# Worksheet # 1: Functions and inverse functions

1. Give the domain and ranges of the following functions.

   (a) $f(x) = \frac{x+1}{x^2+x-2}$

   SOLUTION: a) The domain is $\{x : x \neq -2 \text{ and } x \neq 1\}$ and the range is all real numbers, $\mathbf{R} = (-\infty, \infty)$.

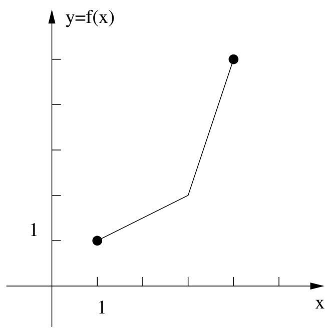
   (b) $g(t) = \frac{1}{\sqrt{t^2-1}}$

   SOLUTION: b) The domain $\{t : -1 < t < 1\} = (-1, 1)$ and the range is $\{t : t \geq 1\} = [1, \infty)$.

2. If $f(x) = 5x+7$ and $g(x) = x^2$, find $f \circ g$ and $g \circ f$. Are the functions $f \circ g$ and $g \circ f$ the same function?
   SOLUTION: The two functions, $f \circ g$ and $g \circ f$, are not the same.

3. Let $f(x) = 2 + \frac{1}{x+3}$. Determine the inverse function of $f$, $f^{-1}$. Give the domain and range of $f$ and the inverse function $f^{-1}$. Verify that $f \circ f^{-1}(x) = x$.

4. Consider the function whose graph appears below.




   (a) Find $f(3)$, $f^{-1}(2)$ and $f^{-1}(f(2))$.

   (b) Give the domain and range of $f$ and of $f^{-1}$.

   (c) Sketch the graph of $f^{-1}$.

5. Let $f(x) = x^2 + 2x + 5$. Find the largest value of $a$ so that $f$ is one to one on the interval $(-\infty, a]$. Let $g$ be the function $f$ with the domain $(-\infty, a]$. Find the inverse function $g^{-1}$. Give the domain and range of $g^{-1}$.

   SOLUTION: *See Written Assignment 1 Solutions*

6. True or False:

   (a) Every function has an inverse.

   SOLUTION: False. *A function must be one-to-one to have an inverse*

   (b) If $f \circ g(x) = x$ for all $x$ in the domain of $g$, then $f$ is the inverse of $g$.

   SOLUTION: False. *The functions must also satisfy $g \circ f(x) = x$ for all $x$ in the domain of $f$ to be inverses*

   (c) If $f \circ g(x) = x$ for all $x$ in the domain of $g$ and $g \circ f(x) = x$ for all $x$ in the domain of $f$, then $f$ is the inverse of $g$.

   SOLUTION: True.

   (d) If $f(x) = 1/(x+2)^3$ and $g$ is the inverse function of $f$, then $g(x) = (x+2)^3$.

   SOLUTION: False. *If the functions where to be inverses we would have $g \circ f(x) = x$ and $f \circ g(x) = x$, but these are not true.*

   (e) The function $f(x) = \sin(x)$ is one to one.

   SOLUTION: False. *The sine function doesn't satisfy the horizontal line test.*

(f) The function $f(x) = 1/(x+2)^3$ is one to one.

SOLUTION: True.

7. Find the slope, $x$-intercept, and $y$-intercept of the line $3x - 2y = 4$.

SOLUTION: The slope is $\frac{3}{2}$, the $x$-intercept is $(\frac{4}{3}, 0)$ and the $y$-intercept is $(0, -2)$

8. Let $f$ be a linear function with slope $m$ with $m \neq 0$. What is the slope of the inverse function $f^{-1}$.

SOLUTION: Since $f$ is a linear function, we have that $f(x) = mx + b$ for some constants $m$ and $b$. To find the inverse of $f$:

$$y = mx + b$$
$$x = my + b$$
$$my = x - b$$
$$y = \frac{1}{m}x - \frac{b}{m}$$

Therefore $f^{-1}(x) = \frac{1}{m}x - \frac{b}{m}$ and the slope is $\frac{1}{m}$.

9. A ball is thrown in the air from ground level. The height of the ball in meters at time $t$ seconds is given by the function $h(t) = -4.9t^2 + 30t$. At what time does the ball hit the ground (be sure to use the proper units)?

SOLUTION: When the ball hits the ground the height of the ball is zero, so

$$h(t) = -4.9t^2 + 30t = 0$$
$$t(-4.9t + 30) = 0$$

This means that either $t = 0$ or $-4.9t + 30 = 0$. When $t = 0$ the ball is just being thrown. This is not the time wanted. The other option gives $t = \frac{30}{4.9}$. Hence the ball hits ground after approximately 6.1224 seconds.

10. We form a box by removing squares of side length $x$ centimeters from the four corners of a rectangle of width 100 cm and length 150 cm and then folding up the flaps between the squares that were removed. a) Write a function which gives the volume of the box as a function of $x$. b) Give the domain for this function.

SOLUTION: *See Written Assignment 1 Solutions*

# Worksheet # 2: Review of Trigonometry

1. Convert the angle $\pi/12$ to degrees and the angle $900°$ to radians. Give exact answers.

2. Suppose that $\sin(\theta) = 5/13$ and $\cos(\theta) = -12/13$. Find the values of $\tan(\theta)$, $\cot(\theta)$, $\csc(\theta)$, and $\sec(\theta)$.

   Find the value of $\tan(2\theta)$.

3. If $\pi/2 \leq \theta \leq 3\pi/2$ and $\tan\theta = 4/3$, find $\sin\theta$, $\cos\theta$, $\cot\theta$, $\sec\theta$, and $\csc\theta$.

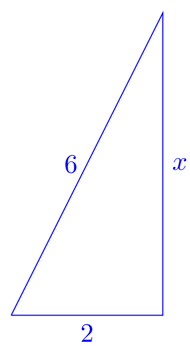
   SOLUTION: The values are $\sin(\theta) = -4/5$, $\cos(\theta) = -3/5$, $\cot(\theta) = 3/4$, $\sec(\theta) = -5/3$, $\csc(\theta) = -5/4$.

4. Find all solutions of the equations a) $\sin(x) = -\sqrt{3}/2$, b) $\tan(x) = 1$.

5. A ladder that is 6 meters long leans against a wall so that the bottom of the ladder is 2 meters from the base of the wall. Make a sketch illustrating the given information and answer the following questions.

   How high on the wall is the top of the ladder located? What angle does the top of the ladder form with the wall?

   SOLUTION:

   If $x$ is how high the top of the ladder is located on the wall, then $x = \sqrt{6^2 - 2^2} = 4\sqrt{2}$. The angle between the top of the ladder and the wall is $\theta = \sin^{-1}(\frac{2}{6}) \approx .3398$.

6. Let $O$ be the center of a circle whose circumference is 48 centimeters. Let $P$ and $Q$ be two points on the circle that are endpoints of an arc that is 6 centimeters long. Find the angle between the segments $OQ$ and $OP$. Express your answer in radians.

   Find the distance between $P$ and $Q$.

7. The center of a clock is located at the origin so that 12 lies on the positive $y$-axis and the 3 lies on the positive $x$-axis. The minute hand is 10 units long and the hour hand is 7 units. Find the coordinates of the tips of the minute hand and hour hand at 9:50 am on Newton's birthday.

   SOLUTION:

   The minute hand is on the 10, which means it forms a $2\pi \cdot 5/12 = 5\pi/6$ angle with the positive $x$ axis. Thus coordinates of the minute hand are $(10\cos(\frac{5\pi}{6}), 10\sin(\frac{5\pi}{6})) = (-5\sqrt{3}, 5)$.
   Since 50 minutes is $5/6$ of an hour, the hand will form an angle of $\pi/6 \cdot 5/6 = \frac{5\pi}{36}$ with the negative $x$-axis. Then the coordinates of the hour hand will be $(7\cos(\frac{31\pi}{36}), 7\sin(\frac{31\pi}{36})) \approx (-6.3442, 2.9583)$.

8. Find all solutions to the following equations in the interval $[0, 2\pi]$. You will need to use some trigonometric identities.

(a) $\sqrt{3}\cos(x) + 2\tan(x)\cos^2(x) = 0$

SOLUTION:
$\sqrt{3}\cos(x) + 2\frac{\sin(x)}{\cos(x)}\cos^2(x) = \cos(x)(\sqrt{3} + 2\sin(x)) = 0$. Setting $\cos(x) = 0$ and $\sin(x) = \frac{-\sqrt{3}}{2}$, we find the solutions are $\frac{\pi}{2}, \frac{3\pi}{2}, \frac{4\pi}{3}, \frac{5\pi}{3}$.

