Exam 3

Name: ______________________________ Section: ______________

Do not remove this answer page — you will return the whole exam. You will be allowed two hours to complete this test. You are allowed to use notes on a single piece of 8.5" x 11" paper, front and back, including formulas and theorems. **You are required to turn this page in with your exam.** You may use a graphing calculator during the exam, but NO calculator with a Computer Algebra System (CAS). Absolutely no communication device use during the exam is allowed.

The exam consists of 10 multiple choice questions and 5 free response questions. Record your answers to the multiple choice questions on this page by filling in the circle corresponding to the correct answer.

Show all work to receive full credit on the free response problems. It will also help you check your answers to show work on multiple choice problems.

Multiple Choice Questions

1 Ⓐ Ⓑ Ⓒ Ⓓ Ⓔ

2 Ⓐ Ⓑ Ⓒ Ⓓ Ⓔ

3 Ⓐ Ⓑ Ⓒ Ⓓ Ⓔ

4 Ⓐ Ⓑ Ⓒ Ⓓ Ⓔ

5 Ⓐ Ⓑ Ⓒ Ⓓ Ⓔ

6 Ⓐ Ⓑ Ⓒ Ⓓ Ⓔ

7 Ⓐ Ⓑ Ⓒ Ⓓ Ⓔ

8 Ⓐ Ⓑ Ⓒ Ⓓ Ⓔ

9 Ⓐ Ⓑ Ⓒ Ⓓ Ⓔ

10 Ⓐ Ⓑ Ⓒ Ⓓ Ⓔ

Multiple Choice	11	12	13	14	15	Total Score
50	10	10	10	10	10	100

1. (5 points) What is the average value of the function $f(x) = (\sin(x))^2 \cos(x)$ on the interval $[0, \frac{\pi}{2}]$?

   A. $\dfrac{2\pi}{3}$

   B. $\dfrac{2}{3\pi}$

   C. $-\dfrac{\pi}{3}$

   D. 0

   E. $\dfrac{3\pi}{2}$

2. (5 points) Recall that the circle of radius 2 centered at the point $(2, 3)$ is the set of points (x, y) which satisfy:
$$(x-2)^2 + (y-3)^2 = 4.$$

   Which of the following is a parametrization of the circle of radius 2 centered at $(2, 3)$?

   A. $x(t) = 2 + 2\sin(5t)$, $y(t) = 3 + 2\cos(5t)$

   B. $x(t) = 2 - \cos(3t)$, $y(t) = 3 - \sin(3t)$

   C. $x(t) = 2 + 2\sin(t)$, $y(t) = 3 - 3\cos(t)$

   D. $x(t) = 1 + 2\cos(t)$, $y(t) = 1 + 3\sin(t)$

   E. $x(t) = 2\sqrt{1-t}$, $y(t) = 3\sqrt{1+t}$

3. (5 points) Three masses are located in the plane: 1 gram at $(0, 2)$, 1 gram at $(-10, 2)$, and 3 grams at $(1, -5)$. Find the center of mass of this system.

   A. $(-1, -2)$

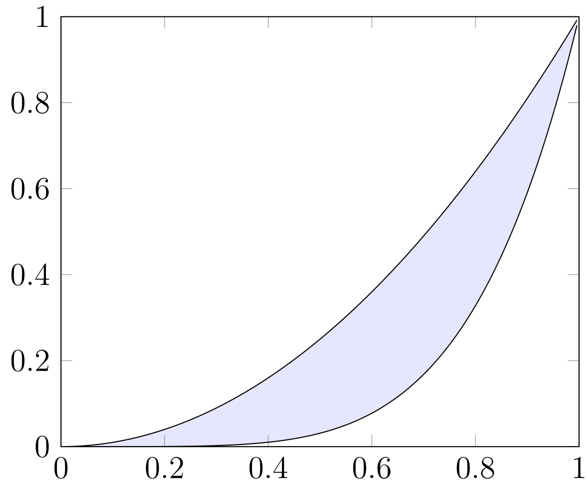
   B. $(0, -\frac{12}{5})$

   C. $(0, -\frac{9}{5})$

   D. $(-\frac{7}{5}, -\frac{11}{5})$

   E. $(\frac{6}{5}, -\frac{14}{5})$

4. (5 points) The region bounded by $y = x^2$ and $y = x^5$ is shown below.



Consider the solid obtained by rotating this region around the **x-axis**. Using the **disks/washers** method, which integral will compute the volume of this solid?

