Name: ______________________________ Section: ______________

Do not remove this answer page — you will return the whole exam. You will be allowed two hours to complete this test. No books or notes may be used. You may use a graphing calculator during the exam, but NO calculator with a Computer Algebra System (CAS) or a QWERTY keyboard is permitted. Absolutely no cell phone use during the exam is allowed.

The exam consists of 10 multiple choice questions and 5 free response questions. Record your answers to the multiple choice questions on this page by filling in the circle corresponding to the correct answer.

Show all work to receive full credit on the free response problems. It will also help you check your answers to show work on multiple choice problems.

Multiple Choice Questions

1 (A) (B) (C) (D) (E)

2 (A) (B) (C) (D) (E)

3 (A) (B) (C) (D) (E)

4 (A) (B) (C) (D) (E)

5 (A) (B) (C) (D) (E)

6 (A) (B) (C) (D) (E)

7 (A) (B) (C) (D) (E)

8 (A) (B) (C) (D) (E)

9 (A) (B) (C) (D) (E)

10 (A) (B) (C) (D) (E)

Multiple Choice	11	12	13	14	15	Total Score
50	10	10	10	10	10	100

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Multiple Choice Questions

1. (5 points) Find all values of x for which the limit $\lim_{n\to\infty} x^n$ exists and is finite.

   A. $(-1,1)$
   B. $[-1,1)$
   C. $[-1,1]$
   D. $(-1,1]$
   E. $[0,1]$

2. (5 points) Suppose that $\{a_n\}$ is a convergent sequence and

$$\lim_{n\to\infty}\left(\frac{n+1}{n+2}+2\,a_n\right)=5.$$

   Find $\lim_{n\to\infty} a_n$.

   A. 1
   B. 2
   C. 3
   D. 4
   E. 5

3. (5 points) Find the sum of the series

$$\sum_{n=1}^{\infty}\left(\frac{1}{n+1}-\frac{1}{n}\right).$$

   A. -1
   B. $-\frac{1}{2}$
   C. 0
   D. 1
   E. $\frac{1}{2}$

4. (5 points) Find the sum of the series $\sum_{n=1}^{\infty} 3^{-n}$.

A. 1/3

B. 1/2

C. 1

D. 3/2

E. 3

5. (5 points) Use the limit comparison test to test the convergence of the series

$$\sum_{n=1}^{\infty} \frac{n^2}{n^3+n+1}.$$

A. Comparing with the series $\sum_{n=1}^{\infty} \frac{1}{n}$ gives that the series diverges.

B. Comparing with the series $\sum_{n=1}^{\infty} \frac{1}{n}$ gives that the series converges.

C. Comparing with the series $\sum_{n=1}^{\infty} \frac{1}{n^3}$ gives that the series converges.

D. Comparing with the series $\sum_{n=1}^{\infty} \frac{1}{n^3}$ gives that the series diverges.

E. No conclusion can be drawn from the limit comparison test.

6. (5 points) Choose the correct statement.

A. The series $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$ is absolutely convergent.

B. The series $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$ is divergent.

C. The series $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ is absolutely convergent.

D. The series $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ is divergent.

E. None of the statements A-D are correct.

7. (5 points) Consider the series $\sum_{n=1}^{\infty} \frac{(-1)^n}{n!}$. Find the limit $\lim_{n\to\infty} \frac{a_{n+1}}{a_n}$.

A. $-\infty$

B. -1

C. 0

D. 1

E. ∞

8. (5 points) Give the radius of convergence for the series $\sum_{n=3}^{\infty} 3^n (x-2)^n$.

A. 1/3

B. 1/2

C. 2

D. 3

E. 6

9. (5 points) If $f(x) = \sum_{n=1}^{\infty} nx^n$, find the coefficient of x^3 in the series for $f'(x)$.

   A. 4

   B. 6

   C. 2

   D. 12

   E. 16

10. (5 points) Find the coefficient of x^4 in the Maclaurin series for $\cos(x^2)$. (The Maclaurin series is another name for the Taylor series centered at 0.)

   A. 1/24

   B. 0

   C. -1/2

   D. 1/2

   E. 1

Free Response Questions

11. Consider the sequence defined recursively by $a_{n+1} = 1 + \frac{a_n}{2}$ and $a_1 = 4$.

(a) (4 points) Find a_2 and a_3.

$$a_2 = 1 + \frac{a_1}{2} = 1 + \frac{4}{2} = 3.$$

$$a_3 = 1 + \frac{a_2}{2} = 1 + \frac{3}{2} = \frac{5}{2}.$$

(b) (3 points) Assume that $A = \lim_{n\to\infty} a_n$ exists and find an equation satisfied by A.

$$A = \lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty} 1 + \frac{a_n}{2} = 1 + \frac{1}{2}\lim_{n\to\infty} a_n = 1 + \frac{1}{2}A.$$

(c) (3 points) Find A.

$$A = 1 + \frac{1}{2}A.$$

$$\frac{1}{2}A = 1$$

$$A = 2.$$

12. (a) (4 points) State the ratio test for convergence of a series $\sum_{n=1}^{\infty} a_n$.

Assume $L = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$ exists.

If $L < 1$, then the series converges absolutely.

If $L > 1$ or $L = \infty$, then the series diverges.

