Exam 2

Name: ______________________________ Section: ______________

Do not remove this answer page — you will return the whole exam. You will be allowed two hours to complete this test. If you find you need scratch paper during the exam, please ask. You may not use any of your own notes, paper or anything else not provided. You may use a graphing calculator during the exam, but NO calculator with a Computer Algebra System (CAS). Absolutely no communication device use during the exam is allowed.

The exam consists of 10 multiple choice questions and 5 free response questions. Record your answers to the multiple choice questions on this page by filling in the circle corresponding to the correct answer.

Show all work to receive full credit on the free response problems. It will also help you check your answers to show work on multiple choice problems.

Multiple Choice Questions

1	Ⓐ	Ⓑ	Ⓒ	Ⓓ	Ⓔ
2	Ⓐ	Ⓑ	Ⓒ	Ⓓ	Ⓔ
3	Ⓐ	Ⓑ	Ⓒ	Ⓓ	Ⓔ
4	Ⓐ	Ⓑ	Ⓒ	Ⓓ	Ⓔ
5	Ⓐ	Ⓑ	Ⓒ	Ⓓ	Ⓔ
6	Ⓐ	Ⓑ	Ⓒ	Ⓓ	Ⓔ
7	Ⓐ	Ⓑ	Ⓒ	Ⓓ	Ⓔ
8	Ⓐ	Ⓑ	Ⓒ	Ⓓ	Ⓔ
9	Ⓐ	Ⓑ	Ⓒ	Ⓓ	Ⓔ
10	Ⓐ	Ⓑ	Ⓒ	Ⓓ	Ⓔ

Multiple Choice	11	12	13	14	15	Total Score
50	10	10	10	10	10	100

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Multiple Choice Questions

1. (5 points) Give the first four terms of the sequence $\{a_n\}_{n=0}^{\infty}$ defined by

$$a_n = \frac{(-1)^n \;\; 2^n}{n!}$$

   A. $\{0,\ 1,\ 2,\ 2\}$
   B. $\{0,\ 1, -2,\ 2\}$
   C. $\{1, -2,\ 2, -4/3\}$
   D. $\{-2,\ 2, -4/3,\ \ 2/3\}$
   E. $\{\text{undefined}, 1, -2,\ 2\}$

2. (5 points) If $\sum_{n=1}^{\infty} a_n$ is a series that has partial sums

$$S_n = \frac{n^2+1}{n^2+n+1},$$

   then what can be said about the series?

   A. The series converges to 0.
   B. The series converges to 1.
   C. The series converges to 2.
   D. The series diverges.
   E. Nothing can be said about the series.

3. (5 points) Determine if the series is convergent and if the series converges, find the sum:

$$\sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n+2} \right)$$

A. The series converges to 0.

B. The series converges to 1/2.

C. The series converges to 1.

D. The series converges to 3/2.

E. The series diverges.

4. (5 points) For the given a_n, determine whether the **sequence** $\{a_n\}$ and the corresponding **series** $\sum_{n=1}^{\infty} a_n$ is convergent or divergent.

$$a_n = \frac{2n-1}{4n+2}$$

A. The sequence $\{a_n\}$ converges to $\frac{1}{2}$ and the series $\sum a_n$ diverges.

B. The sequence $\{a_n\}$ converges to $\frac{1}{2}$ and the series $\sum a_n$ converges to $\frac{1}{2}$.

C. The sequence $\{a_n\}$ diverges and the series $\sum a_n$ diverges.

D. The sequence $\{a_n\}$ converges to 0 and the series $\sum a_n$ converges to $\frac{1}{2}$.

E. The sequence $\{a_n\}$ diverges and the series $\sum a_n$ converges to 0.

5. (5 points) Determine the value of the series

$$\sum_{n=2}^{\infty} \frac{3^n}{3^{n+2}}.$$

A. 1

B. 1/9

C. 1/8

D. 0

E. The series diverges.

6. (5 points) Apply the Ratio Test to decide if the series $\displaystyle\sum_{n=1}^{\infty} \frac{(-2)^n}{(2n+1)!}$ converges or diverges.

