Exam 2

Name: ______________________________ Section: ______________

Do not remove this answer page — you will return the whole exam. You will be allowed two hours to complete this test. If you find you need scratch paper during the exam, please ask. You may not use any of your own notes, paper or anything else not provided. You may use a graphing calculator during the exam, but NO calculator with a Computer Algebra System (CAS). Absolutely no communication device use during the exam is allowed.

The exam consists of 10 multiple choice questions and 5 free response questions. Record your answers to the multiple choice questions on this page by filling in the circle corresponding to the correct answer.

Show all work to receive full credit on the free response problems. It will also help you check your answers to show work on multiple choice problems.

Multiple Choice Questions

1 Ⓐ Ⓑ Ⓒ Ⓓ Ⓔ **6** Ⓐ Ⓑ Ⓒ Ⓓ Ⓔ

2 Ⓐ Ⓑ Ⓒ Ⓓ Ⓔ **7** Ⓐ Ⓑ Ⓒ Ⓓ Ⓔ

3 Ⓐ Ⓑ Ⓒ Ⓓ Ⓔ **8** Ⓐ Ⓑ Ⓒ Ⓓ Ⓔ

4 Ⓐ Ⓑ Ⓒ Ⓓ Ⓔ **9** Ⓐ Ⓑ Ⓒ Ⓓ Ⓔ

5 Ⓐ Ⓑ Ⓒ Ⓓ Ⓔ **10** Ⓐ Ⓑ Ⓒ Ⓓ Ⓔ

Multiple Choice	11	12	13	14	15	Total Score
50	10	10	10	10	10	100

Multiple Choice Questions

1. (5 points) Give the first four terms of the sequence $\{a_1, a_2, a_3, a_4\}$ defined by

$$a_n = \frac{3 \cdot 10^n}{n!}.$$

A. $\{30, 150, 1000, 7500\}$
B. $\{30, 30, 15, 5\}$
C. $\{30, 150, 500, 1250\}$
D. $\{30, 30, 30, 30\}$
E. $\{30, 450, 4500, 33750\}$

2. (5 points) Find the limit of the **sequence** $\{a_1, a_2, a_3, \ldots\}$ defined by

$$a_n = \frac{9n}{15 - 2n}.$$

A. $-\frac{9}{2}$
B. 0
C. $\frac{9}{13}$
D. $\frac{9}{2}$
E. $\frac{3}{5}$

3. (5 points) Find the sum of the geometric series

$$-1 + \frac{-1}{3} + \frac{-1}{9} + \frac{-1}{27} + \cdots$$

A. $-\frac{3}{4}$
B. $-\frac{3}{2}$
C. $\frac{3}{4}$
D. $\frac{3}{2}$
E. $-\frac{4}{3}$

4. (5 points) Does the series $\sum_{n=2}^{\infty} \frac{n+5}{n^4-1}$ converge or diverge?

   A. Converges by the alternating series test.

   B. Converges because $\lim_{n\to\infty} \frac{n+5}{n^4-1} = 0$.

   C. Diverges by the comparison test to $\sum_{n=2}^{\infty} \frac{1}{n^3}$.

   D. Diverges by the ratio test since $\lim_{n\to\infty} \left| \frac{n+6}{(n+1)^4-1} \cdot \frac{n^4-1}{n+5} \right| = 1$.

   E. Converges by the limit comparison test to $\sum_{n=2}^{\infty} \frac{1}{n^3}$.

5. (5 points) What should we compare $\sum_{n=1}^{\infty} \frac{n}{\sqrt{n^3+1}}$ to for a conclusive limit comparison test?

   A. $\sum_{n=1}^{\infty} \frac{1}{n^{\frac{3}{2}}}$

   B. $\sum_{n=1}^{\infty} \frac{1}{n^{\frac{1}{3}}}$

   C. $\sum_{n=1}^{\infty} \frac{1}{n^{\frac{1}{2}}}$

   D. $\sum_{n=1}^{\infty} \frac{1}{n^2}$

   E. $\sum_{n=1}^{\infty} \frac{1}{n^3}$

6. (5 points) Apply the ratio test to decide if the series $\sum_{n=1}^{\infty} \frac{n}{(2n+1)!}$ converges or diverges.

