Exam 1

24 September 2013

Name: KEY

Section: ______________ Instructor or TA: ______________

Do not remove this answer page — you will return the whole exam. You will be allowed two hours to complete this test. No books or notes may be used. You may use a graphing calculator during the exam, but NO calculator with a Computer Algebra System (CAS) or a QWERTY keyboard is permitted. Absolutely no cell phone use during the exam is allowed.

The exam consists of 4 multiple choice questions and 8 free response questions. Record your answers to the multiple choice questions below on this page by filling in the circle corresponding to the correct answer. All other work must be done in the body of the exam.

Multiple Choice Questions

1 (A) ● (C) (D) (E) **3** (A) ● (C) (D) (E)

2 ● (B) (C) (D) (E) **4** (A) (B) (C) (D) ●

SCORE

Multiple Choice	5	6	7	8	9	10	11	12	Total Score
20	10	10	10	10	10	10	10	10	100

Multiple Choice Questions

1. Evaluate $\int x\, f(x)\, dx$.

   A. $x \cdot f(x) - \int x \cdot f'(x)\, dx$
   B. $\frac{x^2}{2} f(x) - \int \frac{x^2}{2} f'(x)\, dx$
   C. $x \cdot f(x) - \frac{x^2}{2} f'(x)$
   D. $x \cdot f(x) - \int f''(x)\, dx$
   E. $\frac{x^2}{2} \int f'(x)\, dx$

2. When making the usual trigonometric substitution in the evaluation of the following integral, which substitution will be made for dx?

$$\int \frac{\sqrt{x^2-3}}{x}\, dx$$

   A. $dx = \sqrt{3} \sec\theta \tan\theta \, d\theta$
   B. $dx = 3 \sec\theta \, d\theta$
   C. $dx = \sqrt{3} \sec^2\theta \, d\theta$
   D. $dx = \sqrt{3} \cos\theta \, d\theta$
   E. $dx = 3 \sec\theta \tan\theta \, d\theta$

3. The region in the first quadrant between the x-axis and the graph of $y = 6x - x^2$ is rotated around the y-axis. The volume of the resulting solid of revolution is given by:

   A. $\displaystyle\int_0^6 \pi \left(6x - x^2\right)^2 \, dx$

   B. $\displaystyle\int_0^6 2\pi x \left(6x - x^2\right) \, dx$

   C. $\displaystyle\int_0^6 \pi x \left(6x - x^2\right)^2 \, dx$

   D. $\displaystyle\int_0^6 \pi \left(3 + \sqrt{9 - y}\right)^2 \, dy$

   E. $\displaystyle\int_0^9 \pi \left(3 + \sqrt{9 - y}\right)^2 \, dy$

4. The average value of $f(x) = x^3$ over the interval $a \leq x \leq b$ is

   A. $3b + 3a$

   B. $b^2 + ab + a^2$

   C. $\dfrac{b^3 + a^3}{2}$

   D. $\dfrac{b^3 - a^3}{2}$

   E. $\dfrac{(b^4 - a^4)}{4(b - a)}$

Free Response Questions

You must show all of your work in these problems to receive credit. Answers without corroborating work will receive no credit.

5. Find $\int_{\pi/4}^{\pi/2} \cot x \, dx$. Show all of your work.

$$= \int_{\pi/4}^{\pi/2} \frac{\cos x}{\sin x} dx$$

$u = \sin x$
$du = \cos x \, dx$
$\sin(\frac{\pi}{4}) = \frac{\sqrt{2}}{2}$
$\sin(\frac{\pi}{2}) = 1$

$$= \int_{\frac{\sqrt{2}}{2}}^{1} \frac{du}{u} = \ln|u| \Big|_{\frac{\sqrt{2}}{2}}^{1}$$

$$= \ln 1 - \ln(\tfrac{\sqrt{2}}{2}) = \ln(\sqrt{2})$$

$$\approx 0.3466$$

6. Evaluate $\int_0^3 \sqrt{9 - x^2} \, dx$. Show all of your work.

Let $x = 3\sin\theta$
$dx = 3\cos\theta \, d\theta$
$\theta = \arcsin(\frac{x}{3})$
$\arcsin(\frac{0}{3}) = 0$
$\arcsin(\frac{3}{3}) = \frac{\pi}{2}$

$$\to = \int_0^{\pi/2} \sqrt{9 - 9\sin^2\theta} \, 3\cos\theta \, d\theta$$

$$= 9\int_0^{\pi/2} \cos^2\theta \, d\theta$$

$$= 9\int_0^{\pi/2} \frac{1 + \cos(2\theta)}{2} d\theta$$

$$= \frac{9}{2}\left(\theta + \frac{\sin(2\theta)}{2}\right)_0^{\pi/2}$$

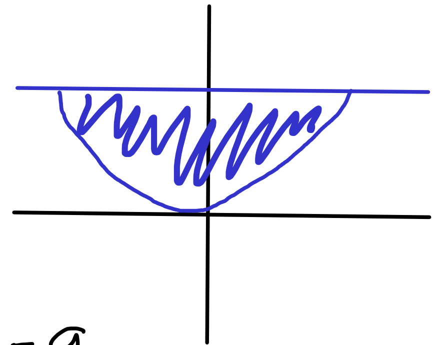
$$= \frac{9}{2}\left[\left(\frac{\pi}{2} + 0\right) - (0 + 0)\right] = \frac{9\pi}{4}$$

$$\approx 7.069$$

7. Find the volume of the solid whose base is the region enclosed by $y = x^2$ and $y = 3$. The vertical cross sections perpendicular to the y-axis are rectangles of height $\sqrt{y}$.