A. The series diverges since $\displaystyle\lim_{n\to\infty} \frac{2n+1}{2} = \infty$.

B. The series converges since $\displaystyle\lim_{n\to\infty} \frac{2}{(2n+3)(2n+2)} = 0$.

C. The series converges since $\displaystyle\lim_{n\to\infty} \frac{2}{(2n+3)(2n+2)(2n+1)} = 0$

D. The series converges since $\displaystyle\lim_{n\to\infty} \frac{2}{2n+2} = 0$

E. The test is inconclusive since $\displaystyle\lim_{n\to\infty} \frac{2\,(1)!}{(2)!} = 1$

7. (5 points) Use the Limit Comparison Test to test the convergence of the series:

$$\sum_{n=1}^{\infty} \frac{n^2}{(n+1)^3}.$$

A. Comparing with the series $\sum_{n=1}^{\infty} \frac{1}{n}$ gives that the series is divergent.

B. Comparing with the series $\sum_{n=1}^{\infty} \frac{1}{n}$ gives that the series is convergent.

C. Comparing with the series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ gives that the series is divergent.

D. Comparing with the series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ gives that the series is convergent.

E. No conclusion can be drawn from the Comparison Test.

8. (5 points) Find the radius of convergence for the series $\sum_{n=1}^{\infty} n(x-2)^n$.

A. $(-2, 2)$

B. $(-1, 1)$

C. 2

D. 1

E. ∞

9. (5 points) If $f(x) = e^{2x}$, find the first three terms of the Taylor Series of $f(x)$ centered at $x = 0$.

   A. $1 + x + x^2$

   B. $1 + 2x + 2x^2$

   C. $1 + 2x + 4x^2$

   D. $1 + x + \frac{1}{2}x^2$

   E. $1 + x^2 + \frac{1}{2}x^4$

10. (5 points) Which power series represents the function $x^3 \cos(2x)$ on the interval $(-\infty, \infty)$?

   A. $\sum_{n=0}^{\infty} \frac{(-1)^n \ x^{2n+3}}{(2n)!}$

   B. $\sum_{n=0}^{\infty} \frac{(-1)^n \ x^{2n+4}}{(2n+1)!}$

   C. $\sum_{n=0}^{\infty} \frac{(-1)^n \ (2x)^{2n}}{(2n)!}$

   D. $\sum_{n=0}^{\infty} \frac{(-1)^n \ (2x)^{2n+1}}{(2n+1)!}$

   E. $\sum_{n=0}^{\infty} \frac{(-1)^n \ 2^{2n} \ x^{2n+3}}{(2n)!}$

Free Response Questions

11. Determine whether the following series converge or diverge. Clearly state your answer and justify it by showing your work and citing any tests used.

(a) (5 points) $\displaystyle\sum_{n=1}^{\infty} \frac{2^n + 5^n}{6^n}$

> **Solution:** Note that $\displaystyle\sum_{n=1}^{\infty} \frac{2^n + 5^n}{6^n} = \sum_{n=1}^{\infty} \left(\frac{2}{6}\right)^n + \sum_{n=1}^{\infty} \left(\frac{5}{6}\right)^n$.
> Since each of these series converge by the geometric series test, their sum converges.

(b) (5 points) $\displaystyle\sum_{n=1}^{\infty} (\sqrt{2})^{-n}$

> **Solution:** Note that $\displaystyle\sum_{n=1}^{\infty} (\sqrt{2})^{-n} = \sum_{n=1}^{\infty} \left(\frac{1}{\sqrt{2}}\right)^n$, which converges by the geometric series test.

12. (a) (6 points) Compute the values of S_3, S_4, and S_5 for $\displaystyle\sum_{n=3}^{\infty}\left(\frac{1}{n}-\frac{1}{n+1}\right)$

Solution:

$$S_3 = (\frac{1}{3}-\frac{1}{4}) + (\frac{1}{4}-\frac{1}{5}) + (\frac{1}{4}-\frac{1}{5}) = \frac{1}{3}-\frac{1}{6}$$
$$S_4 = S_3 + (\frac{1}{6}-\frac{1}{7}) = \frac{1}{3}-\frac{1}{7}$$
$$S_5 = S_4 + (\frac{1}{7}-\frac{1}{8}) = \frac{1}{3}-\frac{1}{8}$$

(b) (2 points) Use your computations from part (a) to determine S_n.

Solution:

$$S_n = \frac{1}{3}-\frac{1}{n+3}$$

(c) (2 points) Use your answer from part (b) to determine the value of $\displaystyle\sum_{n=3}^{\infty}\left(\frac{1}{n}-\frac{1}{n+1}\right)$.

Solution:

$$\sum_{n=3}^{\infty}\left(\frac{1}{n}-\frac{1}{n+1}\right) = \lim_{n\to\infty} S_n = \frac{1}{3}$$

13. Are the series absolutely convergent, conditionally convergent, or divergent? Justify your answer, including citing any tests used.