   A. Converges because $\lim_{n\to\infty} \frac{n+1}{n(2n+3)(2n+2)} = 0$.

   B. Converges because $\lim_{n\to\infty} \frac{n+1}{2n+3} = \frac{1}{2}$.

   C. Diverges because $\lim_{n\to\infty} \frac{2n+1}{n} = 2$.

   D. Diverges because $\lim_{n\to\infty} \frac{(n+1)(2n+1)}{2n+3} = \infty$.

   E. The ratio test is inconclusive for this series.

7. (5 points) Find the smallest value of N so that S_N approximates $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2+1}$ to within an error of at most .03.

   A. $N = 3$

   B. $N = 5$

   C. $N = 7$

   D. $N = 9$

   E. $N = 11$

8. (5 points) Find the interval of convergence of the power series $\displaystyle\sum_{n=1}^{\infty} \frac{3^n}{n!}(x-2)^n$.

   A. $\{2\}$

   B. $[-3, 3]$

   C. $[-1, 6]$

   D. $[\frac{5}{3}, \frac{7}{3}]$

   E. $(-\infty, \infty)$

9. (5 points) Which power series represents the function $f(x) = x^2 e^{3x}$ on the interval $(-\infty, \infty)$?

   A. $\displaystyle\sum_{n=0}^{\infty} \frac{(-3)^n x^n}{n!}$

   B. $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n (x^2)^{3n}}{(3n)!}$

   C. $\displaystyle\sum_{n=0}^{\infty} \frac{3^n x^{n+2}}{n!}$

   D. $\displaystyle\sum_{n=0}^{\infty} \frac{3^n x^2}{n+1}$

   E. $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n x^{3n+2}}{n!}$

10. (5 points) If the Taylor series for $f(x) = \ln(x)$ centered at $a = 5$ is given by $\displaystyle\sum_{n=0}^{\infty} c_n (x-5)^n$, find the coefficients $\{c_0, c_1, c_2, c_3\}$.

    A. $\{\ln(5), \frac{1}{5}, \frac{1}{25}, \frac{1}{125}\}$

    B. $\{\ln(5), -\frac{1}{5}, \frac{1}{10}, -\frac{1}{30}\}$

    C. $\{\ln(5), \frac{1}{5}, -\frac{1}{50}, \frac{1}{375}\}$

    D. $\{\ln(5), \frac{1}{5}, \frac{1}{100}, \frac{1}{500}\}$

    E. $\{1, 1, \frac{1}{2}, \frac{1}{6}\}$

Free Response Questions

11. Find the sum of each series. Clearly show appropriate steps to justify your answer.

(a) (5 points) $\displaystyle\sum_{n=1}^{\infty} \frac{2^n + 3^n}{5^n}$

Solution: First split the fraction; then use the formula for the sum of a geometric series on each term:

$$\sum_{n=1}^{\infty} \frac{2^n + 3^n}{5^n} = \sum_{n=1}^{\infty} \left(\frac{2}{5}\right)^n + \sum_{n=1}^{\infty} \left(\frac{3}{5}\right)^n = \frac{2/5}{1 - 2/5} + \frac{3/5}{1 - 3/5} = \frac{13}{6}$$

(b) (5 points) $\displaystyle\sum_{n=2}^{\infty} \left(\frac{1}{(n+3)^2} - \frac{1}{(n+4)^2}\right)$

Solution: Write out the first few partial sums until you see the pattern: $S_1 = \frac{1}{5^2} - \frac{1}{6^2}$

$S_2 = \left(\frac{1}{5^2} - \frac{1}{6^2}\right) + \left(\frac{1}{6^2} - \frac{1}{7^2}\right)$

$S_3 = \left(\frac{1}{5^2} - \frac{1}{6^2}\right) + \left(\frac{1}{6^2} - \frac{1}{7^2}\right) + \left(\frac{1}{7^2} - \frac{1}{8^2}\right)$

$\vdots$

$S_n = \frac{1}{5^2} - \frac{1}{(n+5)^2}$

Then $S = \lim_{n\to\infty} S_n = \lim_{n\to\infty} \frac{1}{5^2} - \frac{1}{(n+5)^2} = \frac{1}{5^2}$.

