Exam 2

Name: ______________________________ Section: ______________

Do not remove this answer page — you will return the whole exam. You will be allowed two hours to complete this test. No books or notes may be used. You may use a graphing calculator during the exam, but NO calculator with a Computer Algebra System (CAS) or a QWERTY keyboard is permitted. Absolutely no cell phone use during the exam is allowed.

The exam consists of 10 multiple choice questions and 5 free response questions. Record your answers to the multiple choice questions on this page by filling in the circle corresponding to the correct answer.

Show all work to receive full credit on the free response problems. It will also help you check your answers to show work on multiple choice problems.

Multiple Choice Questions

1 (A) (B) (C) (D) (E) **6** (A) (B) (C) (D) (E)

2 (A) (B) (C) (D) (E) **7** (A) (B) (C) (D) (E)

3 (A) (B) (C) (D) (E) **8** (A) (B) (C) (D) (E)

4 (A) (B) (C) (D) (E) **9** (A) (B) (C) (D) (E)

5 (A) (B) (C) (D) (E) **10** (A) (B) (C) (D) (E)

Multiple Choice	11	12	13	14	15	Total Score
50	10	10	10	10	10	100

Trig identities

- $\sin^2(x) + \cos^2(x) = 1$,
- $\sin^2(x) = \frac{1}{2}(1 - \cos(2x))$ and $\cos^2(x) = \frac{1}{2}(1 + \cos(2x))$
- $\sin(x + y) = \sin(x)\cos(y) + \cos(x)\sin(y)$ and $\cos(x + y) = \cos(x)\cos(y) - \sin(x)\sin(y)$

Multiple Choice Questions

1. (5 points) Is the integral $\displaystyle\int_2^{\infty} ye^{-y}dy$ convergent or divergent?

   A. Converges to $\dfrac{3}{e}$.

   B. Converges to $\dfrac{3}{e^2}$.

   C. Converges to $\dfrac{1}{e^2}$.

   D. Diverges.

   E. Converges and can't determine the value it converges to.

Solution: Use integration by parts with $u = y$ and $dv = e^{-y}dy$. Then $du = dy$ and $v = -e^{-y}$.

$$\int ye^{y}dy = -ye^{-y} + \int e^{-y}dy = -ye^{-y} - e^{-y}$$

$$\int_2^{\infty} ye^{-y}dy = \lim_{n\to\infty}\left(\frac{-n}{e^n} - \frac{1}{e^n}\right) + \frac{2}{e^2} + \frac{1}{e^2} = \frac{3}{e^2}$$

2. (5 points) Is the integral $\displaystyle\int_{-1}^{2} \frac{x}{(x+1)^2}dx$ convergent or divergent?

   A. Converges and can't determine the value it converges to.

   B. Converges to 1.

   C. Converges to 0.

   D. Converges to 5.

   E. Diverges.

Solution: Use $u = x + 1$. Then $du = dx$, $x = u - 1$,

$$\int \frac{x}{(x+1)^2} = \int \frac{u-1}{u^2}du = \int \left(\frac{1}{u} - \frac{1}{u^2}\right)du = \ln|u| + \frac{1}{u} + C = \ln|x+1| + \frac{1}{x+1} + C$$

This integral diverges.

3. (5 points) What are the first five terms of the sequence defined by

$$a_1 = 1 \quad a_{n+1} = 3 - \frac{1}{a_n}?$$

A. $\{1, 2, 1, 2, 1\}$

B. $\{1, 2, \frac{5}{2}, \frac{13}{5}, \frac{34}{13}\}$

C. $\{1, \frac{5}{2}, \frac{8}{3}, \frac{11}{4}, \frac{14}{5}\}$

D. $\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}\}$

E. $\{1, 2, \frac{2}{5}, \frac{5}{13}, \frac{13}{34}\}$

Solution: $a_1 = 1$, $a_2 = 3 - 1 = 2$, $a_3 = 3 - \frac{1}{2} = \frac{5}{2}$, $a_4 = 3 - \frac{2}{5} = \frac{13}{5}$, $a_5 = 3 - \frac{5}{13} = \frac{34}{13}$

4. (5 points) Does the sequence

$$a_n = \frac{4^n}{1 + 9^n}$$

converge or diverge?

