Exam 2

Name: ______________________________ Section: ______________

Do not remove this answer page — you will return the whole exam. You will be allowed two hours to complete this test. No books or notes may be used. You may use a graphing calculator during the exam, but NO calculator with a Computer Algebra System (CAS) or a QWERTY keyboard is permitted. Absolutely no cell phone use during the exam is allowed.

The exam consists of 10 multiple choice questions and 5 free response questions. Record your answers to the multiple choice questions on this page by filling in the circle corresponding to the correct answer.

Show all work to receive full credit on the free response problems. It will also help you check your answers to show work on multiple choice problems.

Multiple Choice Questions

1 (A) (B) (C) (D) (E) **6** (A) (B) (C) (D) (E)

2 (A) (B) (C) (D) (E) **7** (A) (B) (C) (D) (E)

3 (A) (B) (C) (D) (E) **8** (A) (B) (C) (D) (E)

4 (A) (B) (C) (D) (E) **9** (A) (B) (C) (D) (E)

5 (A) (B) (C) (D) (E) **10** (A) (B) (C) (D) (E)

Multiple Choice	11	12	13	14	15	Total Score
50	10	10	10	10	10	100

Multiple Choice Questions

1. (5 points) What are the first five terms of the sequence defined by

$$a_0 = 2 \quad a_{n+1} = \frac{a_n}{4} + \frac{3}{4}?$$

   A. $\{2, \frac{5}{4}, \frac{17}{16}, \frac{65}{64}, \frac{257}{256}\}$
   B. $\{2, \frac{5}{4}, \frac{17}{16}, \frac{65}{64}, \frac{129}{128}\}$
   C. $\{2, \frac{5}{4}, \frac{17}{16}, \frac{257}{256}, \frac{1025}{1024}\}$
   D. $\{2, \frac{5}{4}, \frac{129}{128}, \frac{257}{256}, \frac{1025}{1024}\}$
   E. $\{2, 1, 1, 1, 1\}$

2. (5 points) Does the sequence

$$a_n = \frac{3^n}{2 + 8^n}$$

   converge or diverge?

   A. Converges by the squeeze theorem since $0 \leqslant a_n \leqslant b_n = \left(\frac{3}{8}\right)^n$**.**
   B. Converges by the squeeze theorem since $0 \leqslant a_n \leqslant b_n = \frac{1}{8^n}$.
   C. Diverges by the comparison theorem since $0 \leqslant a_n \leqslant b_n = \left(\frac{3}{8}\right)^n$.
   D. Diverges by the comparison theorem since $b_n = \frac{3^n}{1} \leqslant a_n$.
   E. Diverges by the comparison theorem since $b_n = \left(\frac{8}{3}\right)^n \leqslant a_n$.

3. (5 points) Does the series $\sum_{n=1}^{\infty} \frac{3}{2+\left(\frac{4}{5}\right)^n}$ converge or diverge?

A. Diverges because $\lim_{n\to\infty} \frac{3}{2+\left(\frac{4}{5}\right)^n} \neq 0$**.**

B. Converges because $\lim_{n\to\infty} \frac{3}{2+\left(\frac{4}{5}\right)^n} = 0$.

C. Converges because it is a geometric series and $|r| < 1$.

D. Converges by comparison to $\sum_{n=1}^{\infty} \frac{3}{\left(\frac{4}{5}\right)^n}$.

E. Diverges because it is a geometric series and $|r| \geq 1$.

4. (5 points) Which of the following series converge?

A. $\sum_{n=1}^{\infty} \frac{(-1)^n}{3n}$

B. $\sum_{n=1}^{\infty} \frac{5}{\sqrt{n}}$

C. $\sum_{n=1}^{\infty} (-1)^n$

D. $\sum_{n=1}^{\infty} \left(\frac{1000}{999}\right)^n$

E. None of the above series converge.

5. (5 points) Find the sum of the series $\sum_{n=1}^{\infty} \left(\frac{2}{n(n+1)} + \frac{3}{2^n} \right)$

A. 5

B. 0

C. $\frac{2}{3}$

D. $\frac{3}{2}$

E. This series is divergent.

6. (5 points) What would you compare $\sum_{n=1}^{\infty} \frac{\sqrt{n^3+1}}{n^3+n^2+1}$ to for a conclusive comparison test?