If $L = 1$, the Ratio Test is inconclusive.

(b) (6 points) For each of the series below determine if the ratio test gives convergence, divergence or no information.

$$\text{i) } \sum_{n=1}^{\infty} \frac{2^n}{n!}, \qquad \text{ii) } \sum_{n=1}^{\infty} \frac{1}{n^4+1}.$$

(i) $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{2^{n+1}}{(n+1)!} \cdot \frac{n!}{2^n} = \lim_{n\to\infty} \frac{2}{n+1} = 0.$

By the Ratio Test, the series $\sum_{n=1}^{\infty} \frac{2^n}{n!}$ converges (absolutely).

(ii) $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{n^4+1}{(n+1)^4+1} = \lim_{n\to\infty} \frac{1+\frac{1}{n^4}}{(1+\frac{1}{n})^4+\frac{1}{n^4}} = 1.$

The Ratio Test gives no information about the series $\sum_{n=1}^{\infty} \frac{1}{n^4+1}$.

13. (a) (5 points) Find the Maclaurin series for $f(x) = \dfrac{1}{1-x^2}$. Write the answer as an infinite sum and then give the first three non-zero terms of the series. Hint: Recall the sum of a geometric series.

$f(x) = \sum_{n=0}^{\infty} x^{2n}$ for $|x| < 1$.

The first three nonzero terms are $1, x^2, x^4$.

(b) (5 points) Find the Maclaurin series for $\displaystyle\int_0^x \frac{1}{1-t^2}\,dt$. Write the answer as an infinite sum and then give the first three non-zero terms of the series.

Let $f(x) = \int_0^x \frac{1}{1-t^2}\,dt$.

Then $f'(x) = \frac{1}{1-x^2} = \sum_{n=0}^{\infty} x^{2n}$ for $|x|<1$.

So $f(x) = c_0 + \sum_{n=0}^{\infty} \frac{1}{2n+1} x^{2n+1}$ for $|x|<1$, $c_0 \in \mathbb{R}$.

Since $f(0) = \int_0^x \frac{1}{1-t^2}\,dt = 0$, then $c_0 = 0$.

So $f(x) = \sum_{n=0} \frac{1}{2n+1} x^{2n+1}$ for $|x|<1$.

The first three nonzero terms are $x, \frac{1}{3}x^3, \frac{1}{5}x^5$.

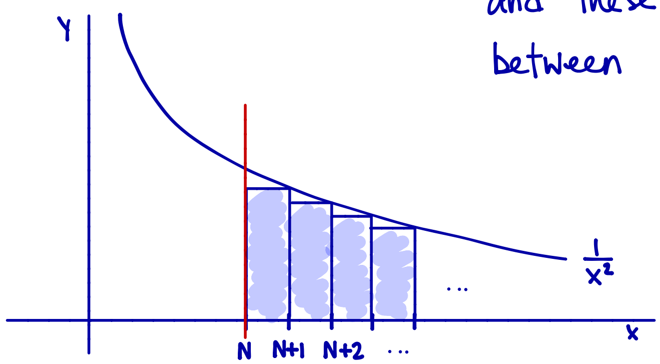
14. (a) (5 points) Find an improper integral which is an upper bound for the sum

$$\sum_{n=N+1}^{\infty} \frac{1}{n^2}.$$

Justify your answer. You may use a sketch as part of the justification.

$$\sum_{n=N+1}^{\infty} \frac{1}{n^2} \leq \int_N^{\infty} \frac{1}{x^2}\, dx.$$

$\sum_{n=N+1}^{\infty} \frac{1}{n^2}$ = the area of the rectangular columns, and these columns lie within the area between $y = \frac{1}{x^2}$ and the x-axis, which is $\int_N^{\infty} \frac{1}{x^2}\, dx$.



(b) (5 points) Use your answer to part a) to find a value N so that $\sum_{n=N+1}^{\infty} \frac{1}{n^2} \leq 0.01$.

$$\sum_{n=N+1}^{\infty} \frac{1}{n^2} \leq \int_N^{\infty} \frac{1}{x^2}\, dx \leq 0.01$$

$$\lim_{b\to\infty} \int_N^{b} \frac{1}{x^2}\, dx \leq 0.01$$

$$\lim_{b\to\infty} -x^{-1}\Big|_N^b \leq 0.01$$

$$\lim_{b\to\infty} \frac{-1}{b} + \frac{1}{N} \leq 0.01$$

$$\frac{1}{N} \leq 0.01$$

So $N \geq 100$ guarantees that $\sum_{n=N+1}^{\infty} \frac{1}{n^2} \leq 0.01$.

15. (10 points) Determine if each of the series converges. Justify your answer.

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{n^2}{(n+1)^2}$

$\displaystyle\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{n^2}{(n+1)^2} = 1 \neq 0.$

By the Divergence Test, the series diverges.

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$ Let $a_n = \frac{1}{\sqrt{n}}$.

$a_n = \frac{1}{\sqrt{n}} > \frac{1}{\sqrt{n+1}} = a_{n+1}.$

$\displaystyle\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{\sqrt{n}} = 0.$

By the Alternating Series Test, the series converges.

(c) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$

$\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} = \sum_{n=1}^{\infty} \frac{1}{n^{1/2}}.$ Since $\frac{1}{2} < 1$, the series diverges.