Exam 2

22 October 2013

KEY

Multiple Choice Questions

1. D
2. B
3. A
4. B
5. D
6. E

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1. Which of the following functions gives the arc length of curve $f(x) = \ln(\cos(x))$ over the interval from $x = 0$ to $x = t$, provided $0 \leq t < \frac{\pi}{2}$?

   A. $s(t) = \int_0^t \cos(x)\,dx$

   B. $s(t) = \int_0^t \sqrt{1 + \frac{\sin(x)}{\cos^2(x)}}\,dx$

   C. $s(t) = \int_0^t \sqrt{1 + \frac{1}{\cos^2(x)}}\,dx$

   D. $s(t) = \int_0^t \sec(x)\,dx$

   E. $s(t) = 1.22619$

Solution: $f'(x) = \frac{d}{dx}\ln(\cos(x)) = -\tan(x)$ so

$$s(t) = \int_0^t \sqrt{1 + (-\tan(x))^2}\,dx = \int_0^t \sqrt{1 + \tan^2(x)}\,dx = \int_0^t \sqrt{\sec^2(x)}\,dx = \int_0^t \sec(x)\,dx$$

2. Which of the following differential equations is **not** separable?

   A. $x(y^2+1)+(x-1)y'=0$

   B. $y'+3y^2=7x$

   C. $xy'+\sqrt{y}=0$

   D. $y'=\cos(y)$

   E. $y'+3y=1$

Solution: Isolating y' we have $y'=7x-3y^2$ and this cannot be written as a product $f(x)g(y)$.

3. Assume that $f(1)=1$, $f'(1)=3$, $f''(1)=2$ and $f'''(1)=4$. Which of the following is the Taylor polynomial $T_3(x)$ centered at $a=1$ for the function $f(x)$?

   A. $T_3(x)=1+3(x-1)+(x-1)^2+\frac{2}{3}(x-1)^3$

   B. $T_3(x)=1+3(x-1)+(x-1)^2+\frac{4}{3}(x-1)^3$

   C. $T_3(x)=3(x-1)+(x-1)^2+\frac{4}{3}(x-1)^3$

   D. $T_3(x)=3(x-1)+2(x-1)^2+4(x-1)^3$

   E. $T_3(x)=1+3(x-1)+2(x-1)^2+4(x-1)^3$

Solution: The nth term in the Taylor polynomial is given by $\dfrac{f^{(n)}(x)}{n!}(x-a)^n$. Therefore,

$$\begin{aligned}
T_3(x) &= f(a)+f'(a)(x-a)+\frac{f''(a)}{2!}(x-a)^2+\frac{f'''(a)}{3!}(x-a)^3 \\
&= f(1)+f'(1)(x-1)+\frac{f''(1)}{2!}(x-1)^2+\frac{f'''(1)}{3!}(x-1)^3 \\
&= 1+3(x-1)+\frac{2}{2!}(x-1)^2+\frac{4}{3!}(x-1)^3 \\
T_3(x) &= 1+3(x-1)+(x-1)^2+\frac{2}{3}(x-1)^3
\end{aligned}$$

so the answer is A.

4. Select the expected form of the partial fraction decomposition of the rational function

$$f(x) = \frac{x^2+3x-4}{(x^2-4)(x^2+4)(x^2+2x)}.$$

A. $\frac{A}{x} + \frac{B}{x-2} + \frac{C}{x+2} + \frac{Dx+E}{(x+2)^2} + \frac{Fx+G}{x^2+4}$

B. $\frac{A}{x} + \frac{B}{x-2} + \frac{C}{x+2} + \frac{D}{(x+2)^2} + \frac{Ex+F}{x^2+4}$

C. $\frac{Ax+B}{x^2+2x} + \frac{Cx+D}{x^2-4} + \frac{Ex+F}{x^2+4}$

D. $\frac{A}{x} + \frac{B}{x-2} + \frac{C}{x+2} + \frac{D}{x+2} + \frac{Ex+F}{x^2+4}$

E. $\frac{A}{x} + \frac{B}{x-2} + \frac{C}{x+2} + \frac{D}{(x+2)^2} + \frac{E}{x^2+4}$

Solution: Note that $(x^2-4)(x^2+4)(x^2+2x) = (x-2)(x+2)(x^2+4)(x)(x+2) = x(x-2)(x+2)^2(x^2+4)$. The form that the partial fractional decomposition will take is

