
Problem 1.
5. **(5 points)** `local/rmb-problems/integral-test-num.pg`

Give the whole number N for which we have $\int_{N+1}^{\infty} \frac{1}{x^2}\,dx \le \sum_{k=6}^{\infty} \frac{1}{k^2} \le \int_{N}^{\infty} \frac{1}{x^2}\,dx.$

$N =$ ______________ .

Evaluate one of the integrals above to find A so that $\sum_{k=6}^{\infty} \frac{1}{k^2} \le A$

$A =$ ______________.

The answers must be correctly rounded to four decimal places or more accurate.
Solution: *(Instructor solution preview: show the student solution after due date.)*

SOLUTION
For a decreasing function, we have $\int_{j}^{j+1} f(x)\,dx \le f(j) \le \int_{j-1}^{j} f(x)\,dx$. Summing these inequalities for $j = 6$ to some large integer M and taking a limit as M approaches ∞ gives

$$\int_{6}^{\infty} f(x)\,dx \le \sum_{j=6}^{\infty} f(j) \le \int_{6-1}^{\infty} f(x)\,dx$$

Thus we can see that we want $N = 5$.
Evaluating the improper integral gives

$$\int_{5}^{\infty} \frac{1}{x^2}\,dx = \frac{1}{5}.$$

Correct Answers:

- `5`
- `0.2`

Problem 2.
7. **(5 points)** `local/rmb-problems/power-series-num.pg`

Consider the function

$$F(x) = \int_{0}^{x} \frac{1}{1+t^3}\,dt.$$

The first three (non-zero) terms of the Mclaurin series for F are $Ax + Bx^4 + Cx^7$. Give the values of A, B, and C.
$A =$ ______________, $B =$ ______________, $C =$ ______________

Your answers must be correctly rounded to four decimal places, or more accurate. Exact answers are preferred.
Solution: *(Instructor solution preview: show the student solution after due date.)*

SOLUTION

We use the geometric series to write $\frac{1}{1+x^3} = \sum_{j=0}^{\infty} (-1)^j x^{3j}$. We integrate this expression term-by-term to find

$$F(x) = \sum_{j=0}^{\infty} \frac{(-1)^j x^{3j+1}}{3j+1}.$$

Writing out the first three terms of the series gives

$$x - \frac{x^4}{4} + \frac{x^7}{7}.$$

Correct Answers:

- `1`
- `-0.25`
- `0.142857`

Problem 3.
1. **(5 points)** `local/rmb-problems/geom-series2-num.pg`

Find the sum of the geometric series $\sum_{k=3}^{\infty} 11 \cdot 3^{-k} =$ __________
Your answer should be correctly rounded to four decimal places, or more accurate.

Solution: *(Instructor solution preview: show the student solution after due date.)*

SOLUTION
The sum of a geometric series is $\frac{a}{1-r}$ where a is the first term and r is the ratio.
For the series in this problem we write a few terms

$$\sum_{k=3}^{\infty} 11 \cdot 3^{-k} = 11 \cdot 3^{-3} + 11 \cdot 3^{-4} + 11 \cdot 3^{-5} + \ldots$$

The first term $a = 11 \cdot 3^{-3}$ and the ratio is $r = 1/3$. This gives the sum of the series is

$$\sum_{k=3}^{\infty} 11 \cdot 3^{-k} = \frac{11 \cdot 3^{-3}}{1 - 1/3}$$

As a decimal, the answer is 0.611111.
Correct Answers:

- `0.611111`

Problem 4.
3. **(5 points)** `local/rmb-problems/alt-series-div-mc.pg`

Select the one correct statement for the series $\sum_{k=1}^{\infty} \frac{(-1)^k k}{4k+7}$?

- A. The series is absolutely convergent.
- B. The series is conditionally convergent.
- C. The series is absolutely convergent, but not convergent.
- D. The series is divergent.
- E. None of the above

Solution: *(Instructor solution preview: show the student solution after due date.)*

SOLUTION
Since $\lim_{k\to\infty} \frac{(-1)^k k}{4k+7} = \frac{1}{4}$ does not exist, the series is divergent.
Correct Answers:

- D

Problem 5.
2. **(5 points)** `local/rmb-problems/telescope-num.pg`

2. **(5 points)** `local/rmb-problems/telescope-num.pg`

Find the value of the finite sum $\sum_{k=5}^{25} \left(\frac{7}{k} - \frac{7}{k+1} \right) =$ __________
Your answer should be correctly rounded to four decimal places, or more accurate.