(b) $3\cot^2(x) = 1$

SOLUTION:

$$3\frac{\cos^2(x)}{\sin^2(x)} = 1$$

$$3\cos^2(x) = 1 - \cos^2(x)$$

$$4\cos^2(x) = 1$$

$$\cos(x) = \pm\frac{1}{2}$$

gives us solutions of $\frac{\pi}{3}, \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{5\pi}{3}$.

(c) $2\cos(x) + \sin(2x) = 0$

SOLUTION: $2\cos(x) + \sin(2x) = 2\cos(x) + 2\sin(x)\cos(x) = 2\cos(x)(1 + \sin(x)) = 0$. Setting $\cos(x) = 0$ and $\sin(x) = -1$, we find the solutions are $\frac{\pi}{2}, \frac{3\pi}{2}$.

9. A function is said to be periodic with period $T$ if $f(x) = f(x+T)$ for any $x$. Find the smallest, positive period of the following trigonometric functions. Assume that $\omega$ is positive.

(a) $|\sin t|$

SOLUTION: $T = \pi$

(b) $\sin(3t)$.

SOLUTION: $T = 2\pi/3$

(c) $\sin(\omega t) + \cos(\omega t)$.

SOLUTION: The period of $\sin(t) + \cos(t)$ is $2\pi$. Thus the period of $\sin(\omega t) + \cos(\omega t)$ is $2\pi/\omega$.

(d) $\tan^2(\omega t)$.

SOLUTION: The period of $\tan^2(t)$ is $\pi$. Then the period of $\tan^2(\omega t)$ is $\pi/\omega$.

10. Find a quadratic function $p(x)$ so that the graph $p$ has $x$-intercepts at $x = 2$ and $x = 5$ and the $y$-intercept is $y = -2$.

SOLUTION: Since $p$ has $x$-intercepts, $x = 2$ and $x = 5$, we can write $p(x) = a(x-2)(x-5)$ Since the $y$-intercept is $-2$, $-2 = p(0) = 10a$. Hence $a = -\frac{1}{5}$ and so $p(x) = -\frac{1}{5}(x-2)(x-5)$ is the solution.

11. Find the exact values of the following expressions. Do not use a calculator.

(a) $\tan^{-1}(1)$

SOLUTION: Solving $\tan^{-1}(1)$ is equivalent to solving $\tan(\theta) = 1$ for $\theta$ in $[-\frac{\pi}{2}, \frac{\pi}{2}]$. Our solution is $\frac{\pi}{4}$.

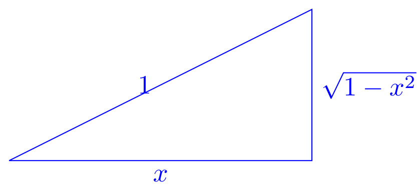
(b) $\tan(\tan^{-1}(10))$

(c) $\sin^{-1}(\sin(7\pi/3))$

(d) $\tan(\sin^{-1}(0.8))$

12. Give a simple expression for $\sin(\cos^{-1}(x))$.

SOLUTION: Using $x = \cos(\theta)$, we have that $\sin(\cos^{-1}(x)) = \sin(\theta)$. Since $\cos(\theta) = \frac{adjacent}{hypotenuse} = x$, we can sketch a picture and solve for $\theta$ using the Pythagorean Theorem. If $\theta$ is the bottom left angle, we see that $\sin(\theta) = \sqrt{1-x^2}$, where we chose the positive root because the range of $\cos^{-1}(x)$ is $[0, \pi]$ and on this domain sine is positive.




Hence

$$\sin(\cos^{-1}(x) = \sqrt{1-x^2}$$

13. Let $f$ be the function with domain $[\pi/2, 3\pi/2]$ with $f(x) = \sin(x)$ for $x$ in $[\pi/2, 3\pi/2]$. Since $f$ is one to one, we may let $g$ be the inverse function of $f$. Give the domain and range of $g$. Find $g(1/2)$.

SOLUTION: The domain of $g$ is $[-1, 1]$ and the range of $g$ is $[\pi/2, 3\pi/2]$. Now if $g(1/2) = \theta$, then $\sin(\theta) = \frac{1}{2}$. This occurs at $\theta = \frac{\pi}{6}$ and $\frac{5\pi}{6}$. Since we know the domain of $g$ is $[\pi/2, 3\pi/2]$, the answer is $g(\frac{1}{2}) = \frac{5\pi}{6}$.

# Worksheet # 3: The Exponential Function and the Logarithm

1. (a) Graph the functions $f(x) = 2^x$ and $g(x) = 2^{-x}$ and give the domains and range of each function.
   (b) Determine if each function is one-to-one. Determine if each function is increasing or decreasing.
   (c) Graph the inverse function to $f$. Give the domain and range of the inverse function.

   SOLUTION: Each function has domain $(-\infty, \infty)$ and range $(0, \infty)$.

   Both functions are one to one. The function $f$ is increasing and the function $g$ is decreasing.

   The domain of $f$ is $(0, \infty)$ and the range is $(-\infty, \infty)$. This function is the logarithm with base 2, $\log_2(x)$.

2. (a) Solve $10^{2x+1} = 100$.
   (b) Solve $2^{(x^2)} = 16$.
   (c) Solve $2^x = 4^{x+2}$.
   (d) Find $\log_2(8)$.
   (e) Find $\ln(e^2)$.
   (f) Solve $e^{3x} = 3$.
   (g) Solve $4^x = e$.
   (h) Solve $\ln(x+1) + \ln(x-1) = \ln(3)$. Be sure to check your answer.

3. Evaluate the expressions $4^{(3^2)}$ and $(4^3)^2$. Are they equal?

   SOLUTION: $4^{(3^2)} = 4^9 = 262144$, but $(4^3)^2 = 64^2 = 4096$, so the two are not equal.

4. Suppose $a$ and $b$ are positive real numbers and $\ln(ab) = 3$ and $\ln(ab^2) = 5$. Find $\ln(a)$, $\ln(b)$, and $\ln(a^3/\sqrt{b})$.

   SOLUTION: We need the logarithm property $\ln(ab) = \ln(a) + \ln(b)$ for this problem. Using this, we can solve for $\ln(a)$ in both cases and set the answers equal to each other.

$$\begin{aligned} \ln(ab) = \ln(a) + \ln(b) = 3 \quad &\rightarrow \quad \ln(a) = 3 - \ln(b) \\ \ln(ab^2) = \ln(a) + \ln(b^2) = 5 \quad &\rightarrow \quad \ln(a) = 5 - \ln(b^2) \end{aligned}$$

$$3 - \ln(b) = 5 - \ln(b^2)$$

   Now we use the logarithm property $\ln(b^x) = x\ln(b)$ to solve for $\ln(b)$, and then plug our answer back into our of the original equations to find $\ln(a)$.

$$\begin{aligned} 3 - \ln(b) &= 5 - \ln(b^2) \\ 3 - \ln(b) &= 5 - 2\ln(b) \\ 2\ln(b) - \ln(b) &= 5 - 3 \\ \ln(b) &= 2 \end{aligned}$$

$$\begin{aligned} \ln(a) + \ln(b) &= 3 \\ \ln(a) + 2 &= 3 \\ \ln(a) &= 1 \end{aligned}$$

   Finally, to find $\ln(a^3/\sqrt{b})$, we rewrite it as $\ln(a^3 b^{1/2})$ and use our previous answers.

$$\ln(a^3 b^{1/2}) = \ln(a^3) + \ln(b^{1/2}) = 3\ln(a) + (1/2)\ln(b) = 3(1) + (1/2)(2) = 4 + 1 = 5$$

5. Consider the function $f(x) = 1 + \ln(x)$. Determine the inverse function of $f$. Give the domain and range of $f$ and of the inverse function $f^{-1}$.

6. Suppose that a population doubles every two hours. If we have one hundred critters at 12 noon, how many will there be after 1 hour? after 2 hours? How many were there at 11am? Give a formula for the number of critters at $t$ hours after 12 noon.

   SOLUTION:

   We use the fact that $e^x$ and $\ln(x)$ are inverses to get $4^x = e^{\ln(4^x)}$, and we use properties of logarithms to get $e^{\ln(4^x)} = e^{\ln(4)x}$.

7. Let $f$ be the function $f(x) = 4^x$. Write the function $f$ in the form $f(x) = e^{kx}$.

8. Let $f$ be the function $f(x) = 5 \cdot 3^x$. Write the function in the form $Ae^{kx}$.

9. Suppose that $f$ is a function of the form $f(x) = Ae^{kx}$. If $f(2) = 20$ and $f(5) = 10$, will we have $k > 0$ or $k < 0$? Find $A$ and $k$ so that $f(2) = 20$ and $f(5) = 10$.