$$\frac{x^2+3x-4}{(x^2-4)(x^2+4)(x^2+2x)} = \frac{x^2+3x-4}{x(x-2)(x+2)^2(x^2+4)}$$
$$= \frac{A}{x} + \frac{B}{x-2} + \frac{C}{x+2} + \frac{D}{(x+2)^2} + \frac{Ex+F}{x^2+4}$$

5. Which of the following statements is false?

A. $\int_1^\infty \frac{300}{x^2}\,dx$ converges.

B. $\int_0^1 x^{-2/5}\,dx$ converges.

C. $\int_4^\infty \frac{1}{x-\sqrt{3}}\,dx$ diverges.

D. $\int_3^6 \frac{1}{(x-3)^2}\,dx$ **converges**

E. $\int_2^\infty \frac{e^x}{x-1}\,dx$ diverges

Solution: A. converges by the p-series test since $2 > 1$. Likewise C diverges by the p-series test. E diverges since e^x grows faster than any power of x. B converges since this integral is equal to $\int_1^\infty x^{-5/2}\,dx + 1$ and $5/2 > 1$. D diverges.

6. Which of the following is a possible solution to the differential equation $y' = 2(y - 5)$?

   A. $y = 2 + 7e^{5t}$

   B. $y = 5 + 3e^{-2t}$

   C. $y = 2 - 3e^{5t}$

   D. $y = 5 - 2e^{3t}$

   E. $y = 5 + 7e^{2t}$

Solution: There are multiple ways to solve this. A student could plug each of the proposed solutions into the differential equation and see which one holds. A student could also remember the general form for the solutions of these types of equations. Either way, E is the correct choice.

Free Response Questions

You must show all of your work in these problems to receive credit. Answers without corroborating work will receive no credit.

7. Let $f(x) = \dfrac{18}{x^2 - 4x - 5}$.

(a) Find $\displaystyle\int \frac{18}{x^2 - 4x - 5}\,dx$.

Solution: Use partial fractions to decompose the integrand into two simpler pieces.

$$\frac{18}{x^2 - 4x - 5} = \frac{18}{(x-5)(x+1)} = \frac{A}{x-5} + \frac{B}{x+1}$$
$$18 = A(x+1) + B(x-5)$$

Substituting $x = 5$ in the above formula, we find that $18 = 6A$ or $A = 3$. Likewise substituting $x = -1$ gives us that $18 = -6B$ or $B = -3$. Thus,

$$\int \frac{18}{x^2 - 4x - 5}\,dx = \int \left(\frac{3}{x-5} - \frac{3}{x+1}\right)dx = 3\left(\ln|x-5| - \ln|x+1|\right) + C$$
$$= 3\ln\left|\frac{x-5}{x+1}\right| + C$$

(b) Find $\displaystyle\int_6^\infty \frac{18}{x^2 - 4x - 5}\,dx$.

Solution:

$$\int_6^\infty \frac{18}{x^2 - 4x - 5}\,dx = \lim_{M\to\infty} \int_6^M \frac{18}{x^2 - 4x - 5}\,dx$$
$$= \lim_{M\to\infty} 3\ln\left|\frac{x-5}{x+1}\right|\Bigg|_6^M$$
$$= \lim_{M\to\infty} \left(3\ln\left|\frac{M-5}{M+1}\right| - 3\ln\left|\frac{1}{7}\right|\right)$$
$$= 0 - 3\ln\frac{1}{7} = 3\ln 7 = \ln\left(7^3\right) = \ln(343) \approx 5.8377$$

8. (a) Calculate the arc length of the curve $y = \frac{2}{3}x^{3/2}$ over the interval $[0, 8]$.