12. Decide if the series converges or diverges. Clearly state which test(s) are used, and show all steps.

(a) (5 points) $\displaystyle\sum_{n=2}^{\infty} \frac{(n^2+5)}{(n+3)(4n-1)}$

Solution:
Consider

$$\lim_{n\to\infty} \frac{(n^2+5)}{(n+3)(4n-1)} = \lim_{n\to\infty} \frac{n^2}{4n^2} = \frac{1}{4} \neq 0.$$

Hence the series diverges by the divergence test.

(b) (5 points) $\displaystyle\sum_{n=1}^{\infty} \frac{3^n}{n!}$

Solution:
Apply the Ratio Test:

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{3^{n+1}}{(n+1)!}\frac{n!}{3^n}\right| = \lim_{n\to\infty} \left|\frac{3}{n+1}\right| = 0$$

Since $0 < 1$ the series converges by the ratio test.

13. (10 points) Are the series absolutely convergent, conditionally convergent, or divergent? Clearly state which test(s) are used, and show all steps.

(a) (5 points) $\displaystyle\sum_{n=1}^{\infty} \frac{\cos(n)}{n^3+1}$

Solution: We have
$$\left|\frac{\cos(n)}{n^3+1}\right| \leq \frac{1}{n^3+1} < \frac{1}{n^3}$$
Note that the series $\sum \frac{1}{n^3}$ converges, as it is a p-series with $p = 3 > 1$. Then we conclude that $\sum \left|\frac{\cos(n)}{n^3+1}\right|$ converges by the comparison test to $\sum \frac{1}{n^3}$. Thus, $\sum_{n=1}^{\infty} \frac{\cos(n)}{n^3+1}$ converges absolutely.

(b) (5 points) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$

Solution: This series converges by the alternating series test: it alternates, and $\{\frac{1}{\sqrt{n}}\}$ decreases to 0.

On the other hand, $\sum \left|\frac{(-1)^n}{\sqrt{n}}\right| = \sum \frac{1}{\sqrt{n}}$ diverges, as it is a p-series with $p = \frac{1}{2} \leq 1$. Thus, $\sum \frac{(-1)^n}{\sqrt{n}}$ converges conditionally.

14. Find the interval of convergence for the series. Hint: show clearly where you test the endpoints of your interval

$$\sum_{n=0}^{\infty} 5^n (x-3)^n$$

Solution: Use the ratio test

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{5^{n+1}(x-3)^{n+1}}{5^n(x-3)^n}\right| = \lim_{n\to\infty}|5(x-3)| = |5(x-3)|$$

Then $-1 < 5(x-3) < 1$ implies $\frac{14}{5} < x < \frac{16}{5}$. To test the endpoints we do the following:

If $x = \frac{14}{5}$, then the series becomes $\sum 5^n \left(\frac{14}{5} - 3\right)^n = \sum 5^n \left(\frac{-1}{5}\right)^n = \sum (-1)^n$ which diverges by the divergence test.

If $x = \frac{16}{5}$, then the series becomes $\sum 5^n \left(\frac{16}{5} - 3\right)^n = \sum 5^n \left(\frac{1}{5}\right)^n = \sum 1$ which also diverges by the divergence test.

Then the interval of convergence is $\left(\frac{14}{5}, \frac{16}{5}\right)$.

15. (a) (4 points) Write a Taylor series centered at $a = 0$ (i.e., a Maclaurin series) for $f(x) = \dfrac{x^3}{1+2x^2}$.

Solution:

$$x^3 \frac{1}{1-(-2x^2)} = x^3 \sum_{n=0}^{\infty} (-2x^2)^n = x^3 \sum_{n=0}^{\infty} (-2)^n x^{2n} = \sum_{n=0}^{\infty} (-2)^n x^{2n+3}$$

(b) (6 points) Write a Taylor series centered at $a = 0$ (i.e., a Maclaurin series) for $f(x) = \ln(1-3x^9)$ and state the Interval of Convergence.

Solution:
$\ln(1-3x^9) = \ln(1+(-3x^9)) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}(-3x^9)^n}{n} = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}(-3)^n x^{9n}}{n}$
Referring to the template we see that the series converges for $-1 < -3x^9 \leq 1$, so dividing by -3 and taking 9-th root we obtain $\sqrt[9]{1/3} > x \geq \sqrt[9]{-1/3}$. Thus, the interval of convergence is $\left[-\sqrt[9]{1/3}, \sqrt[9]{1/3}\right)$.