Exam 2

Name: ______________________________ Section: ______________

Do not remove this answer page — you will return the whole exam. You will be allowed two hours to complete this test. If you find you need scratch paper during the exam, please ask. You may not use any of your own notes, paper or anything else not provided. You may use a graphing calculator during the exam, but NO calculator with a Computer Algebra System (CAS). Absolutely no communication device use during the exam is allowed.

The exam consists of 10 multiple choice questions and 5 free response questions. Record your answers to the multiple choice questions on this page by filling in the circle corresponding to the correct answer.

Show all work to receive full credit on the free response problems. It will also help you check your answers to show work on multiple choice problems.

Multiple Choice Questions

1 (A) (B) (C) (D) (E) **6** (A) (B) (C) (D) (E)

2 (A) (B) (C) (D) (E) **7** (A) (B) (C) (D) (E)

3 (A) (B) (C) (D) (E) **8** (A) (B) (C) (D) (E)

4 (A) (B) (C) (D) (E) **9** (A) (B) (C) (D) (E)

5 (A) (B) (C) (D) (E) **10** (A) (B) (C) (D) (E)

Multiple Choice	11	12	13	14	15	Total Score
50	10	10	10	10	10	100

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Multiple Choice Questions

1. (5 points) Determine the first four terms $\{a_1, a_2, a_3, a_4\}$ of the sequence defined by

$$a_n = \frac{2n+1}{n!}.$$

A. $\left\{3, \frac{5}{2}, \frac{7}{6}, \frac{9}{24}\right\}$

B. $\left\{0, \frac{5}{1}, \frac{7}{2}, \frac{9}{3}\right\}$

C. $\left\{1, 3, \frac{5}{2}, \frac{7}{6}\right\}$

D. $\left\{3, \frac{5}{2}, \frac{7}{3}, \frac{9}{4}\right\}$

E. $\left\{1, \frac{3}{2}, \frac{5}{6}, \frac{7}{24}\right\}$

2. (5 points) Calculate the value of the series:

$$\sum_{n=0}^{\infty} \left(\frac{1}{n+1} - \frac{1}{n+2} \right)$$

A. 2.5

B. 1.5

C. 0

D. 1

E. The series diverges.

3. (5 points) For the given a_n, determine whether the **sequence** $\{a_n\}$ and the corresponding **series** $\sum_{n=1}^{\infty} a_n$ is convergent or divergent.

$$a_n = \frac{3n+2}{5n-1}$$

A. The sequence $\{a_n\}$ diverges and the series $\sum a_n$ diverges.

B. The sequence $\{a_n\}$ converges to 0 and the series $\sum a_n$ converges to $\frac{3}{5}$.

C. The sequence $\{a_n\}$ converges to $\frac{3}{5}$ and the series $\sum a_n$ diverges.

D. The sequence $\{a_n\}$ converges to $\frac{3}{5}$ and the series $\sum a_n$ converges to $\frac{3}{5}$.

E. The sequence $\{a_n\}$ diverges and the series $\sum a_n$ converges to 0.

4. (5 points) Which one of the following series **converges**? There is only one correct answer.

A. $\sum_{n=1}^{\infty} \frac{1}{\sqrt[3]{n}}$

B. $\sum_{n=1}^{\infty} \frac{3n-4}{2n+7}$

C. $\sum_{n=1}^{\infty} \frac{n}{\cos(n\pi)}$

D. $\sum_{n=1}^{\infty} \frac{3^{n+2}}{5^n}$

E. None of the series converge.

5. (5 points) Determine the value of the series: $\displaystyle\sum_{n=2}^{\infty} \frac{1}{2^{n+2}}$

A. $\frac{1}{8}$

B. $\frac{1}{16}$

C. $\frac{1}{4}$

D. $\frac{1}{2}$

E. The series diverges.

6. (5 points) Which test should be used to determine the convergence of the series $\displaystyle\sum_{n=1}^{\infty} \frac{2^n}{n^2 \cdot n!}$?

A. Geometric Series Test

B. Alternating Series Test

C. Ratio Test

D. p-Series Test

E. Telescoping Series Test

7. (5 points) Does the series $\sum_{n=0}^{\infty} \frac{2^n - 1}{4^n}$ converge or diverge?