Solution: $f'(x) = \frac{2}{3}\left(\frac{3}{2}x^{1/2}\right) = x^{1/2}$, so $\sqrt{1 + [f'(x)]^2} = \sqrt{1+x}$.

$$L = \int_0^8 \sqrt{1+x}\, dx = \frac{2}{3}(1+x)^{3/2}\Big|_0^8 = \frac{52}{3} \approx 17.333$$

(b) Now, take the curve $y = \frac{1}{3}x^3$ and rotate it about the x-axis. Calculate the surface area of the solid of rotation defined by this curve, $y = \frac{1}{3}x^3$ over the interval $[0, 3]$.

Solution: $f'(x) = \frac{1}{3}(3x^2) = x^2$, so $\sqrt{1 + [f'(x)]^2} = \sqrt{1 + (x^2)^2}$.

$$S = \int_0^3 2\pi \frac{1}{3}x^3 \sqrt{1 + (x^2)^2}\, dx = \frac{\pi}{9}(x^4+1)^{3/2}\Big|_0^3 = \frac{\pi}{9}(82\sqrt{82} - 1) \approx 258.847$$

9. In the following we use Simpson's rule for integral approximation, S_N on N subintervals.

(a) Compute S_4 for the integral $\int_0^1 e^{-2x} dx$.

Solution: $$\frac{1}{12}\left(e^{-2(0)} + 4e^{-2(.25)} + 2e^{-2(.5)} + 4e^{-2(.75)} + e^{-2(1)}\right) \approx .43248$$

(b) The error bound for using S_n for $x \in [a, b]$ is given by

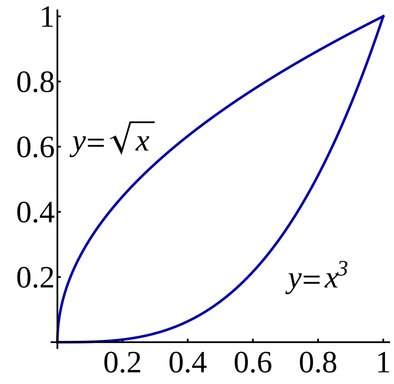
$$\text{Error}(S_N) \le \frac{K_4(b-a)^5}{180N^4},$$

where K_4 is the maximum value of $|f^{(4)}(x)|$ for all $x \in [a, b]$. Find K_4 and calculate the error bound for S_4.

Solution: $|f^{(4)}(x)| = 16e^{-2x}$ which is decreasing so, K_4 should be made 16.

$$\text{Bound} = \frac{16\,(1-0)^5}{180(4)^4} = \frac{1}{2880} \approx .00035$$

10. Determine the y-coordinate of the center of mass for the region bounded by $y = x^3$ and $y = \sqrt{x}$.



Solution: $y_{CM} = \dfrac{M_x}{M}$, so

$$y_{CM} = \frac{M_x}{M} = \frac{\rho \int_0^1 y\left(\sqrt[3]{y} - y^2\right)\,dy}{\rho \int_0^1 \left(\sqrt[3]{y} - y^2\right)\,dy} = \frac{\int_0^1 \left(y^{4/3} - y^3\right)\,dy}{\int_0^1 \left(y^{1/3} - y^2\right)\,dy}$$

$$= \frac{\left(\frac{3}{7}y^{7/3} - \frac{1}{4}y^4\right|_0^1}{\left(\frac{3}{4}y^{4/3} - \frac{1}{3}y^3\right|_0^1} = \frac{\frac{5}{28}}{\frac{5}{12}} = \frac{3}{7}$$