   SOLUTION:

   We know that $f(x)$ is of the form $f(x) = Ae^{kx}$. Also, we have that $k > 0$ if and only if $f(x)$ is increasing, and $k < 0$ if and only if it is decreasing. Since $f(2) = 20$ and $f(5) = 10$, $f(x)$ is getting smaller as $x$ increases, so $f(x)$ is decreasing, and therefore $k < 0$.

   To solve for $A$ and $k$ exactly, note that we're given $f(2) = Ae^{2k} = 20$ and $f(5) = Ae^{5k} = 10$. If we multiply each side of the second equation by 2, we get $2Ae^{5k} = 20$, so we can set it equal to $Ae^{2k}$ and solve for $k$:

$$\begin{aligned} 2Ae^{5k} &= Ae^{2k} \\ 2e^{3k} &= 1 && \text{(division by } Ae^{2k}\text{)} \\ e^{3k} &= \frac{1}{2} \\ \ln(e^{3k}) &= \ln(1/2) \\ 3k &= -\ln(2) && \text{(because } \ln(1/2) = \ln(2^{-1}) = -\ln(2)\text{)} \\ k &= \frac{-\ln(2)}{3} \approx -0.23 \end{aligned}$$

   And we can plug this $k$ value into $Ae^{2k} = 20$ to get our answer for $A$:

$$\begin{aligned} Ae^{2\cdot -.23} &= 20 \\ A &= \frac{20}{e^{2\cdot -.23}} \approx 31.68 \end{aligned}$$

10. Let $f$ be the function given by $f(x) = \tan(x)$ with $x$ in the interval $(\pi/2, 3\pi/2)$. Find $f^{-1}(0)$ and $f^{-1}(\sqrt{3})$.

    Sketch the graphs of $f$ and $f^{-1}$. Give a formula which expresses $f^{-1}$ in terms of the function $\tan^{-1}$ (or arctan).

    SOLUTION: We have $f^{-1}(x) = \pi + \arctan(x)$, so $f^{-1}(0) = \pi + \arctan(0) = \pi$ and

$$f^{-1}(\sqrt{3}) = \pi + \arctan(\sqrt{3}) = \pi + \pi/3 = 4\pi/3.$$

# Worksheet # 4: Limits: A Numerical and Graphical Approach, the Limit Laws

1. Comprehension check:

   (a) In words, describe what "$\lim_{x\to a} f(x) = L$" means.

   (b) In words, what does "$\lim_{x\to a} f(x) = \infty$" mean?

   (c) Suppose $\lim_{x\to 1} f(x) = 2$. Does $f(1) = 2$?

   (d) Suppose $f(1) = 2$. Does $\lim_{x\to 1} f(x) = 2$?

2. Let $p(t)$ denote the distance (in meters) to the right of the origin of a particle at time $t$ minutes after noon. If $p(t) = p(t) = t^3 - 45t$, give the average velocity on the intervals $[2, 2.1]$ and $[2, 2.01]$. Use this information to guess a value for the instantaneous velocity of particle at 12:02pm.

3. Consider the circle $x^2 + y^2 = 25$ and verify that the point $(-3, 4)$ lies on the circle. Find the slope of the secant line that passes through the points with $x$-coordinates $-3$ and $-3.1$. Find the slope of the secant line that passes through the points with $x$-coordinates $-3$ and $-2.99$.

   Use this information to guess the slope of the tangent line to the circle at $(-3, 4)$. Write the equation of the tangent line and use a graphing calculator to verify that you have found the tangent line.

4. Compute the value of the following functions near the given $x-$value. Use this information to guess the value of the limit of the function (if it exists) as $x$ approaches the given value.

   (a) $f(x) = \frac{4x^2-9}{2x-3}$, $x = \frac{3}{2}$

   (b) $f(x) = \frac{x}{|x|}$, $x = 0$

   (c) $f(x) = \frac{\sin(2x)}{x}$, $x = 0$

   (d) $f(x) = \sin(\pi/x)$, $x = 0$

   Solution:

   (a)

   (b) $f(x) = \frac{x}{|x|}$, $x = 0$

   | | |
   |---|---|
   | $x = 0.1, f(x) = 1$ | $x = -0.1, f(x) = -1$ |
   | $x = 0.01, f(x) = 1$ | $x = -0.01, f(x) = -1$ |
   | $x = 0.001, f(x) = 1$ | $x = -0.001, f(x) = -1$ |

   This indicates that $\lim_{x\to 0^-} = -1$, but $\lim_{x\to 0^+} = 1$. The left and right limits are not equal, and so the limit does not exist.

   (c) $f(x) = \frac{\sin(2x)}{x}$, $x = 0$

   | | |
   |---|---|
   | $x = 0.1, f(x) = 1.98$ | $x = -0.1, f(x) = 1.98$ |
   | $x = 0.01, f(x) = 1.999$ | $x = -0.01, f(x) = 1.999$ |
   | $x = 0.001, f(x) = 1.99999$ | $x = -0.001, f(x) = 1.99999$ |

   As $x$ approaches 0 from both the positive and negative directions, $f(x)$ approaches 2, so $\lim_{x\to 0} \dfrac{\sin(2x)}{x} = 2$.

   (d)

5. Let $f(x) = \begin{cases} x^2 & \text{if } x \leq 0 \\ x-1 & \text{if } 0 < x \text{ and } x \neq 2 \\ -3 & \text{if } x = 2 \end{cases}$.

   (a) Sketch the graph of $f$.

   (b) Compute the following:

   i. $\lim_{x \to 0^-} f(x)$

   ii. $\lim_{x \to 0^+} f(x)$

   iii. $\lim_{x \to 0} f(x)$

   iv. $f(0)$

   v. $\lim_{x \to 2^-} f(x)$

   vi. $\lim_{x \to 2^+} f(x)$

   vii. $\lim_{x \to 2} f(x)$

   viii. $f(2)$

6. Compute the following limits or explain why they fail to exist:

   (a) $\lim_{x \to -3^+} \frac{x+2}{x+3}$

   (b) $\lim_{x \to -3^-} \frac{x+2}{x+3}$

   (c) $\lim_{x \to -3} \frac{x+2}{x+3}$

   (d) $\lim_{x \to 0^-} \frac{1}{x^3}$

7. Given $\lim_{x \to 2} f(x) = 5$ and $\lim_{x \to 2} g(x) = 2$, use limit laws to compute the following limits or explain why we cannot find the limit. Note when working through a limit problem that your answers should be a chain of true equalities. Make sure to keep the $\lim_{x \to a}$ operator until the very last step.

   (a) $\lim_{x \to 2} (2f(x) - g(x))$

   (b) $\lim_{x \to 2} (f(x)g(2))$

   (c) $\lim_{x \to 2} \frac{f(x)g(x)}{x}$

   (d) $\lim_{x \to 2} f(x)^2 + x \cdot g(x)^2$

   (e) $\lim_{x \to 2} [f(x)]^{\frac{3}{2}}$

   (f) $\lim_{x \to 2} \frac{f(x) - 5}{g(x) - 2}$

   Solution:

   (a) $\lim_{x \to 2} (2f(x) - g(x)) = \lim_{x \to 2} 2f(x) - \lim_{x \to 2} g(x) = 2 \lim_{x \to 2} f(x) - \lim_{x \to 2} g(x) = 2(5) - 2 = 8$

   (b) We cannot find the limit $\lim_{x \to 2} (f(x)g(2))$, because we do not know the value of $g(2)$, or even whether $g(2)$ exists; we can have $g(2)$ undefined while having $\lim_{x \to 2} g(x) = 2$.

   (c) $$\lim_{x \to 2} \frac{f(x)g(x)}{x} = \frac{\lim_{x \to 2} f(x)g(x)}{\lim_{x \to 2} x} = \frac{\left(\lim_{x \to 2} f(x)\right)\left(\lim_{x \to 2} g(x)\right)}{\lim_{x \to 2} x} = \frac{(5)(2)}{2} = 5$$

   Note that we can only take this first step because $\lim_{x \to 2} x \neq 0$; see part (f).

   (d)

   (e)

   (f) We cannot use the rule about quoetients of limits on $\lim_{x \to 2} \frac{f(x) - 5}{g(x) - 2}$, because the denominator has a limit of 0.

8. Let $f(x) = \begin{cases} 2x+2 & \text{if } x > -2 \\ a & \text{if } x = -2 \\ kx & \text{if } x < -2 \end{cases}$. Find $k$ and $a$ so that $\lim_{x \to -2} f(x) = f(-2)$.