Solution: *(Instructor solution preview: show the student solution after due date.)*

SOLUTION
This is a telescoping series. If we write out a few terms, we see that second fraction in a term cancels with the first fraction in the next term. After canceling we are left with the first fraction from the first term and second fraction from the last term.

$$\sum_{k=5}^{25} \left(\frac{7}{k} - \frac{7}{k+1} \right) = \frac{7}{5} - \frac{7}{5+1} + \frac{7}{5+1} - \frac{7}{5+2} + \cdots + \frac{7}{25-1} - \frac{7}{25}\frac{7}{25} - \frac{7}{25+1}$$

After canceling we are left with the first fraction from the first term and second fraction from the last term which gives the value

$$\frac{7}{5} - \frac{7}{25+1}$$

We recommend you enter the exact answer, but you may also round to obtain the value for the answer, 1.13077.
Correct Answers:

- 1.13077

Problem 6.
4. **(5 points)** `local/rmb-problems/limit-comp-mc.pg`

Consider the series $\sum_{j=1}^{\infty} \frac{1}{3^j + 17j + 2}$ and select the correct statement.

- A. Using the comparison test to compare with the series $\sum_{j=1}^{\infty} \frac{1}{3^j}$ establishes of the divergence of the series.
- B. Using the comparison test to compare with the series $\sum_{j=1}^{\infty} \frac{1}{3^j}$ establishes convergence of the series.
- C. The series converges conditionally, but we cannot use a comparison test to establish conditional convergence.
- D. Using the comparison test to compare with the series $\sum_{j=1}^{\infty} \frac{1}{17j}$ establishes divergence of the series.

- E. Using the comparison test to compare with the series $\sum_{j=1}^{\infty} \frac{1}{17j}$ establishes convergence of the series.

Correct Answers:

- `B`

Problem 7.

6. **(5 points)** `local/rmb-problems/ratio-test-num.pg`

Consider the power series $\sum_{k=1}^{\infty} \frac{5^k x^k}{k!}$ with terms $a_k = \frac{5^k x^k}{k!}$.

Find the limit of the ratio $\lim_{k\to\infty} \frac{a_{k+1}}{a_k}$ = ____________________.

Give the radius of convergence of the series $\sum_{k=1}^{\infty} \frac{5^k x^k}{k!}$.

- A. ∞
- B. $1/5$
- C. 0
- D. 1
- E. 5

Solution: *(Instructor solution preview: show the student solution after due date.)*

SOLUTION

We simplify the ratio $\frac{a_{k+1}}{a_k} = \frac{5^{k+1} x^{k+1} k!}{5^k x^k (k+1)!} = \frac{5x}{k+1}$. The limit of this ratio is $\lim_{k\to\infty} \frac{5x}{k+1} = 0$.

For the series to converge, we must choose x so that the limit above is less than 1. But since this limit is 0, it is always less than 1. Thus the radius of convergence is ∞.

Correct Answers:

- `0`
- `A`

Problem 8.

8. **(5 points)** `local/rmb-problems/cond-abs-conv-mc.pg`

Determine if each series converges absolutely, converges conditionally, or diverges.

$$\sum_{k=1}^{\infty} \frac{(-1)^k}{k^2}$$

- A. The series diverges.
- B. The series converges conditionally.
- C. The series converges absolutely.

$$\sum_{k=1}^{\infty} (-1)^k 2^k$$

- A. The series converges absolutely.
- B. The series converges conditionally.
- C. The series diverges.

Correct Answers:

- `C`
- `C`

This is the free response part of Exam 2. There are 3 questions, each worth 20 points. Please write your solutions in full, clearly indicating each step leading to the final answer. Omitting details will result in a lower grade.

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Question 1. Determine the **radius** and the **interval** of convergence of the power series

$$\sum_{n=0}^{\infty} \frac{(-1)^n(2n+3)(5n+11)x^n}{7^n}.$$

Be sure to test the endpoints, and clearly label your final answers.

SOLUTION: Writing

$$a_n = \frac{(-1)^n(2n+3)(5n+11)x^n}{7^n},$$

we have

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{(2n+5)(5n+16)\,|x|}{7(2n+3)(5n+11)} = \frac{|x|}{7}.$$ (10)

(4)

It follows that the radius of convergence is $R = 7$ and the series converges in $(-7, 7)$ and diverges outside of $[-7, 7]$. For both $x = 7$ and $x = -7$, we have

$$|a_n| = (2n+3)(5n+11)$$

and thus $\{a_n\}$ does not converge to zero. Therefore the series diverges at $x = \pm 7$ and the interval of convergence is $(-7, 7)$. ★

end-points (6)

using something else: −2 for any of the three series

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Question 2. For each of the following series, use the limit comparison test to determine if the series converges.