11. Use separation of variables to find the general solution to $5y' + 6x^2y^2 = 0$.

Solution:

$$\begin{aligned} 5y' + 6x^2y^2 &= 0 \\ 5y' &= -6x^2y^2 \\ \frac{y'}{y^2} &= -\frac{6}{5}x^2 \\ -\frac{1}{y} &= -\frac{2}{5}x^3 + C \\ y(x) &= \frac{1}{\frac{2}{5}x^3 + C} \end{aligned}$$

12. Let $T_n(x)$ be the nth Taylor polynomial for $f(x) = e^{-x}$ centered at $a = 0$.

(a) Find $T_4(x)$.

Solution: We need $f(0)$, $f'(0)$, $f''(0)$, $f'''(0)$, and $f^{(4)}(0)$.

$$\begin{aligned} f(x) &= e^{-x} \Rightarrow f(0) = e^0 = 1 \\ f'(x) &= -e^{-x} \Rightarrow f'(0) = -e^0 = -1 \\ f''(x) &= e^{-x} \Rightarrow f''(0) = e^0 = 1 \\ f'''(x) &= -e^{-x} \Rightarrow f''''(0) = -e^0 = -1 \\ f^{(4)}(x) &= e^{-x} \Rightarrow f^{(4)}(0) = e^0 = 1 \\ T_4(x) &= f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \frac{f^{(4)}(0)}{4!}x^4 \\ &= 1 - x + \frac{x^2}{2} - \frac{x^3}{3!} + \frac{x^4}{4!} \end{aligned}$$

(b) The error bound for using $T_n(x)$ centered at $x = a$ to approximate $f(x)$ is given by

$$|f(x) - T_n(x)| \le K\frac{|x-a|^{n+1}}{(n+1)!}, \text{ where } K = \max\left\{|f^{(n+1)}(u)| : u \text{ is between } a \text{ and } x\right\}.$$

Use this formula to find the error bound of $|f(1) - T_4(1)|$.

Solution: The fifth derivative of $f(x)$ is $f^{(5)}(x) = -e^{-x}$. On the interval $[0, 1]$ the largest absolute value of this derivative will be e so let $K = 1$. Then

$$|f(1) - T_4(1)| \le K\frac{|x-1|^5}{5!} \le 1\frac{|0-1|^5}{5!} = \frac{1}{5!}.$$

13. Newton's Law of Cooling predicts that the temperature $y(t)$ of a cooling object satisfies the differential equation

$$y' = -k(y - T_0),$$

where $k > 0$ is a constant and T_0 is the temperature of the environment around the object.

A bowl of soup is served at $70°C$. Assuming a cooling constant of $k = 1\,\text{min}^{-1}$ and an ambient temperature of $T_0 = 20°\text{C}$

(a) Use the information given above to set up the precise differential equation needed to solve this problem.

Solution: $y' = -1(y - 20)$

(b) What is the initial condition?

Solution: $y_0 = 70$.

(c) Find a formula for $y(t)$

Solution:

$$\begin{aligned} y' &= -1(y-20) = 20 - y \\ \frac{y'}{y-20} &= -1 \\ \ln|y-20| &= -t + C, \\ \text{When } t = 0,\ y = 70, \quad \ln|70-20| &= C \\ C &= \ln 50 \\ \ln|y-20| &= \ln(50) - t \\ y(t) - 20 &= e^{\ln(50)-t} = 50e^{-t} \\ y(t) &= 20 + 50e^{-t} \end{aligned}$$

(d) Use this to predict how long it will take for the soup to cool to $50°C$.

Solution: Find t so that $y(t) = 50$.

$$\begin{aligned} y(t) &= 20 + 50e^{-t} \\ 50 &= 20 + 50e^{-t} \\ e^{-t} &= \frac{3}{5} \\ t &= -\ln\frac{3}{5} \\ &= 0.51 \text{ min} \end{aligned}$$