A. $\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$

B. $\sum_{n=1}^{\infty} \frac{1}{n}$

C. $\sum_{n=1}^{\infty} \ln n$

D. $\sum_{n=1}^{\infty} \frac{1}{n^3}$

E. The comparison test can't be used to understand convergence for this series.

7. (5 points) If $\sum_{k=1}^{\infty} a_k$ is a series that has partial sums $s_N = \sum_{k=1}^{N} a_k = \frac{3N^2+4}{5N^2+N+6}$, then what can be said about the series?

A. The series converges to $\frac{3}{5}$.

B. The series converges to 0.

C. The series converges to 1.

D. The series diverges since $\lim_{N\to\infty} \frac{3N^2+4}{5N^2+N+6} \neq 0$.

E. The series diverges by the comparison test.

8. (5 points) When using the ratio test to decide whether the following series converges or diverges, which of the following limits would you need to compute?

$$\sum_{n=1}^{\infty} \frac{3^n}{(2n)!}$$

A. $\lim_{n\to\infty} \frac{3}{(2n+2)(2n+1)}$

B. $\lim_{n\to\infty} \frac{3}{(2n+1)}$

C. $\lim_{n\to\infty} \sqrt[n]{\frac{6}{(2n+1)}}$

D. $\lim_{n\to\infty} \frac{6}{(2n+1)}$

E. $\lim_{n\to\infty} \frac{3}{(2n+1)!}$

9. (5 points) What is the interval of convergence of the power series $\sum_{n=1}^{\infty} \frac{(x-5)^n}{n}$?

A. $[4, 6)$

B. $(-5, 5)$

C. $[-1, 1)$

D. $(-\infty, \infty)$

E. $\{5\}$

10. (5 points) Find the first 4 terms of the Taylor series for $f(x) = \sin(-2x)$ centered at 0.

A. $0 + \frac{2}{1!}x + 0 - \frac{8}{3!}x^3$

B. $0 - \frac{2}{1!}x - 0 + \frac{8}{3!}x^3$

C. $0 + \frac{2}{1!}x + 0 + \frac{2}{3!}x^3$

D. $0 - \frac{1}{1!}x + 0 + \frac{1}{2!}x^4$

E. $0 + \frac{2}{2!}x^2 + 0 + \frac{8}{4!}x^4$

Free Response Questions

11. (a) (4 points) If $\dfrac{1}{(n+5)(n+6)} = \dfrac{A}{n+5} + \dfrac{B}{n+6}$ find A and B.

Solution:

$$
\begin{aligned}
1 &= A(n+6) + B(n+5) \\
1 &= A(-5+6) & A &= 1 \\
1 &= A(-6+6) + B(-6+5) & B &= -1
\end{aligned}
$$

(b) (4 points) Use part (a) to find a simpler expression for

$$\sum_{n=1}^{k} \frac{1}{(n+5)(n+6)}$$

Solution:

$$\sum_{n=1}^{k} \frac{1}{(n+5)(n+6)} = \sum_{n=1}^{k} \left(\frac{1}{n+5} - \frac{1}{n+6} \right) = \frac{1}{1+5} - \frac{1}{k+6}$$

(c) (2 points) What is the sum of the series

$$\sum_{n=1}^{\infty} \frac{1}{(n+5)(n+6)}?$$

Solution:

$$\lim_{k \to \infty} \frac{1}{1+5} - \frac{1}{k+6} = \frac{1}{6}$$

12. Are the series below absolutely convergent, conditionally convergent, or divergent?

(a) (5 points)

$$\sum_{n=1}^{\infty} \frac{n^2}{5n^2+4}$$

Solution: Divergence Test:

$$\lim_{n\to\infty} a_n = \frac{1}{5} \neq 0$$

Thus, the series $\displaystyle\sum_{n=1}^{\infty} \frac{n^2}{5n^2+4}$ diverges.

(b) (5 points)

$$\sum_{n=1}^{\infty} \frac{4+3n}{(1+n^2)^2}$$

Solution: Limit comparison test:

$$\lim_{n\to\infty} \frac{\frac{4+3n}{(1+n^2)^2}}{\frac{1}{n^3}} = \lim_{n\to\infty} \frac{4+3n}{(1+n^2)^2} n^3 = \lim_{n\to\infty} \frac{4n^3+3n^4}{(1+n^2)^2}$$
$$= \lim_{n\to\infty} \frac{4n^3+3n^4}{1+2n^2+n^4} = \lim_{n\to\infty} \frac{\frac{4}{n}+3}{\frac{1}{n^4}+\frac{2}{n^2}+1} = 3$$

Since $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^3}$ converges, the original series converges. And since the terms of the original series are all positive, it is absolutely convergent.