   A. It converges by the geometric series test since $\frac{2^n - 1}{4^n} = -\frac{1}{2^n}$

   B. It converges by the direct comparison test with $\sum_{n=0}^{\infty} \frac{1}{4^n}$.

   C. It diverges by the divergence test since $\lim_{n\to\infty} \frac{2^n - 1}{4^n} = \frac{1}{2}$.

   D. It converges by the geometric series test since $\frac{2^n - 1}{4^n} = \frac{2^n}{4^n} - \frac{1}{4^n}$

   E. It converges by the root test since $\lim_{n\to\infty} \sqrt[n]{\frac{2^n - 1}{4^n}} = \frac{1}{4}$.

8. (5 points) Which should we compare $\sum_{n=1}^{\infty} \frac{n^2 + 2n - 4}{\sqrt{n^6 - 7n^4 - 12}}$ to for a conclusive limit comparison test?

   A. $\sum_{n=1}^{\infty} \frac{1}{n^2}$

   B. $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$

   C. $\sum_{n=1}^{\infty} \frac{1}{n^3}$

   D. $\sum_{n=1}^{\infty} \frac{1}{n}$

   E. $\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$

9. (5 points) Find the first four nonzero terms of the Taylor polynomial of $f(x) = \ln(2x)$ centered at $x = 1$.

A. $(x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3 - \frac{1}{4}(x-1)^4$

B. $\ln(2) + (x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3$

C. $\ln(2) + (2x-1) - \frac{1}{2}(2x-1)^2 + \frac{1}{3}(2x-1)^3$

D. $(2x-1) - \frac{1}{2}(2x-1)^2 + \frac{1}{3}(2x-1)^3 - \frac{1}{4}(2x-1)^4$

E. $2x \cdot (x-1) - \frac{2x}{2}(x-1)^2 + \frac{2x}{3}(x-1)^3 - \frac{2x}{4}(x-1)^4$

10. (5 points) The power series $\displaystyle\sum_{n=0}^{\infty} \frac{2^n\ x^{n+3}}{n!}$ represents which function?

A. $2x^3 e^x$

B. $x^2 \cos(\frac{1}{2}x)$

C. $x^3 e^{2x}$

D. $2x^2 \cos(x)$

E. e^{2x+3}

Free Response Questions

11. Determine whether the following series converge or diverge. Clearly state your answer and justify it by showing your work and citing any tests used.

(a) (5 points) $\displaystyle\sum_{n=1}^{\infty} \frac{\cos^2(n)}{n^4}$

Solution: $\cos^2(n) \leq 1 \implies \dfrac{\cos^2(n)}{n^4} \leq \dfrac{1}{n^4}$

Since $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^4}$ converges by the p-Series Test, $\displaystyle\sum_{n=1}^{\infty} \frac{\cos^2(n)}{n^4}$ converges by the Direct Comparison Test.

(b) (5 points) $\displaystyle\sum_{n=1}^{\infty} \frac{4n^7}{12n^9}$

Solution: $\displaystyle\sum_{n=1}^{\infty} \frac{4n^7}{12n^9} = \sum_{n=1}^{\infty} \frac{1}{3n^2}$, which converges by the p-Series Test.

12. Determine the value of the following series. If the series diverges, write "Diverges." Clearly state your answer and justify it by showing your work and citing any tests used.

(a) (5 points) $\displaystyle\sum_{n=1}^{\infty} \frac{2^n + 5^n}{6^n}$.

Solution: By the Geometric Series Test,

$$\sum_{n=1}^{\infty} \frac{2^n + 5^n}{6^n} = \sum_{n=1}^{\infty} \left(\frac{2}{6}\right)^n + \sum_{n=1}^{\infty} \left(\frac{5}{6}\right)^n$$
$$= \frac{(2/6)}{1-(2/6)} + \frac{(5/6)}{1-(5/6)} = \frac{11}{2}$$

(b) (5 points) $\displaystyle\sum_{n=1}^{\infty} \ln\left(\frac{n+1}{n+2}\right)$. (Hint: $\ln\left(\frac{n+1}{n+2}\right) = \ln(n+1) - \ln(n+2)$.)

Solution: $\displaystyle\sum_{n=1}^{\infty} \ln\left(\frac{n+1}{n+2}\right) = \sum_{n=1}^{\infty} \ln(n+1) - \ln(n+2)$.

$S_3 = (\ln(2) - \ln(3)) + (\ln(3) - \ln(4)) + (\ln(4) - \ln(5)) = \ln(2) - \ln(5)$

$S_4 = S_3 + (\ln(5) - \ln(6)) = \ln(2) - \ln(6)$

$\vdots$

$S_n = \ln(2) - \ln(n+2)$

Since $\displaystyle\sum_{n=1}^{\infty} \ln\left(\frac{n+1}{n+2}\right) = \lim_{n\to\infty} S_n = -\infty$, the series diverges.