9. In the theory of relativity, the mass of a particle with velocity $v$ is:

$$m = \frac{m_0}{\sqrt{1 - \dfrac{v^2}{c^2}}}$$

where $m_0$ is the mass of the particle at rest and $c$ is the speed of light. What happens as $v \to c^-$?

# Worksheet # 5: Continuity

1. Comprehension check:

   (a) Define what it means for $f(x)$ to be continuous at the point $x = a$. What does it mean if $f(x)$ is continuous on the interval $[a, b]$? What does it mean to say $f(x)$ is continuous?

   (b) There are three distinct ways in which a function will fail to be continuous at a point $x = a$. Describe the three types of discontinuity. Provide a sketch and an example of each type.

   (c) True or false? Every function is continuous on its domain.

   (d) True or false? The sum, difference, and product of continuous functions are all continuous.

   (e) If $f(x)$ is continuous at $x = a$, what can you say about $\lim_{x \to a^+} f(x)$?

2. Using the definition of continuity and properties of limits, show that the following functions are continuous at the given point $a$.

   (a) $f(x) = \pi$, $a = 1$

   (b) $f(x) = \dfrac{x^2 + 3x + 1}{x + 3}$, $a = -1$

   (c) $f(x) = \sqrt{x^2 - 9}$, $a = 4$

3. Give the intervals of continuity for the following functions.

   (a) $f(x) = \dfrac{x+1}{x^2 + 4x + 3}$

   (b) $f(x) = \dfrac{x}{x^2 + 1}$

   (c) $f(x) = \sqrt{2x - 3} + x^2$

   (d) $f(x) = \begin{cases} x^2 + 1 & \text{if } x \le 0 \\ x + 1 & \text{if } 0 < x < 2 \\ -(x-2)^2 & \text{if } x \ge 2 \end{cases}$

4. If $\lim_{h \to 3}(hf(h)) = 15$, can you find $\lim_{h \to 3} f(h)$? Use the limit laws to justify your answer.

5. Let $c$ be a number and consider the function $f(x) = \begin{cases} cx^2 - 5 & \text{if } x < 1 \\ 10 & \text{if } x = 1 \\ \dfrac{1}{x} - 2c & \text{if } x > 1 \end{cases}$.

   (a) Find all numbers $c$ such that $\lim_{x \to 1} f(x)$ exists.

   (b) Is there a number $c$ such that $f(x)$ is continuous at $x = 1$? Justify your answer.

6. Find parameters $a$ and $b$ so that $f(x) = \begin{cases} 2x^2 + 3x & \text{if } x \le -4 \\ ax + b & \text{if } -4 < x < 3 \\ -x^3 + 4x^2 - 5 & \text{if } 3 \le x \end{cases}$ is continuous.

7. Suppose that $f(x)$ and $g(x)$ are continuous functions where $f(2) = 5$ and $g(6) = 1$. Compute the following:

   (a) $\lim_{x \to 2} \dfrac{[f(x)]^2 + x}{3x + 2}$.

   (b) $\lim_{x \to 6} \dfrac{g(x) + 4x}{f\left(\frac{x}{3}\right) - g(x)}$

8. Suppose that: $f(x) = \begin{cases} \frac{x-6}{|x-6|} & \text{for } x \neq 6, \\ 1 & \text{for } x = 6 \end{cases}$

   Determine the points at which the function $f(x)$ is discontinuous and state the type of discontinuity.

# Worksheet # 6: Algebraic Evaluation of Limits, Trigonometric Limits

1. For each limit, evaluate the limit or or explain why it does not exist. Use the limit rules to justify each step. It is good practice to sketch a graph to check your answers.

   (a) $\lim_{x\to 2} \frac{x+2}{x^2-4}$

   (b) $\lim_{x\to 2} \frac{x-2}{x^2-4}$

   (c) $\lim_{x\to 2} \frac{x-2}{x^2+4}$

   (d) $\lim_{x\to 2} \left(\frac{1}{x-2} - \frac{3}{x^2-x-2}\right)$

   (e) $\lim_{x\to 9} \frac{x-9}{\sqrt{x}-3}$

   (f) $\lim_{x\to 0} \frac{(2+x)^3-8}{x}$

2. Let $f(x) = \sqrt{x}$

   (a) Let $g(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$ and find $g(x)$.

   (b) What is the geometric meaning of $g(4)$?

   (c) What is the domain of $g(x)$?

3. Let $f(x) = 1+x^2\sin(1/x)$ for $x \neq 0$. Find two simpler functions $g$ and $h$ so that we can use the squeeze theorem to show $\lim_{x\to 0} f(x) = \lim_{x\to 0} g(x) = \lim_{x\to 0} h(x)$. Give the common value of the limits.

   Use your calculator to produce a graph that illustrates that the squeeze theorem applies.

4. Evaluate the limits

   (a) $\lim_{t\to 0} \frac{\sin(2t)}{t}$

   (b) $\lim_{t\to 0} \frac{\tan(2t)}{t}$

   (c) $\lim_{t\to 0} \frac{\cos(t)\tan(2t)}{t}$

   (d) $\lim_{t\to 0} \csc(3t)\tan(2t)$

   (e) $\lim_{t\to 0} \frac{1-\cos(2t)}{t}$

   (f) $\lim_{t\to 0} \frac{1-\cos(2t)}{\sin^2(2t)}$ **Hint**: Multiply by $\frac{1+\cos(2t)}{1+\cos(2t)}$.

   (g) $\lim_{t\to \pi/2} \frac{1-\cos(t)}{t}$

   The following identity may be useful for the next problems.

   $$\cos(x+y) = \cos(x)\cos(y) - \sin(x)\sin(y) \tag{1}$$

5. Use equation (1), to simplify the limit

   $$\lim_{h\to 0} \frac{\cos(x+h)-\cos(x)}{h}$$

6. Evaluate the following limits

   (a) $\lim_{t\to 0} \frac{t^2}{\sin t}$

   (b) $\lim_{t\to 0} \frac{\cos(5t)-\cos^2(5t)}{t}$

   (c) $\lim_{x\to 0} \frac{\tan(11x)}{5x}$

   (d) $\lim_{x\to 0} \frac{\cos(2x)-1}{\cos(x)-1}$ **Hint:** Use equation (1)

   (e) $\lim_{x\to 0} \frac{1-\cos(3x)}{x^2}$ **Hint:** Multiply and divide by $1+\cos(3x)$

   (f) $\lim_{x\to 0} \frac{\cos x - \cos(4x)}{x^2}$ **Hint:** Use equation (1) to rewrite $\cos(4x)$ as $\cos(x+3x)$

# Worksheet # 7: The Intermediate Value Theorem

1. (a) State the Intermediate Value Theorem.
   (b) Show that $f(x) = x^3 + x - 1$ has a zero in the interval $[0, 1]$.

2. Let $f(x) = e^x/(e^x - 2)$. Show that $f(0) < 1 < f(\ln(4))$.

   Can you use the intermediate value theorem on the interval $[0, \ln(4)]$ to conclude that there is a solution of $f(x) = 1$?

   Solution: No, the function $f$ is not continuous on the interval $[0, \ln(4)]$.

   Can you find a solution to $f(x) = 1$?

   Solution: No. If we try to solve
   $$\frac{e^x}{e^x - 2} = 1$$
   we obtain the equation $e^x = e^x - 2$ or $0 = -2$ which is not true.

3. Use the Intermediate Value Theorem to find an interval of length 1 in which a solution to the equation $2x^3 + x = 5$ must exist.

4. Show that there is some $a$ with $0 < a < 2$ such that $a^2 + \cos(\pi a) = 4$.

5. Show that the equation $xe^x = 2$ has a solution in the interval $(0, 1)$. Solution: The function $xe^x$ is a product of the continuous functions $x$ and $e^x$ and hence continuous. We have $f(0) = 0$ and $f(1) = e \approx 2.78$. Since $0 < 2 < e$, the intermediate value theorem tells us there is a solution to the equation $f(x) = 2$.

   Determine if the solution lies in the interval $(0, 1/2)$ or $(1/2, 1)$.

   Continue to find an interval of length $1/8$ which contains a solution of the equation $xe^x = 2$.

6. Find an interval for which the equation $\ln(x) = e^{-x}$ has a solution. Use the intermediate value theorem to check your answer.

7. Let $f(x) = \begin{cases} 0 & \text{if } x \leq 0 \\ 1 & \text{if } x > 0 \end{cases}$ be a piecewise function.
   Although $f(-1) = 0$ and $f(1) = 1$, $f(x) \neq 1/2$ for all $x$ in its domain. Why doesn't this contradict to the Intermediate Value Theorem?