$$\text{Volume} = \int_0^3 A(y)\,dy$$

$$A(y) = \text{base} \cdot \text{height} = (2 \cdot \sqrt{y}) \cdot \sqrt{y}$$
$$= 2y$$



$$\text{Volume} = \int_0^3 2y\,dy = y^2 \Big|_0^3 = 9 - 0 = 9$$

8. Evaluate $\int \sin^3 x \cos^2 x \, dx$. Show all of your work.

$$= \int \sin x (1 - \cos^2 x) \cos^2 x \, dx$$
$$= \int \sin x (\cos^2 x - \cos^4 x) \, dx$$

$u = \cos x$
$du = -\sin x \, dx$

$$= -\int u^2 - u^4 \, du = \int u^4 - u^2 \, du$$

$$= \frac{u^5}{5} - \frac{u^3}{3} + C = \frac{\cos^5 x}{5} - \frac{\cos^3 x}{3} + C$$

9. Evaluate $\int x^2 \ln x \, dx$. Show all of your work.

$u = \ln(x)$ $\quad dv = x^2 dx$

$du = \frac{dx}{x}$ $\quad v = \frac{x^3}{3}$

$$= \frac{x^3 \ln(x)}{3} - \frac{1}{3}\int x^3 \cdot \frac{1}{x} dx$$

$$= \frac{x^3 \ln(x)}{3} - \frac{1}{3}\int x^2 dx = \frac{x^3 \ln(x)}{3} - \frac{x^3}{9} + C$$

10. A rod of length 7 meters has the linear density $\rho(x) = x(1+x)$ kg/m for $0 \leq x \leq 7$. Find the total mass M of this rod.

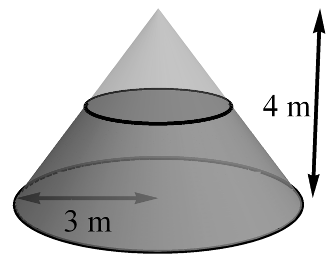
$$\text{mass} = \int \rho(x) dx$$

$$= \int_0^7 x(1+x) dx = \int_0^7 x + x^2 \, dx$$

$$= \frac{x^2}{2} + \frac{x^3}{3} \Big|_0^7 = \frac{49}{2} + \frac{343}{3}$$

$$= \frac{833}{6} \approx 138.8$$

11. A conical tank of height 4m and base radius 3m sits on its base and is filled to half its height with water. Find the work done in emptying the tank by pumping the water out the top. The acceleration due to gravity is 9.8 m/sec^2 and the density of water is 1,000 kg/m^3.



work on layer at height y

$= \text{mass} \cdot 9.8 \cdot (4-y)$

$= \pi r^2 \cdot \Delta y \cdot 1000 \cdot 9.8 \cdot (4-y)$

$= \pi \frac{9}{16}(4-y)^2 \Delta y \; 9800 \; (4-y)$

$\frac{\text{height}}{\text{base}} = \frac{4}{3} = \frac{4-y}{r}$

$r = \frac{3}{4}(4-y)$

Total work $= \int_0^2 \frac{9\pi}{16} \cdot 9800 \; (4-y)^3 dy$

$= \frac{11{,}025\pi}{2} \int_0^2 (4-y)^3 dy$

$u = 4-y$
$du = -dy$

$= -\frac{11{,}025\pi}{2} \int_4^2 u^3 du$

$= \frac{11{,}025\pi}{2} \int_2^4 u^3 du$

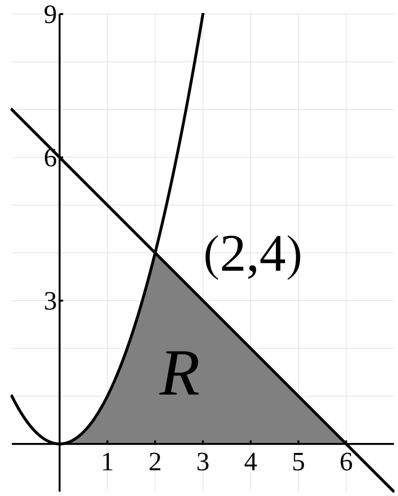
$= \frac{11{,}025\pi}{2} \cdot \frac{u^4}{4}\Big|_2^4 = \frac{11{,}025\pi}{2}\left(\frac{256-16}{4}\right)$

$= \frac{11{,}025\pi}{2} \cdot 60 = 330{,}750\pi \text{ J}$

$\approx 1039 \text{ kJ}$

12. The region R is the region bounded by the curves $f(x) = x^2$ and $g(x) = 6 - x$ and the x-axis for $x \geq 0$.
    (a) The region R is rotated around the x-axis. Set up the integral to be used to find the volume of this solid.

    (b) Find the volume of the solid in (b)



Disk method:

a) $V = \int_0^2 \pi (x^2)^2 \, dx + \int_2^6 \pi (6-x)^2 \, dx$

b) $\int_0^2 \pi (x^2)^2 dx = \pi \int_0^2 x^4 dx = \pi \frac{x^5}{5}\Big|_0^2 = \frac{32\pi}{5}$

$\int_2^6 \pi (6-x)^2 dx = \pi \int_2^6 (6-x)^2 dx$ $\quad u = 6-x$, $du = -dx$

$= -\pi \int_4^0 u^2 du = \pi \int_0^4 u^2 du = \pi \frac{u^3}{3}\Big|_0^4 = \frac{64\pi}{3}$

Volume $= \frac{32\pi}{5} + \frac{64\pi}{3} = \frac{416\pi}{15} \approx 87.13$