A. Converges by the squeeze theorem since $0 \leqslant a_n \leqslant b_n = \left(\frac{4}{9}\right)^n$**.**

B. Converges by the squeeze theorem since $0 \leqslant a_n \leqslant b_n = \frac{1}{9}\left(\frac{4}{9}\right)^n$.

C. Diverges by the comparison theorem since $0 \leqslant a_n \leqslant b_n = \left(\frac{4}{9}\right)^n$.

D. Diverges by the comparison theorem since $b_n = \frac{4^n}{1} \leqslant a_n$.

E. Diverges by the comparison theorem since $b_n = \left(\frac{1}{9}\right)^n \leqslant a_n$.

Solution: Since $9^n < 1 + 9^n$,

$$0 < \frac{4^n}{1 + 9^n} < \frac{4^n}{9^n} = \left(\frac{4}{9}\right)^n$$

The sequence $\left(\frac{4}{9}\right)^n$ converges to 0.

5. (5 points) Does the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{1 + \left(\frac{2}{3}\right)^n}$ converge or diverge?

A. Converges because $\displaystyle\lim_{n\to\infty} \frac{1}{1 + \left(\frac{2}{3}\right)^n} = 0$.

B. Converges because it is a geometric series and $|r| < 1$.

C. Converges by comparison to $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\left(\frac{2}{3}\right)^n}$.

D. Diverges because it is a geometric series and $|r| \geq 1$.

E. Diverges because $\displaystyle\lim_{n\to\infty} \frac{1}{1+\left(\frac{2}{3}\right)^n} \neq 0$.

Solution: Since $\displaystyle\lim_{n\to\infty} \left(\frac{2}{3}\right)^n = 0$

$$\lim_{n\to\infty} \frac{1}{1+\left(\frac{2}{3}\right)^n} = 1$$

and the series diverges.

6. (5 points) Which of the following series converge?

A. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$

B. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$

C. $\displaystyle\sum_{n=1}^{\infty} (1.05)^n$

D. All of the above series converge.

E. None of the above series converge.

Solution: The first series converges by the alternating series test. The second and third diverge by the integral test.

7. (5 points) Which if the following series converge absolutely?

A. $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{n^2+1}$

B. $\displaystyle\sum_{n=1}^{\infty} \left(\frac{-1}{2}\right)^n$

C. $\displaystyle\sum_{n=1}^{\infty} \frac{\sin n}{n^3}$

D. All of the series above converge absolutely.

E. None of the series above converge absolutely.

Solution: $\sum_{n=0}^{\infty} \frac{1}{n^2+1}$ converges by comparison to $\sum_{n=0}^{\infty} \frac{1}{n^2}$. $\sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n$ converges because it is a geometric series for a number smaller than 1. $\sum_{n=1}^{\infty} \frac{|\sin n|}{n^3}$ converges by comparison to $\sum_{n=1}^{\infty} \frac{1}{n^3}$

8. (5 points) What would you compare $\displaystyle\sum_{n=1}^{\infty} \frac{n^2+2}{n^4+5}$ to for a conclusive comparison test?

A. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$

B. $\displaystyle\sum_{n=1}^{\infty} \ln n$

C. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$

D. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^4}$

E. The comparison test can't be used to understand convergence for this series.

9. (5 points) If $\sum_{k=1}^{\infty} a_k$ is a series that has partial sums $\sum_{k=1}^{N} a_k = s_N = \frac{2N^2+1}{3N^2+N+1}$, then what can be said about the series?

A. The series converges to 0.

B. The series converges to $\frac{2}{3}$.

C. The series converges to 1.

D. The series diverges since $\lim_{N\to\infty} \frac{2N^2+1}{3N^2+N+1} \neq 0$.

E. The series diverges by the comparison test.

10. (5 points) Which of the following is the limit would you need to compute if you used the ratio test to decide if

$$\sum_{n=2}^{\infty} \frac{n^2}{(2n-1)!}$$

converges?