8. Prove that $x^4 = -1$ has no solution.

9. Determine if the following are true or false.

   (a) If $f$ is continuous on $[-1, 1]$, $f(-1) = -2$ and $f(1) = 2$, then $f(0) = 0$.
   (b) If $f$ is continuous on $[-1, 1]$, $f(-1) = -2$ and the equation $f(x) = 1$ has no solution, then there is no solution to $f(x) = 0$.
   (c) If $f$ is continuous on $[-1, 1]$, $f(-1) = 1$, and the equation $f(x) = 1$ has no solution, then there is no solution to $f(x) = 2$.
   (d) If $f$ is a function with domain $[-1, 1]$, $f(-1) = -2$, $f(1) = 2$, and for each real number $y$, we can find a solution to the equation $f(x) = y$, then $f$ is continuous.

10. (Review) A particle moves along a line and its position after time $t$ seconds is $p(t) = 3t^3 + 2t$ meters to the right of the origin. Find the instantaneous velocity of the particle at $t = 2$.

# Worksheet # 8: Review for Exam I

1. Find all real numbers of the constant $a$ and $b$ for which the function $f(x) = ax + b$ satisfies:

   (a) $f \circ f(x) = f(x)$ for all $x$.

   (b) $f \circ f(x) = x$ for all $x$.

2. Simplify the following expressions.

   (a) $\log_5 125$

   (b) $(\log_4 16)(\log_4 2)$

   (c) $\log_{15} 75 + \log_{15} 3$

   (d) $\log_x(x(\log_y y^x))$

   (e) $\log_\pi(1 - \cos x) + \log_\pi(1 + \cos x) - 2\log_\pi \sin x$

3. Suppose that $\tan(x) = 3/4$ and $-\pi < x < 0$. Find $\cos(x)$, $\sin(x)$, and $\sin(2x)$.

4. (a) Solve the equation $3^{2x+5} = 4$ for $x$. Show each step in the computation.

   (b) Express the quantity $\log_2(x^3 - 2) + \dfrac{1}{3}\log_2(x) - \log_2(5x)$ as a single logarithm. For which $x$ is the resulting identity valid?

5. Calculate the following limits using the limit laws. Carefully show your work and use only one limit law per step.

   (a) $\lim\limits_{x \to 0}(2x - 1)$

   SOLUTION:

   $\lim\limits_{x \to 0}(2x - 1) = \lim\limits_{x \to 0}(2x) - \lim\limits_{x \to 0} 1$

   $\lim\limits_{x \to 0}(2x - 1) = 2\lim\limits_{x \to 0}(x) - 1$

   $\lim\limits_{x \to 0}(2x - 1) = 2(0) - 1 = -1$

   (b) $\lim\limits_{x \to 0} \dfrac{\sqrt{x+4} - 2}{x}$

   SOLUTION:

   $$\lim_{x \to 0} \frac{\sqrt{x+4} - 2}{x} = \lim_{x \to 0}\left(\frac{\sqrt{x+4} - 2}{x}\frac{\sqrt{x+4} + 2}{\sqrt{x+4} + 2}\right)$$

   $$\lim_{x \to 0} \frac{\sqrt{x+4} - 2}{x} = \lim_{x \to 0}\left(\frac{x + 4 - 4}{x\sqrt{x+4} + 2}\right)$$

   $$\lim_{x \to 0} \frac{\sqrt{x+4} - 2}{x} = \lim_{x \to 0}\left(\frac{x}{x\sqrt{x+4} + 2}\right)$$

   $$\lim_{x \to 0} \frac{\sqrt{x+4} - 2}{x} = \lim_{x \to 0} \frac{1}{\sqrt{x+4} + 2}$$

   $$\lim_{x \to 0} \frac{\sqrt{x+4} - 2}{x} = \frac{1}{\lim\limits_{x \to 0}(\sqrt{x+4} + 2)}$$

   $$\lim_{x \to 0} \frac{\sqrt{x+4} - 2}{x} = \frac{1}{(\sqrt{\lim\limits_{x \to 0}(x) + 4} + 2)}$$

   $$\lim_{x \to 0} \frac{\sqrt{x+4} - 2}{x} = \frac{1}{(\sqrt{(0) + 4} + 2)} = \frac{1}{4}$$

6. Determine if the following limits exist and calculate the limit when possible.

(a) $\lim_{x\to 1} \frac{x-2}{\frac{1}{x}-\frac{1}{2}}$

(b) $\lim_{x\to 2} \frac{x-2}{\frac{1}{x}-\frac{1}{2}}$

(c) $\lim_{x\to 2} \frac{x^2}{x-2}$

(d) $\lim_{x\to 4} \frac{\sqrt{x}-2}{x^2-16}$

(e) $\lim_{x\to 2} \frac{x+1}{x-2}$

7. Suppose that the height of an object at time $t$ is $h(t) = 5t^2 + 40t$.

   (a) Find the average velocity of the object on the interval $[3, 3.1]$.

   (b) Find the average velocity of the object on the interval $[a, a+h]$.

   (c) Find the instantaneous velocity of the object time $a$.

8. Use the fact that $\lim_{x\to 0} \frac{\sin(x)}{x} = 1$ to find $\lim_{x\to 0} \frac{x}{\sin(3x)}$.

9. (a) State the Squeeze Theorem.

   (b) Use the Squeeze Theorem to find the limit $\lim_{x\to 0} x \sin \frac{1}{x^2}$

10. Use the Squeeze Theorem to find $\lim_{x\to\frac{\pi}{2}} \cos x \cos(\tan x)$

    Solution: First we know that

$$-1 \leq \cos(\tan(x)) \leq 1$$

since the range of cosine is $[-1, 1]$. Since $\cos(x)$ is positive, I can multiply it through the inequality without flipping the inequality directions. This gives

$$-\cos(x) \leq \cos(x)\cos(\tan(x)) \leq \cos(x)$$

Next note $\lim_{x\to \pi/2} \cos(x) = \cos(\frac{\pi}{2}) = 0$. Similarly, $\lim_{x\to\frac{\pi}{2}} -\cos(x) = -\cos(\frac{\pi}{2}) = 0$

Therefore the squeeze theorem gives that $\lim_{x\to \pi/2} \cos(x)\cos(\tan(x)) = 0$

11. If $f(x) = \frac{|x-3|}{x^2-x-6}$, find $\lim_{x\to 3^+} f(x)$, $\lim_{x\to 3^-} f(x)$ and $\lim_{x\to 3} f(x)$.

12. (a) State the definition of the continuity of a function $f(x)$ at $x = a$.

    Solution: A function $f$ is continuous at $x = a$ if $\lim x \to af(x) = f(a)$

    (b) Find the constant $a$ so that the function is continuous on the entire real line.

$$f(x) = \begin{cases} \frac{x^2-a^2}{x-a} & \text{if } x \neq a \\ 8 & \text{if } x = a \end{cases}$$

Solution: First since each function of the piecewise defined function, $\frac{x^2-a^2}{x-a}$ and 8 are continuous where they are defined, we only need to worry about continuity at the point where the two function meet up, $x = a$. For $f$ to be continuous at $x = a$, we need

$$\lim x \to af(x) = f(a)$$

. Hence,

$$\lim x \to a\frac{x^2-a^2}{x-a} = 8$$

$$\lim x \to a\frac{(x-a)(x+a)}{x-a} = 8$$
$$\lim x \to a(x+a) = 8$$
$$2a = 8$$

Therefore $a = 4$ makes $f$ continuous.

13. Complete the following statements:

   (a) A function $f(x)$ passes the horizontal line test, if the function $f$ is ...............

   SOLUTION: one-to-one

   (b) If $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$ exist, then

   $$\lim_{x\to a}\frac{f(x)}{g(x)} = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)} \quad \text{provided ..................}$$

   SOLUTION: $\lim_{x\to a} g(x) \neq 0$

   (c) $\lim_{x\to a^+} f(x) = \lim_{x\to a^-} f(x) = L$ if and only if ..................

   SOLUTION: $\lim_{x\to a} f(x) exists$

   (d) Let $g(x) = \begin{cases} x & \text{if } x \neq 2 \\ 1 & \text{if } x = 2 \end{cases}$ be a piecewise function.

   The function $g(x)$ is NOT continuous at $x = 2$ since ................................

   SOLUTION: $\lim_{x\to 2} g(x) \neq g(2)$.

   (e) Let $f(x) = \begin{cases} x^2 & \text{if } x < 0 \\ 1 & \text{if } x = 0 \\ x & \text{if } x > 0 \end{cases}$ be a piecewise function.

   The function $f(x)$ is NOT continuous at $x = 0$ since ................................