(a) (5 points) $\displaystyle\sum_{n=1}^{\infty} \frac{\cos(\frac{\pi}{n})}{n^2}$

Solution: Consider

$$\sum_{n=1}^{\infty} \frac{\cos(\pi/n)}{n^2}.$$

Since

$$\left|\frac{\cos(\pi/n)}{n^2}\right| \leq \frac{1}{n^2}$$

for all n, and since

$$\sum_{n=1}^{\infty} \frac{1}{n^2}$$

converges (by the p-Series Test), the given series converges absolutely by the (Direct) Comparison Test.

(b) (5 points) $\displaystyle\sum_{n=1}^{\infty} \frac{n+1}{3n+2}$

Solution: Consider

$$\sum_{n=1}^{\infty} \frac{n+1}{3n+2}.$$

We check the nth term:

$$\lim_{n\to\infty} \frac{n+1}{3n+2} = \frac{1}{3} \neq 0.$$

Since the terms do not approach 0, the series diverges by the Test for Divergence (Divergence Test).

14. (10 points) Find the Interval of Convergence for the series. Be sure to clearly show where you test the endpoints of your interval and cite any tests used.

$$\sum_{n=1}^{\infty} \frac{(x-2)^n}{n^2\, 5^n}$$

Solution: (Root Test) $\lim_{n\to\infty} \sqrt[n]{\left|\frac{(x-2)^n}{n^2 5^n}\right|} = \lim_{n\to\infty} \frac{|x-2|}{5} = \frac{|x-2|}{5}$. Thus the series converges for

$$\frac{|x-2|}{5} < 1 \quad \Longrightarrow \quad |x-2| < 5,$$

so the initial interval is

$$-3 < x < 7.$$

Now test the endpoints.

At $x = -3$,

$$\sum_{n=1}^{\infty} \frac{(-5)^n}{n^2\, 5^n} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2},$$

which converges absolutely since

$$\sum_{n=1}^{\infty} \frac{1}{n^2}$$

converges.

At $x = 7$,

$$\sum_{n=1}^{\infty} \frac{5^n}{n^2\, 5^n} = \sum_{n=1}^{\infty} \frac{1}{n^2},$$

which converges.

Therefore the interval of convergence is

$$[-3, 7].$$

15. (a) (5 points) Find the Maclaurin Series for $f(x) = \dfrac{x^3}{1-9x^7}$ and state the Interval of Convergence.

Solution: Starting from the geometric series formula,

$$\frac{1}{1-r} = \sum_{n=0}^{\infty} r^n \qquad (|r| < 1),$$

take $r = 9x^7$. Then the Maclaurin series is

$$\frac{x^3}{1-9x^7} = x^3 \sum_{n=0}^{\infty} (9x^7)^n = \sum_{n=0}^{\infty} 9^n x^{7n+3}.$$

For convergence we need

$$|9x^7| < 1 \quad \Longrightarrow \quad |x| < 9^{-1/7}.$$

At $x = \pm 9^{-1/7}$, the terms do not approach 0, so the endpoints are not included. Therefore the interval of convergence is $\left(-9^{-1/7}, 9^{-1/7}\right)$.

(b) (5 points) Write the first four nonzero terms of the Taylor Series of $f(x) = \ln(x)$ centered at $x = 2$.

Solution: To find the Taylor series of $f(x) = \ln x$ centered at $x = 2$, we use

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(2)}{n!}(x-2)^n.$$

First compute derivatives:

$$f(x) = \ln x, \qquad f'(x) = \frac{1}{x}, \qquad f''(x) = -\frac{1}{x^2}, \qquad f^{(3)}(x) = \frac{2}{x^3}$$

Now evaluate at $x = 2$:

$$f(2) = \ln 2, \qquad f'(2) = \frac{1}{2}, \qquad f''(2) = -\frac{1}{4}, \qquad f^{(3)}(2) = \frac{2}{8} = \frac{1}{4}$$

Substituting into the Taylor formula gives

$$\ln x = f(2) + f'(2)(x-2) + \frac{f''(2)}{2!}(x-2)^2 + \frac{f^{(3)}(2)}{3!}(x-2)^3 + \cdots$$

$$= \ln 2 + \frac{x-2}{2} - \frac{(x-2)^2}{8} + \frac{(x-2)^3}{24} + \cdots.$$