13. (10 points) Determine if the following series converges absolutely, converges conditionally, or diverges. Justify your answer by showing your work and citing any tests used.

$$\sum_{n=1}^{\infty} \frac{(-1)^n\, n^2}{4n^3+1}$$

Solution: Begin by determining the convergence of the absolute value of the series: $\displaystyle\sum_{n=1}^{\infty} \left|\frac{(-1)^n\, n^2}{4n^3+1}\right| = \sum_{n=1}^{\infty} \frac{n^2}{4n^3+1}$.

Limit compare with $\displaystyle\sum_{n=1}^{\infty} \frac{1}{4n}$, which diverges by the p-Series Test.

$$\lim_{n\to\infty} \left(\frac{n^2}{4n^3+1}\right)\left(\frac{4n}{1}\right) = 1$$

Therefore, $\displaystyle\sum_{n=1}^{\infty} \left|\frac{(-1)^n\, n^2}{4n^3+1}\right|$ diverges by the Limit Comparison Test.

Now, determine the convergence of $\sum_{n=1}^{\infty} \frac{(-1)^n\, n^2}{4n^3+1}$. We'll use the Alternating Series Test.

- $\displaystyle\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{n^2}{4n^3+1} = 0$

- $\displaystyle(a_n)' = \frac{2n-4n^4}{(4n^3+1)^2} < 0 \implies 2n - 4n^3 < 0 \implies n > 2^{1/3} \approx 1.256.$

  So, by the first derivative test, the sequence is decreasing for all $n > 2$.

  (Note that any valid argument that a_n is decreasing may be sufficient here.)

Therefore, $\sum_{n=1}^{\infty} \frac{(-1)^n\, n^2}{4n^3+1}$ converges by the Alternating Series Test. So $\sum_{n=1}^{\infty} \frac{(-1)^n\, n^2}{4n^3+1}$ is conditionally convergent.

14. (10 points) Find the interval of convergence for the series. Be sure to clearly show where you test the endpoints of your interval and cite any tests used.

$$\sum_{n=0}^{\infty} \frac{(-1)^n \cdot n \cdot x^{2n}}{4^n}$$

Solution: By the Root Test,

$$\lim_{n\to\infty} \sqrt[n]{\left|\frac{(-1)^n\ n\ x^{2n}}{4^n}\right|} = \lim_{n\to\infty} \frac{\sqrt[n]{n}\ x^2}{4} = \frac{x^2}{4} < 1 \implies -2 < x < 2.$$

Testing the endpoints gives:

$\underline{x = -2:}$ $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n \cdot n \cdot (-2)^{2n}}{4^n} = \sum_{n=0}^{\infty} (-1)^n\ n,$

which diverges by the Divergence Test.

$\underline{x = 2}$ $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n \cdot n \cdot (2)^{2n}}{4^n} = \sum_{n=0}^{\infty} (-1)^n\ n,$ which diverges by the Divergence Test.

Therefore, the interval of convergene is $(-2, 2)$.

15. (a) (5 points) Compute the first **four** *nonzero* terms of the Taylor series for $f(x) = \sqrt{x}$ centered at $x = 1$.

Solution:

$$f(x) = \sqrt{x}, \qquad f'(x) = \frac{1}{2}x^{-1/2}, \qquad f''(x) = -\frac{1}{4}x^{-3/2}, \qquad f'''(x) = \frac{3}{8}x^{-5/2}$$

So, using $c_n = \dfrac{f^{(n)}(a)}{n!}$, we have:

$$c_0 = 1, \qquad c_1 = \frac{1}{2}, \qquad c_2 = -\frac{1}{8}, \qquad c_3 = \frac{1}{16}$$

Therefore, $p_4 = 1 + \frac{1}{2}(x-1) - \frac{1}{8}(x-1)^2 + \frac{1}{16}(x-1)^3$.

(b) (5 points) Using your answer from part (a), determine the first **three** *nonzero* terms of the Taylor Series for $f(x) = \dfrac{1}{\sqrt{x}}$ centered at $x = 1$.
(Hint: Recall that $\frac{d}{dx}\sqrt{x} = \frac{1}{2\sqrt{x}}$.)

Solution: Taking the derivative of the above answer,

$$\frac{1}{2\sqrt{x}} = \frac{1}{2} - \frac{2}{8}(x-1) + \frac{3}{16}(x-1)^2 + \ldots$$

Multiply both sides by 2:

$$\frac{1}{\sqrt{x}} = 1 - \frac{1}{2}(x-1) + \frac{3}{8}(x-1)^2 + \ldots$$