13. Are the series below absolutely convergent, conditionally convergent, or divergent?

(a) (5 points) $\displaystyle\sum_{n=1}^{\infty}\left(\frac{5-n}{3+2n}\right)^n$

Solution: Root test:

$$\sqrt[n]{\left|\left(\frac{5-n}{3+2n}\right)^n\right|} = \frac{|5-n|}{3+2n} = \frac{|\frac{n}{n}-1|}{\frac{3}{n}+2} \mapsto \frac{1}{2}$$

Since $\frac{1}{2} < 1$, the series absolutely converges.

(b) (5 points) $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^{n-1}n^3}{n^4+2}$

Solution: Alternating Series Test:

$$b_n = \frac{n^3}{n^4+2} \text{ then } \lim_{n\to\infty} b_n = 0$$

and $b_{n+1} \le b_n$. Thus, the series converges. To show that the series is not absolutely convergent, we use the Limit comparison test:

$$\begin{aligned}\lim_{n\to\infty}\frac{\frac{n^3}{n^4+2}}{\frac{1}{n}} &= \lim_{n\to\infty}\frac{n^4}{n^4+2}\\ &= \lim_{n\to\infty}\frac{1}{1+\frac{2}{n^4}} = 1 > 0\end{aligned}$$

Since $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n}$ diverges, the original series is not absolutely convergent. Thus, the original series is conditionally convergent.

14. (a) (5 points) What is the radius of convergence of the power series $\sum_{n=1}^{\infty} \frac{n}{9^n} x^n$? What is the interval of convergence?

Solution:

$$\lim_{n\to\infty} \left| \frac{\frac{n+1}{9^{n+1}} x^{n+1}}{\frac{n}{9^n} x^n} \right| = \lim_{n\to\infty} \left| \frac{n+1}{9^{n+1}} x^{n+1} \frac{9^n}{n x^n} \right| = \lim_{n\to\infty} \left| \frac{9^n}{9^{n+1}} \frac{x^{n+1}}{x^n} \frac{n+1}{n} \right|$$
$$= \left| \frac{x}{9} \right| \lim_{n\to\infty} \frac{n+1}{n} = \left| \frac{x}{9} \right|$$

The radius of convergence is 9.

Since $\lim_{n\to\infty} n \neq 0$, $\sum_{n=1}^{\infty} \frac{n}{9^n} 9^n = \sum_{n=1}^{\infty} n$ and $\sum_{n=1}^{\infty} \frac{n}{9^n} (-9)^n = \sum_{n=1}^{\infty} (-1)^n n$ both diverge and the interval of convergence is $(-9, 9)$.

(b) (5 points) What is the radius of convergence of the power series $\sum_{n=1}^{\infty} n!(x-5)^n$? What is the interval of convergence?

Solution:

$$\lim_{n\to\infty} \left| \frac{(n+1)!(x-5)^{n+1}}{n!(x-5)^n} \right| = \lim_{n\to\infty} |(n+1)(x-5)| = |(x-5)| \lim_{n\to\infty} (n+1)$$

The radius of convergence is 0 and the interval of convergence is $\{5\}$.

15. (a) (3 points) Find the first 6 terms of the Taylor series for the function $f(x) = \ln(x+1)$ centered at 0.

Solution:

$$f(x) = \ln(x+1) \qquad f'(x) = (x+1)^{-1} \qquad f''(x) = -(x-1)^{-2}$$
$$f'''(x) = 2(x-1)^{-3} \qquad f^{(iv)}(x) = -1(3!)(x+1)^{-4}$$
$$f^{(v)}(x) = (4)!(x+1)^5$$

So the first 6 terms of the Taylor series is

$$T_5 = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5}$$

(b) (3 points) Find the Taylor series centered at 0 for the function $f(x) = \ln(x+1)$.

Solution:

$$f^{(n)}(x) = (-1)^{n-1}(n-1)!(x+1)^{-n} \text{ and } f^{(n)}(0) = (n-1)!(-1)^{n-1}$$

So the Taylor series is $f(x) = \ln(x+1) = \sum_{n=0}^{\infty} (-1)^{n-1} \frac{x^n}{n}$.

(c) (4 points) Use your answer from part (b) to calculate the sum of $\sum_{n=0}^{\infty} (-1)^{n-1} \frac{1}{n}$

Solution: When $x = 1$, we have

$$\sum_{n=0}^{\infty} (-1)^{n-1} \frac{1}{n} = \ln(2).$$