   SOLUTION: $0 = \lim_{x\to 0^-} f(x) = \lim_{x\to 0^+} f(x) \neq f(0) = 1$

14. If $g(x) = x^2 + 5^x - 3$, use the Intermediate Value Theorem to show that there is a number $a$ such that $g(a) = 10$.

   SOLUTION: Since

   $$3 = g(1) < 10 < g(2) = 26$$

   and g is a continuous function, the Intermediate Value Theorem says that there is some number $c \in (1, 2)$ such that $g(c) = 10$

# Worksheet # 9: The Derivative

1. Comprehension check:

   (a) What is the definition of the derivative $f'(a)$ at a point $a$?

   SOLUTION: Students should learn the definition as in the textbook.

   The derivative $f'(a)$ gives the slope of the line that is tangent to the graph of $f$ at the point $a$.

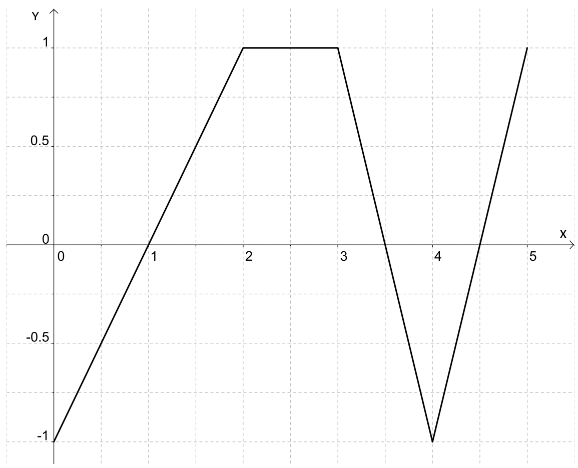
   (b) What is the geometric meaning of the derivative $f'(a)$ at a point $a$?

   (c) True or false: If $f(1) = g(1)$, then $f'(1) = g'(1)$?

   SOLUTION:

   False. Consider $f(x) = x^2$ and $g(x) = x$.

2. Consider the graph below of the function $f(x)$ on the interval $[0, 5]$.




   (a) For which $x$ values would the derivative $f'(x)$ not be defined?

   (b) Sketch the graph of the derivative function $f'$.

3. Find a function $f$ and a number $a$ so that the following limit represents a derivative $f'(a)$.

$$\lim_{h \to 0} \frac{(4+h)^3 - 64}{h}$$

   SOLUTION:

   According to the definition of the derivative, an obvious choice is $f(x) = x^3$ and $a = 4$.

4. Let $f(x) = |x|$. Find $f'(1)$, $f'(0)$ and $f'(-1)$ or explain why the derivative does not exist.

   SOLUTION:

   Looking at the graph of $|x|$, we can see the slope of the tangent lines to get $f'(1) = 1$ and $f'(-1) = -1$, but $f'(0)$ does not exist as the slope is different on each side of $x = 0$. This means left and right limits are not equal.

5. Find the specified derivative for each of the following.

   (a) If $f(x) = 1/x$, find $f'(2)$.

   (b) If $g(x) = \sqrt{x}$, find $g'(2)$.

   (c) If $h(x) = x^2$, find $h'(s)$.

(d) If $f(x) = x^3$, find $f'(-2)$.

(e) If $g(x) = 1/(2-x)$, find $g'(t)$.

SOLUTION:

$$g'(t) = \lim_{h\to 0} \frac{\frac{1}{2-(t+h)} - \frac{1}{2-t}}{h} = \lim_{h\to 0} \frac{2-t-2+t+h}{(2-(t+h))(2-t)h} = \lim_{h\to 0} \frac{1}{(2-(t+h))(2-t)} = \frac{1}{(2-t)^2}$$

6. Let

$$g(t) = \begin{cases} at^2 + bt + c & \text{if } t \le 0 \\ t^2 + 1 & \text{if } t > 0 \end{cases}.$$

Find all values of $a$, $b$, and $c$ so that $g$ is differentiable at $t = 0$.

SOLUTION:

In order for $g$ to be differentiable at $t = 0$, $g$ must be continuous at $t = 0$. Then $\lim_{t\to 0} g(t) = f(0)$. Since $f(0) = c$ and $\lim_{t\to 0^+} g(t) = \lim_{t\to 0^+} t^2 + 1 = 1$, then $c = 1$. We also must check that the limit definition of the derivative exists $t = 0$. We need to look at the left and right hand limits separately We have

$$\lim_{h\to 0^-} \frac{f(0+h) - f(0)}{h} = \lim_{h\to 0^-} \frac{a(0+h)^2 + b(0+h) + c - (a(0)^2 + b(0) + c)}{h} = \lim_{h\to 0^-} \frac{ah^2 + bh}{h} = \lim_{h\to 0^-} ah + b = b.$$

We compare this to the limit from the right,

$$\lim_{h\to 0^+} \frac{(0+h)^2 + 1 - (0^2 + 1)}{h} = \lim_{h\to 0} h = 0.$$

Thus we must have $b = 0$. Notice that no restriction on the value of $a$ came up, so $a$ can be anything. So we have $b = 0$, $c = 1$, and $a$ is any real number.

7. Let $f(x) = e^x$ and estimate the derivative $f'(0)$ by considering difference quotients $(f(h) - f(0))/h$ for small values of $h$.

8. Suppose that $f'(0)$ exists. Does the limit

$$\lim_{h\to 0} \frac{f(h) - f(-h)}{h}$$

exist? Can you express the limit in terms of $f'(0)$?

SOLUTION:

Since $f'(0)$ exists, then $\lim_{h\to 0} \frac{f(h)-f(0)}{h}$ exists and equals $f'(0)$. Since $h$ is approaching 0 from both sides of 0, we also have that $\lim_{h\to 0} \frac{f(-h)-f(0)}{-h} = f'(0)$. Then

$$\lim_{h\to 0} \frac{f(h) - f(0)}{h} + \lim_{h\to 0} \frac{f(-h) - f(0)}{-h} = \lim_{h\to 0} \left( \frac{f(h) - f(0)}{h} - \frac{f(-h) - f(0)}{h} \right) = \lim_{h\to 0} \frac{f(h) - f(-h)}{h}.$$

Thus we conclude that the limit does exists and it equals $2f'(0)$.

9. Can you find a function $f$ so that

$$\lim_{h\to 0} \frac{f(h) - f(-h)}{h}$$

exists, but $f$ is not differentiable at 0?

SOLUTION: Yes, for an even function the limit will be zero. The function $f(x) = |x|$ is even but not differentiable.

10. Find $A$ and $B$ so that the limit

$$\lim_{x \to 1} \frac{x^2 + 2x - (Ax + B)}{(x-2)^2}$$

is finite. Give the value of the limit.

# Worksheet # 10: The Derivative as Function, Product and Quotient Rules

1. Comprehension check:
   (a) True or false: If $f'(x) = g'(x)$ for all $x$, then $f = g$?
   (b) True or false: If $f(x) = g(x)$ for all $x$, then $f' = g'$?
   (c) True or false: $(f+g)' = f' + g'$
   (d) True or false: $(fg)' = f'g'$
   (e) How is the number $e$ defined?
   (f) Are differentiable functions also continuous? Are continuous functions also differentiable?

2. Show by way of example that, in general,
$$\frac{d}{dx}(f \cdot g) \neq \frac{df}{dx} \cdot \frac{dg}{dx}$$
and
$$\frac{d}{dx}\left(\frac{f}{g}\right) \neq \frac{\frac{df}{dx}}{\frac{dg}{dx}}.$$

3. Calculate the derivatives of the following functions in the two ways that are described.
   (a) $f(r) = r^3/3$
      i. using the constant multiple rule and the power rule
      ii. using the quotient rule and the power rule

      Which method should we prefer?
   (b) $f(x) = x^5$
      i. using the power rule
      ii. using the product rule by considering the function as $f(x) = x^2 \cdot x^3$
   (c) $g(x) = (x^2+1)(x^4-1)$
      i. first multiply out the factors and then use the power rule
      ii. by using the product rule

4. State the quotient and product rule and be sure to include all necessary hypotheses.

5. Compute the first derivative of each of the following:
   (a) $f(x) = (3x^2 + x)e^x$
   (b) $f(x) = \dfrac{\sqrt{x}}{x-1}$
   (c) $f(x) = \dfrac{e^x}{2x^3}$
   (d) $f(x) = (x^3 + 2x + e^x)\left(\dfrac{x-1}{\sqrt{x}}\right)$
   (e) $f(x) = \dfrac{2x}{4+x^2}$
   (f) $f(x) = \dfrac{ax+b}{cx+d}$
   (g) $f(x) = \dfrac{(x^2+1)(x^3+2)}{x^5}$
   (h) $f(x) = (x-3)(2x+1)(x+5)$

6. Let $f(x) = (3x-1)e^x$. For which $x$ is the slope of the tangent line to $f$ positive? Negative? Zero?

7. Find an equation of the tangent line to the given curve at the specified point. Sketch the curve and the tangent line to check your answer.

