y = \mp 2$. Thus the points with slope 1 are

$$(-2, 1) \qquad \text{and} \qquad (2, -1).$$

b) For each tangent line we need a point and a slope. At the point $(-2, 1)$, the slope is 1 (as required in part a). The tangent line will be $y - 1 = 1(x + 2)$ or $y = x + 3$ in slope-intercept form.

For the point $(2, -1)$, the slope is again 1 and the tangent line is $y + 1 = 1(x - 2)$ or $y = x - 3$.

The Desmos page `https://www.desmos.com/calculator/opiuxi7rjo` shows that our answers are correct.

2. a) The derivative $f'(x) = 2 + \sin(x)$. If $f'(x) = 0$, then $\sin(x) = -2$ and this equation has no solution since $-1 \leq \sin(x) \leq 1$ for all x.

b) The function f is differentiable and continuous on the real line, so if $f(a) = f(b) = 0$ for $a \neq b$, then the mean value theorem tells us there is a point c between a and b, so that $0 = f(b) - f(a) = f'(c)(b-a)$. Since $b - a \neq 0$, we would have $f'(c) = 0$.

c) In part b), we saw that if there are two solutions of $f(x) = 0$, then there is a solution to $f'(x) = 0$. In part a), we saw that there is no solution to $f'(x) = 0$, so there cannot be two solutions of $f(x) = 0$.

d) The function $f(x) = 2x - \cos(x)$ is continuous on the real line, so we may apply the intermediate value theorem on any closed interval. We consider the interval $[0, \pi/2]$. We have $f(0) = -\cos(0) = -1$ and $f(\pi/2) = \pi - \cos(\pi/2) = \pi$. Since 0 is between -1 and π, the intermediate value theorem tells us there is a solution to $f(x) = 0$ in the interval $[0, \pi/2]$.

Grading: 1a) Find $y' = -x/2y$ (1 point), solve for points (x, y) (1 point), b) Give tangent lines (1 point).

2a) Observe range of sin is $[-1, 1]$ and conclude derivative is not zero (1 point).

b) Use Mean value theorem or Rolle's theorem (1 point).

c) Use a) and b) to conclude that we cannot have two solutions to $f(x) = 0$. (1 point)

d) Give an interval and use intermediate value theorem to verify that it contains one solution to $f(x) = 0$. (1 point)

Give one additional point if they check the hypotheses continuity and differentiability of MVT in part 2b) or the hypothesis of continuity for IVT in part 2d).

Comment: Some students may try to use that f is increasing in part c) to conclude $f(x) = 0$ can have at most one solution. This is part of Corollary 2.13.12 in the textbook CLP1 and should be accepted.