A. $\lim_{n\to\infty} \frac{n^2}{(2n-1)!}$

B. $\lim_{n\to\infty} \sqrt[n]{\frac{n^2}{(2n-1)!}}$

C. $\lim_{n\to\infty} \frac{(n+1)^2 n^2}{(2n+1)!(2n-1)!}$

D. $\lim_{n\to\infty} \frac{(n+1)^2}{2n^3(2n+1)}$

E. $\lim_{n\to\infty} \frac{(n+1)}{n(2n+1)}$

Solution:

$$\begin{aligned}\frac{(n+1)^2}{(2(n+1)-1)!}\frac{(2n-1)!}{n^2} &= \left(\frac{n+1}{n}\right)^2 \frac{(2n-1)!}{(2n+2-1)!}\\ &= \left(\frac{n+1}{n}\right)^2 \frac{1}{(2n+1)2n} = \frac{(n+1)^2}{2n^3(2n+1)}\end{aligned}$$

Free Response Questions

11. (a) (2 points) Find $\int \frac{1}{x^2} dx$.

Solution:

$$\int \frac{1}{x^2} dx = \int x^{-2} dx = -x^{-1} + C$$

Suggested grading rubric:
(1 point) Say something relevant but not totally correct.

(b) (3 points) Does $\int_1^\infty \frac{1}{x^2} dx$ converge or diverge?

Solution: An antiderivative for x^{-2} is $-x^{-1}$. Then

$$\int_1^\infty \frac{1}{x^2} dx = \lim_{k \to \infty} \frac{-1}{k} + \frac{1}{1} = 0 + 1$$

Suggested grading rubric:
(1 point) Find a correct antiderivative - or copy from (a).
(1 point) Know what limit to compute.
(1 point) Compute limit correctly.

(c) (3 points) Use the comparison theorem to determine if the integral $\int_1^\infty \frac{x}{x^3+1} dx$ converges or diverges.

Solution: $0 < \frac{x}{x^3+1} < \frac{1}{x^2}$ since $xx^2 < x^3 + 1$ so this integral converges.

Suggested grading rubric:
(2 points) Correct comparison to $\frac{1}{x^2}$
(1 point) Correct conclusion.

(d) (2 points) Given that the inequality $\frac{x}{x^3+1} \leqslant \frac{1}{x^2}$ is true for $0 < x < \infty$, what is wrong with using the comparison test to conclude that the improper integral $\int_0^\infty \frac{x}{x^3+1} dx$ converges?

Solution: The integral $\int_0^\infty \frac{1}{x^2} dx$ doesn't converge.

12. (a) (2 points) Find the first three terms of the sequence defined by

$$a_n = \frac{1-n}{2+n}$$

Solution: $a_1 = \dfrac{1-1}{2+1} = 0$, $a_2 = \dfrac{1-2}{2+2} = \dfrac{-1}{4}$, $a_3 = \dfrac{1-3}{2+3} = \dfrac{-2}{5}$

Suggested grading rubric:
(1 point) Say something relevant but not totally correct.

(b) (5 points) Is this sequence bounded? Justify your answer!

Solution:

$$\begin{aligned} 1-n &< 0 = 0 * (2+n) \\ \frac{1-n}{2+n} &< 0 \end{aligned}$$

$$\begin{aligned} -2-n &< 1-n \\ -1(2+n) &< 1-n \\ -1 &< \frac{1-n}{2+n} \end{aligned}$$

Suggested grading rubric:
(3 points) If correctly shows bounded above or below but not both.
take off 1 point for arithmetic errors.

(c) (3 points) What is the limit?

Solution: $\lim\limits_{n\to\infty} \dfrac{1-n}{2+n} = \lim\limits_{n\to\infty} \dfrac{\frac{1}{n}-1}{\frac{2}{n}+1} = -1$

L'Hopital and dividing top and bottom by n are both acceptable here.

13. (a) (4 points) If $\dfrac{1}{(n+2)(n+3)} = \dfrac{A}{n+2} + \dfrac{B}{n+3}$ find A and B.

Solution:

$$\begin{aligned} 1 &= A(n+3) + B(n+2) & \\ 1 &= A(-2+3) & A = 1 \\ 1 &= A(-3+3) + B(-3+2) & B = -1 \end{aligned}$$

Suggested grading rubric:
take off one point for arithmetic error.