   (a) $y = x^2 + \dfrac{e^x}{x^2+1}$ at the point $x = 3$.

   (b) $y = 2xe^x$ at the point $x = 0$.

8. Suppose that $f(2) = 3$, $g(2) = 2$, $f'(2) = -2$, and $g'(2) = 4$. For the following functions, find $h'(2)$.

   (a) $h(x) = 5f(x) + 2g(x)$

   (b) $h(x) = f(x)g(x)$

   (c) $h(x) = \dfrac{f(x)}{g(x)}$

   (d) $h(x) = \dfrac{g(x)}{1 + f(x)}$

# Worksheet # 11: Rates of Change

1. Given that $G(t) = 4t^2 - 3t + 42$ find the instantaneous rate of change when $t = 3$.

2. A particle moves along a line so that its position at time $t$ is $p(t) = 3t^3 - 12t$ where $p(t)$ represents the distance to the right of the origin.

   (a) Find the velocity and speed at time $t = 1$.

   (b) Find the acceleration at time $t = 1$.

   (c) Is the velocity increasing or decreasing when $t = 1$?

   (d) Is the speed increasing or decreasing when $t = 1$?

3. An object is thrown upward so that its height at time $t$ seconds after being thrown is $h(t) = -4.9t^2 + 20t + 25$ meters. Give the position, velocity, and acceleration at time $t$.

4. An object is thrown upward with an initial velocity of 5 m/s from an initial height of 40 m. Find the velocity of the object when it hits the ground. Assume that the acceleration of gravity is 9.8 m/s$^2$.

5. An object is thrown upward so that it returns to the ground after 4 seconds. What is the initial velocity? Assume that the acceleration of gravity is 9.8 m/s$^2$.

6. Suppose that height of a triangle is equal to its base $b$. Find the instantaneous rate of change in the area respect to the base $b$ when the base is 7.

7. The cost in dollars of producing $x$ bicycles is $C(x) = 4000 + 210x - x^2/1000$.

   (a) Explain why $C'(40)$ is a good approximation for the cost of the 41st bicycle.

   (b) How can you use the values of $C(40)$ and $C'(40)$ to approximate the cost of 42 bicycles?

   (c) Explain why the model for $C(x)$ is not a good model for cost. What happens if $x$ is very large?

   Solution: The tangent line to the function $C(x)$ a t $x = 40$ is $C(40) + C'(40)(x - 40)$. If $x = 41$, then we have $C(41)$ is approximately $C(40) + C'(40)$.

   When $x$ is large, the cost becomese negative.

8. Suppose that $f$ is a function with $f(7) = 34$ and $f'(7) = 4$. Explain why we can use 42 as an approximation for $f(9)$. What is an approximate value for $f(6)$?

9. (Review) Suppose that the tangent line to the graph of $f$ at $(2, f(2))$ is $y = 3x + 4$. Find the tangent line to the graph of $xf(x)$.

10. (Review) Differentiate the following functions.

    (a) $f(x) = (x + 1)(x - 2)$

    (b) $g(x) = \frac{x^2-1}{x^2+1}$

    (c) $h(x) = x \sin(x)$

# Worksheet # 12: Higher Derivatives and Trigonometric Functions

1. Calculate the indicated derivative:

   (a) $f^{(4)}(1)$, $\qquad f(x) = x^4$
   (b) $g^{(3)}(5)$, $\qquad g(x) = 2x^2 - x + 4$
   (c) $h^{(3)}(t)$, $\qquad h(t) = 4e^t - t^3$
   (d) $s^{(2)}(w)$, $\qquad s(w) = \sqrt{w}e^w$

2. Calculate the first three derivatives of $f(x) = xe^x$ and use these to guess a general formula for $f^{(n)}(x)$, the $n$-th derivative of $f$.

3. Let $f(t) = t + 2\cos(t)$.

   (a) Find all values of $t$ where the tangent line to $f$ at the point $(t, f(t))$ is horizontal.
   (b) What are the largest and smallest values for the slope of a tangent line to the graph of $f$?

4. Differentiate each of the following functions:

   (a) $f(t) = \cos(t)$
   (b) $g(u) = \dfrac{1}{\cos(u)}$
   (c) $r(\theta) = \theta^3 \sin(\theta)$
   (d) $s(t) = \tan(t) + \csc(t)$
   (e) $h(x) = \sin(x)\csc(x)$
   (f) $f(x) = x^2 \sin(x)$
   (g) $g(x) = \sec(x) + \cot(x)$

5. Calculate the first five derivatives of $f(x) = \sin(x)$. Then determine $f^{(8)}$ and $f^{(37)}$

6. Calculate the first 5 derivatives of $f(x) = 1/x$. Can you guess a formula for the $n$th derivative, $f^{(n)}$?

7. A particle's distance from the origin (in meters) along the $x$-axis is modeled by $p(t) = 2\sin(t) - \cos(t)$, where $t$ is measured in seconds.

   (a) Determine the particle's speed (speed is defined as the absolute value of velocity) at $\pi$ seconds.
   (b) Is the particle moving towards or away from the origin at $\pi$ seconds? Explain.
   (c) Now, find the velocity of the particle at time $t = \dfrac{3\pi}{2}$. Is the particle moving toward the origin or away from the origin?
   (d) Is the particle speeding up at $\frac{\pi}{2}$ seconds?

8. Find an equation of the tangent line at the point specified:

   (a) $y = x^3 + \cos(x)$, $\qquad x = 0$
   (b) $y = \csc(x) - \cot(x)$, $\qquad x = \frac{\pi}{4}$
   (c) $y = e^\theta \sec(\theta)$, $\qquad \theta = \frac{\pi}{4}$

9. Comprehension check for derivatives of trigonometric functions:

   (a) True or False: If $f'(\theta) = -\sin(\theta)$, then $f(\theta) = \cos(\theta)$.
   (b) True or False: If $\theta$ is one of the non-right angles in a right triangle and $\sin(\theta) = \dfrac{2}{3}$, then the hypotenuse of the triangle must have length 3.

# Worksheet # 13: Chain Rule

1. (a) Carefully state the chain rule using complete sentences.

   (b) Suppose $f$ and $g$ are differentiable functions so that $f(2) = 3$, $f'(2) = -1$, $g(2) = \dfrac{1}{4}$, and $g'(2) = 2$. Find each of the following:

   i. $h'(2)$ where $h(x) = \sqrt{[f(x)]^2 + 7}$.

   ii. $l'(2)$ where $l(x) = f(x^3 \cdot g(x))$.

2. Suppose that $k(x) = \sqrt{\sin^2(x) + 4}$. Find three functions $f$, $g$, and $h$ so that $k(x) = f(g(h(x)))$.

3. Given the following functions: $f(x) = \sec(x)$, and $g(x) = x^3 - 2x + 1$. Find:

   (a) $f(g(x))$

   (b) $f'(x)$

   (c) $g'(x)$

   (d) $f'(g(x))$

   (e) $(f \circ g)'(x)$

4. Differentiate each of the following and simplify your answer.

   (a) $f(x) = \sqrt[3]{2x^3 + 7x + 3}$

   (b) $g(t) = \tan(\sin(t))$

   (c) $h(u) = \sec^2(u) + \tan^2(u)$

   (d) $f(x) = xe^{(3x^2+x)}$

   (e) $g(x) = \sin(\sin(\sin(x)))$

5. Find an equation of the tangent line to the curve at the given point.

   (a) $f(x) = x^2 e^{3x}$, $x = 2$

   (b) $f(x) = \sin(x) + \sin^2(x)$, $x = 0$

6. Compute the derivative of $\frac{x}{x^2+1}$ in two ways:

   (a) Using the quotient rule.

   (b) Rewrite the function $\frac{x}{x^2+1} = x(x^2 + 1)^{-1}$ and use the product and chain rule.

   Check that both answers give the same result.

7. If $h(x) = \sqrt{4 + 3f(x)}$ where $f(1) = 7$ and $f'(1) = 4$, find $h'(1)$.

8. Let $h(x) = f \circ g(x)$ and $k(x) = g \circ f(x)$ where some values of $f$ and $g$ are given by the table

| $x$ | $f(x)$ | $g(x)$ | $f'(x)$ | $g'(x)$ |
|---|---|---|---|---|
| -1 | 4 | 4 | -1 | -1 |
| 2 | 3 | 4 | 3 | -1 |
| 3 | -1 | -1 | 3 | -1 |
| 4 | 3 | 2 | 2 | -1 |

   Find: $h'(-1), h'(3)$ and $k'(2)$.