(2)

(a) $\sum_{k=1}^{\infty} \frac{1}{3^k} \left(\frac{4k^2+5}{2k^2+1}\right)$

SOLUTION: Writing

$$a_k = \frac{1}{3^k}\left(\frac{4k^2+5}{2k^2+1}\right), \qquad b_k = \frac{1}{3^k}, \quad (2)$$

we have

$$\lim_{n\to\infty} \frac{a_k}{b_k} = \lim_{n\to\infty} \frac{4k^2+5}{2k^2+1} = 2 \neq 0. \quad (2)$$

Since

$$\sum_{k=1}^{\infty} b_k = \sum_{k=1}^{\infty} \frac{1}{3^k} \quad (2)$$

is a geometric series with common ratio $1/3 < 1$, it converges. Thus $\sum_{k=1}^{\infty} a_k$ converges by the limit comparison test.

(b) $\sum_{k=1}^{\infty} \left(\frac{k^2+2k-1}{3k^3+5k}\right)$

SOLUTION: Writing

$$a_k = \frac{k^2+2k-1}{3k^3+5k}, \qquad b_k = \frac{1}{k}, \quad (2)$$

we have

$$\lim_{n\to\infty} \frac{a_k}{b_k} = \lim_{n\to\infty} \frac{k^2+2k-1}{3k^2+5} = \frac{1}{3} \neq 0. \quad (2)$$

Since the harmonic series $\sum_{k=1}^{\infty} \frac{1}{k}$ diverges, the series $\sum_{k=1}^{\infty} a_k$ diverges by the limit comparison test.

(2)

(third series on next page)

(c) $\sum_{k=1}^{\infty}\left(\frac{2k^3+3k^2}{5k^7+7k^5+12}\right)$

Solution: Writing

$$a_k = \frac{2k^3+3k^2}{5k^7+7k^5+12}, \quad b_k = \frac{1}{k^4}, \quad (2)$$

we have

$$\lim_{n\to\infty}\frac{a_k}{b_k} = \lim_{n\to\infty}\frac{2k^7+3k^6}{5k^7+7k^5+12} = \frac{2}{5} \neq 0. \quad (2)$$

Since the p-series $\sum_{k=1}^{\infty}\frac{1}{k^4}$ converges, the series $\sum_{k=1}^{\infty} a_k$ converges by the limit comparison test.

(2)

★

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Question 3. Let $f(x) = \ln(3-2x)$.

(a) Find the derivative $f'(x)$ of $f(x)$, and then find a power series centered at 0 for $f'(x)$. Give the radius of convergence for the power series.

SOLUTION: We have, using a geometric series with common ratio $2x/3$,

$$\begin{aligned} f'(x) &= -\frac{2}{3-2x} \quad \text{(2)} \\ &= -\frac{2}{3}\left(\frac{1}{1-(2x/3)}\right) \quad \text{(4)} \\ &= -\frac{2}{3}\left(1+\frac{2x}{3}+\left(\frac{2x}{3}\right)^2+\cdots\right) \quad \text{(4) (3)} \\ &= -\sum_{n=0}^{\infty}\left(\frac{2}{3}\right)^{n+1} x^n. \end{aligned}$$

do not insist on having this written

The series converges when $|2x/3| < 1$. Therefore the radius of convergence is $R = 3/2$. (1)

(b) Use your answer from part (a) to find a power series centered at 0 for $f(x)$. Find the radius of convergence for the power series.

SOLUTION: Integrating the series obtained in part (a), we get

$$\ln(3-2x) = C - \sum_{n=0}^{\infty}\left(\frac{2}{3}\right)^{n+1}\frac{x^{n+1}}{n+1} \quad \text{(5)}$$

and since $\ln(3-2x) = \ln 3$ for $x = 0$, we find $C = \ln 3$. Thus

$$\ln(3-2x) = \ln 3 - \sum_{n=0}^{\infty}\left(\frac{2}{3}\right)^{n+1}\frac{x^{n+1}}{n+1}. \quad \text{(8) (4)}$$

Term-by-term integration does not change the radius of convergence. Therefore $R = 3/2$. (1)

It is OK if the student writes

$$\ln(3-2x) = \ln 3 - \left(\frac{2}{3}\right)x - \left(\frac{2}{3}\right)^2\frac{x^2}{2} - \left(\frac{2}{3}\right)^3\frac{x^3}{3} - \cdots$$

(end of exam questions)