(b) (4 points) Use part (a) to find a simpler expression for

$$\sum_{n=1}^{k} \frac{1}{(n+2)(n+3)}$$

Solution:

$$\sum_{n=1}^{k} \frac{1}{(n+2)(n+3)} = \sum_{n=1}^{k} \left(\frac{1}{n+2} - \frac{1}{n+3} \right) = \frac{1}{1+2} - \frac{1}{k+3}$$

(c) (2 points) What is the sum of the series

$$\sum_{n=1}^{\infty} \frac{1}{(n+2)(n+3)}?$$

Solution:

$$\lim_{k\to\infty} \frac{1}{1+2} - \frac{1}{k+3} = \frac{1}{3}$$

Suggested grading rubric:
(3 points) Use trig substitution to simplify into a form integrated directly
(3 points) integrate
(2 points) substitute back in

14. Are the series below absolutely convergent, conditionally convergent, or divergent?

(a) (5 points)

$$\sum_{n=1}^{\infty} \frac{9^n}{3+10^n}$$

Solution: Comparison test:

$$\begin{aligned} 10^n &< (3+10^n) \\ 9^n 10^n &< 9^n(3+10^n) \\ \frac{9^n}{3+10^n} &< \frac{9^n}{10^n} \end{aligned}$$

The series $\sum_{n=1}^{\infty} \frac{9^n}{10^n}$ converges since it is a geometric series for $\frac{9}{10} < 1$.

Suggested grading rubric:

(3 points) Correct comparison (with justification!)

(2 points) Correct convergence for series compared to.

(b) (5 points)

$$\sum_{n=1}^{\infty} \frac{5+2n}{(1+n^2)^2}$$

Solution: Limit comparison test:

$$\begin{aligned} \lim_{n\to\infty} \frac{\frac{5+2n}{(1+n^2)^2}}{\frac{1}{n^3}} &= \lim_{n\to\infty} \frac{5+2n}{(1+n^2)^2} n^3 = \lim_{n\to\infty} \frac{5n^3+2n^4}{(1+n^2)^2} \\ &= \lim_{n\to\infty} \frac{5n^3+2n^4}{1+2n^2+n^4} = \lim_{n\to\infty} \frac{\frac{5}{n}+2}{\frac{1}{n^4}+\frac{2}{n^2}+1} = 2 \end{aligned}$$

Since $\sum_{n=1}^{\infty} \frac{1}{n^3}$ converges the original series converges.

Suggested grading rubric:

(3 points) Correct comparison (with justification!)

(2 points) Correct convergence for series compared to.

15. Are the series below absolutely convergent, conditionally convergent, or divergent?

(a) (5 points) $\displaystyle\sum_{n=1}^{\infty}\left(\frac{1-n}{2+3n}\right)^n$

Solution: Root test:

$$\sqrt[n]{\left|\left(\frac{1-n}{2+3n}\right)^n\right|} = \frac{|1-n|}{2+3n} = \frac{|\frac{1}{n}-1|}{\frac{2}{n}+3} \mapsto \frac{1}{3}$$

So the series is absolutely converges.

Suggested grading rubric:

1 point for choosing a reasonable test
3 points for correct simplification
1 points for correct interpretation

(b) (5 points) $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^{n-1}3^n}{2^n n^3}$

Solution: Ratio test

$$\left|\frac{\frac{(-1)^n 3^{n+1}}{2^{n+1}(n+1)^3}}{\frac{(-1)^{n-1}3^n}{2^n n^3}}\right| = \left|\frac{(-1)^n 3^{n+1}}{2^{n+1}(n+1)^3}\frac{2^n n^3}{(-1)^{n-1}3^n}\right| = \left|\frac{(-1)3n^3}{2(n+1)^3}\right| = \frac{3}{2}\left|\frac{n}{n+1}\right|^3$$

$$\lim_{n\to\infty}\left|\frac{\frac{(-1)^n 3^{n+1}}{2^{n+1}(n+1)^3}}{\frac{(-1)^{n-1}3^n}{2^n n^3}}\right| = \frac{3}{2}\left|\lim_{n\to\infty}\frac{n}{n+1}\right|^3 = \frac{3}{2}$$

So the series is divergent.

Suggested grading rubric:

1 point for choosing a reasonable test
3 points for correct simplification
1 points for correct interpretation