9. Find all $x$ values so that $f(x) = 2\sin(x) + \sin^2(x)$ has a horizontal tangent at $x$.

10. Suppose that at the instant when the radius of a circle of a circle is 5 cm, the radius is decreasing at a rate of 3 cm/sec. Find the rate of change of the area of the circle when the the radius is 5 cm.

    Solution: We have $A = \pi r^2$ and $\frac{dA}{dt} = 2\pi r \frac{dr}{dt}$. If $r = 5cm$ and $dr/dt = -3$ cm/s, then we have $dA/dt = 2\pi(5)(-3)\text{cm}^2/\text{s} = -30\pi\text{cm}^2/\text{s}$.

11. Differentiate both sides of the double-angle formula for the cosine function, $\cos(2x) = \cos^2(x) - \sin^2(x)$. Do you obtain a familiar identity?

# Worksheet # 14: Implicit Differentiation and Inverse Functions

1. Find the derivative of $y$ with respect to $x$:

   (a) $\ln(xy) = \cos(y^4)$.

   (b) $x^{\frac{2}{3}} + y^{\frac{2}{3}} = \pi^{\frac{2}{3}}$.

   (c) $\sin(xy) = \ln\left(\frac{x}{y}\right)$.

   SOLUTION:

   - a): $\ln(xy) - \cos(y^4) = 0$. We start by replacing $y$ by $g(x)$ to see how the chain rule is involved. This leaves us with
   $$\ln(xg(x)) - \cos(g(x)^4) = 0$$
   Then implicit differentiation gives us
   $$\frac{1}{xg(x)}\left(g(x) + xg'(x)\right) + \sin(g(x)^4) \cdot 4 \cdot g(x)^3 g'(x) = 0.$$
   Now we can replace $g(x)$ by $y$ and we thus get
   $$\frac{1}{xy}\left(y + xy'\right) + \sin(y^4) \cdot 4 \cdot y^3 y' = 0.$$
   Solving for $y'$ gives us
   $$y' = \frac{-y}{4xy^4 \sin(y^4) + x}.$$
   - b): Differentiate $x^{\frac{2}{3}} + y^{\frac{2}{3}} = \pi^{\frac{2}{3}}$ with respect to $x$. We proceed as before and get
   $$\frac{2}{3}x^{\frac{-1}{3}} + \frac{2}{3}y^{\frac{-1}{3}}y' = 0.$$
   Solving for $y'$ shows
   $$y' = -\sqrt[3]{\frac{y}{x}}.$$
   - We have to differentiate $\sin(xy) - \ln(\frac{x}{y}) = 0$. Since this expression won't be as compact as before, we should do it step by step. Differentiating $\sin(xy)$ gives us
   $$\cos(xy)\left(y + xy'\right),$$
   and differentiating $\ln(\frac{x}{y})$ gives us
   $$\frac{y}{x}\left(\frac{y - xy'}{y^2}\right)$$
   Since the right-hand side stays 0, we can easily solve for $y'$ and hence we get
   $$y' = \frac{1 - xy\cos(xy)}{(1 + xy\cos(xy))}.$$

2. Consider the ellipse given by the equation $\frac{(x-2)^2}{25} + \frac{(y-3)^2}{81} = 1$.

   (a) Find the equation of the tangent line to the ellipse at the point $(u, v)$ where $u = 4$ and $v > 0$.

   (b) Sketch the ellipse and the line to check your answer.

3. Find the derivative of $f(x) = \pi^{\tan^{-1}(\omega x)}$, where $\omega$ is a constant.

4. Let $(a, b)$ be a point in the circle $x^2 + y^2 = 144$. Use implicit differentiation to find the slope of the tangent line to the circle at $(a, b)$.

5. Let $f(x)$ be an invertible function such that $g(x) = f^{-1}(x)$, $f(3) = \sqrt{5}$ and $f'(3) = -\frac{1}{2}$. Using only this information find the equation of the tangent line to $g(x)$ at $x = \sqrt{5}$.

6. Let $y = f(x)$ be the unique function satisfying $\dfrac{1}{2x} + \dfrac{1}{3y} = 4$. Find the slope of the tangent line to $f(x)$ at the point $(\frac{1}{2}, \frac{1}{9})$. SOLUTION: We want to find the slope of the tangent line at $x = \frac{1}{2}$. For that we find an explicit representation of $f(x)$, differentiate this representation and plug in $\frac{1}{2}$. It is not too hard to see that

$$f(x) = \frac{2x}{3(8x-1)}.$$

Differentiating (using the quotient rule) shows that

$$f'(x) = \frac{-6}{(3(8x-1))^2},$$

and thus $f'(\frac{1}{2}) = -\frac{2}{27}$.

7. The equation of the tangent line to $f(x)$ at the point $(2, f(2))$ is given by the equation $y = -3x + 9$. If $G(x) = \dfrac{x}{4f(x)}$, find $G'(2)$.

SOLUTION: This is a review problem. Since the tangent to the graph of $f$ at 2 is given by $y = -3x+9$, we know that $f'(2) = -3$, the slope of the tangent line and $f(2) = -3 \cdot 2 + 9 = 3$.

Now to find $G'(2)$, we differentiate $G$ with quotient rule and substitute $x = 2$ to find

$$G'(2) = \frac{1 \cdot 4f(2) - 2 \cdot 4f'(2)}{f(2)^2} = \frac{12+24}{6} = 6.$$

8. Differentiate both sides of the equation, $V = \dfrac{4}{3}\pi r^3$, with respect to $V$ and find $\dfrac{dr}{dV}$ when $r = 8\sqrt{\pi}$.

9. Use implicit differentiation to find the derivative of $\arctan(x)$. Thus if $x = \tan(y)$, use implicit differentiation to compute $dy/dx$. Can you simplify to express $dy/dx$ in terms of $x$?

10. (a) Compute $\dfrac{d}{dx}\arcsin(\cos(x))$.

    (b) Compute $\frac{d}{dx}(\arcsin(x) + \arccos(x))$. Give a geometric explanation as to why the answer is 0.

    (c) Compute $\dfrac{d}{dx}\left(\tan^{-1}\left(\dfrac{1}{x}\right) + \tan^{-1}(x)\right)$ and simplify to show that the derivative is 0. Give a geometric explanation of your result.

11. Consider the line through $(0, b)$ and $(2, 0)$. Let $\theta$ be the directed angle from the $x$-axis to this line so that $\theta > 0$ when $b < 0$. Find the derivative of $\theta$ with respect to $b$.

12. Let $f$ be defined by $f(x) = e^{-x^2}$.

    (a) For which values of $x$ is $f'(x) = 0$

    (b) For which values of $x$ is $f''(x) = 0$

Solution: Let $f(x) = e^{-x^2}$. Then we immediately get (by applying the chain rule) $f'(x) = -2xe^{-x^2}$. So $f'(x) = 0$ if and only if $x = 0$, since the $e^{-x^2} > 0$ for all $x$. This solves a). For part b) we differentiate $f'$ which gives

$$f''(x) = -2\left(e^{-x^2} - 2x^2e^{-x^2}\right) = -2e^{-x^2}\left(1 - 2x^2\right).$$

Hence, $f'' = 0$ if and only if $1 - 2x^2 = 0$, if and only if $x = \pm\sqrt{\frac{1}{2}}$.

13. The notation $\tan^{-1}(x)$ is ambiguous. It is not clear if the exponent $-1$ indicates the reciprocal or the inverse function. If we allow both interpretations, how many different ways can you (correctly) compute the derivative $f'(x)$ for

$$f(x) = (\tan^{-1})^{-1}(x)?$$

In order to avoid this ambiguity, we will generally use $\cot(x)$ for the reciprocal of $\tan(x)$ and $\arctan(x)$ for the inverse of the tangent function restricted to the domain $(-\pi/2, \pi/2)$.

Solution: If both of the exponents $-1$ indicate the inverse function, then $f(x) = \tan(x)$ and $f'(x) = \sec^2(x)$.

If both exponents indicate the reciprocal, then again $f(x) = \tan(x)$ and $f'(x) = \sec^2(x)$.

If the first exponent (reading left to right) denotes the inverse function and the second exponent denotes the reciprocal and we can write $f(x) = 1/\arctan(x)$ and $f'(x) = \frac{-1}{(1+x^2)(\arctan(x))^2}$.

Finally, if the first exponent denotes the reciprocal and the second exponent denotes the inverse function, then $f(x) = \operatorname{arccot}(x)$ and $f'(x) = -1/(1 + x^2)$.