MA 113 Calculus I, Fall 2023
Written Assignment #5

Instructions: The purpose of this assignment is to develop your ability to formulate and communicate mathematical arguments. *Unreadable work may receive no credit.*

You should provide well-written, complete answers to each of the questions. We will look for correct mathematical arguments, well-written explanations, and correct use of English. Your solution should be formulated in complete sentences. As appropriate, you may want to include diagrams or equations written out on a separate line. You may read your textbook to find examples of how we communicate mathematics. You should also look at the section on "Expectations for student work" in the syllabus web page.

Students may use word processing software, a writing app on a tablet or pencil and paper to prepare their solutions. It may be simpler to draw graphs and mathematical expressions by hand. The final solution *must* be prepared as a single pdf and uploaded to Canvas. For those that write their solutions on paper, a tablet or phone can be used to scan the work into a pdf file. Scanning functionality is built into Google Drive and the Files app on Apple products. Since you are submitting this work to Canvas, there is no need to put your name on your work. We suggest that you not include your name so that the instructors have the option Canvas to grade anonymously.

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1. Consider the function $f(x) = x^4 + x$ for $-\infty < x < +\infty$.
   (a) Find the smallest value of f or explain why f has no smallest value.
   (b) Find the largest value of f or explain why f has no largest value.
2. (a) Suppose that a material has a half-life of T years. How long will it take for 10 grams of the material to decay to 3 grams? Your answer will be an expression involving T.
   (b) Suppose 10 grams decays to 3 grams in 221 years. What is the half-life?
   (c) Suppose a quantity $m(t)$ is decaying exponentially with $m(0) = 10$ and $m(221) = 3$. What is the differential equation $m' = km$ that m solves? That is, find the value of k for which $m' = km$.

`Solution:`

1) a) We have $f(x) = x^4 + x$ and $f'(x) = 4x^3 + 1$. Solving $f'(x) = 0$ we find $f(-1/\sqrt[3]{4}) = 0$. Also, $f'(x) > 0$ for $x > -1/\sqrt[3]{4}$. Thus using Corollary 2.13.12b), f is increasing to the right of $x = -1/\sqrt[3]{4}$ which implies $f(x) \geq f(-1/\sqrt[3]{4})$ if $x > -1/\sqrt[3]{4}$. Also, $f'(x) < 0$ for $x < -1/\sqrt[3]{4}$ and Corollary 2.13.12d) implies that $f(x) \geq f(-1/\sqrt[3]{4})$ for $x < -1/\sqrt[3]{4}$.

Together we have $f(x) \geq f(-1/\sqrt[3]{4})$ for all x and thus f has a minimum value of $f(-1/\sqrt[3]{4}) = ((-1/\sqrt[3]{4})^3 + 1)(-1/\sqrt[3]{4}) = -3 \cdot 4^{-4/3}$ at $x = -1/\sqrt[3]{4}$.

b) Since $f(x)$ is a polynomial of even degree and the highest degree term is $+x^4$, we have $\lim_{x\to+\infty} f(x) = +\infty$ and $\lim_{x\to-\infty} f(x) = +\infty$. If $f(x_0)$ is a value of f, then since $\lim_{x\to\infty} f(x) = +\infty$, we may find x with $f(x) > f(x_0)$. Thus f has no maximum value.

2a) If T is the half-life, then we know $e^{-rT} = 1/2$ and solving this equation for r, we have $-rT = \ln(1/2) = -\ln(2)$ or $r = \ln(2)/T$.

If we start with 10 grams, then we know the mass $m(t)$ as a function of time is $m(t) = 10e^{-rt}$. The problem asks to find S so that $m(S) = 3$. This requires solving the equation $10e^{-rS} = 3$. The solution is $S = -\ln(0.3)/r$ and using the expression above for k we have $S = -\ln(0.3)T/\ln(2)$.

b) We are given that $10e^{-r\cdot 221} = 3$. Using the expression from part a) relating S and the half-life T, we have that $221 = -\ln(3/10)T/\ln(2)$ or $T = -221\ln(2)/\ln(3/10)$. Using a calculator, $T \approx 127.23$ years.

c) From part a) we know that $m(t) = 10e^{-rt}$ where r and the half-life T are related by $r = \ln(2)/T$. Using the expression for T in part b), $T = -221\ln(2)/\ln(3/10)$ we find that

$$k = -\ln(3/10)/221 \approx -5.448 \times 10^{-3} = -0.005448.$$

Note that in part c), we are writing $m(t) = Ae^{kt}$ and need k to be a negative value.

Grading: 1a) Find critical point $x = -1/\sqrt[3]{4}$ (1 point). $f(-1/\sqrt[3]{4})$ is smallest value (1 point). Do not give credit for an approximate minimum from a calculator without some evidence that the student used calculus to find the minimum.

b) f has no largest value (1 point), since $\lim_{x\to\pm\infty} f(x) = +\infty$ (1 point). Accept an argument that f has arbitrarily large values whether or not it involves taking a limit.

2a) Formula relating r and the half-life T (1 point).

b) Equation for 3/10-life, $e^{-rS} = 3/10$ (1 point). Expression for S in terms of T (1 point).

c) Expression for $k = -r$ (1 point).

If majority of answers are not in complete sentences, remind students to write out their work in sentences. But this is not assigned points.