A. $\int_0^{\sqrt{3}} 2\pi \left(x^2 - x^5\right)^2 dx$.

B. $\int_0^1 \pi \left(x^4 - x^{10}\right) dx$**.**

C. $\int_0^1 \pi \left(x^2 - x^5\right) dx$.

D. $\int_0^1 \pi \left(x^2 - x^5\right)^2 dx$.

E. $\int_0^1 2\pi \left(x^3 - x^6\right) dx$.

5. (5 points) Which integral below computes the length of the curve $y = f(x)$ where $f(x) = \cos(x) + \sin(x)$, and $0 \leq x \leq \pi$?

A. $\int_0^\pi \sqrt{1 + \cos^2(x) - \sin^2(x)}\, dx$.

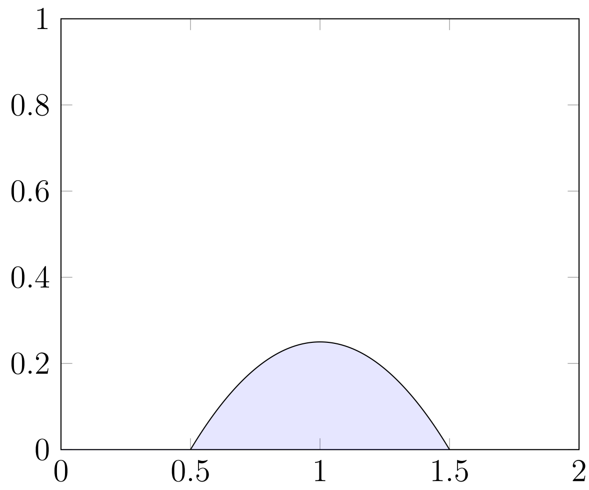
B. $\int_0^\pi \sqrt{t^2 + [\cos(x) - \sin(x)]^2}\, dx$.

C. $\int_0^\pi \sqrt{1 + [2\cos(x))]^2}\, dx$.

D. $\int_0^\pi \sqrt{2 - 2\sin(x)\cos(x)}\, dx$**.**

E. $\int_0^\pi 2\pi t^2 \sqrt{1 + [\cos(x) + \sin(x)]^2}\, dx$.

6. (5 points) The region bounded by the curve $y = -x^2 + 2x - \frac{3}{4}$ and the x-axis is shown below.



Consider the solid obtained by rotating this region about the **y-axis**. Using the **shell** method, which integral will compute the volume of this solid?

A. $2\pi \int_{\frac{1}{2}}^{\frac{3}{2}} (-x^3 + 2x^2 - \frac{3}{4}x)\ dx$.

B. $\pi \int_{\frac{1}{2}}^{\frac{3}{2}} (-x^3 + 2x^2 - \frac{3}{4}x)^2\ dx$.

C. $\pi \int_{\frac{1}{2}}^{\frac{3}{2}} (-x^3 + 2x^2 - \frac{3}{4}x)\ dx$.

D. $2\pi \int_{\frac{1}{2}}^{\frac{3}{2}} (-x^6 + 2x^4 - \frac{3}{4}x^2)\ dx$.

E. $2\pi \int_{\frac{1}{2}}^{\frac{3}{2}} (-x^2 + 2x - \frac{3}{4})\ dx$.

7. (5 points) Find the slope of the tangent line to the curve parametrized by $x(t) = t^2, \quad y(t) = t^3 - 2t$ at the point $(x, y) = (9, 21)$.

A. -5

B. $\frac{27}{5}$

C. $\frac{27}{2}$

D. $-\frac{6}{25}$

E. $\frac{25}{6}$

8. (5 points) The line $y = 2x$ for $1 \leqslant x \leqslant 2$ is rotated about the **x-axis**. What is the **surface area** of the resulting surface?

   A. 30π

   B. $6\sqrt{5}\pi$

   C. $18\sqrt{3}\pi$

   D. 20π

   E. $\sqrt{31}\pi$

9. (5 points) Let R be the region bounded by $f(x) = e^x$, $y = e$ and the y-axis; and form a solid by revolving R about the **y-axis**. Which integral represents the volume of this solid using the **shell method**?

   A. $2\pi \int_0^1 e^{2x}\, dx$

   B. $2\pi \int_0^1 \ln(x)\, dx$

   C. $2\pi \int_1^e x\sqrt{1 + e^{2x}}\, dx$

   D. $2\pi \int_0^1 xe^{2x}\, dx$

   E. $2\pi \int_0^1 x(e - e^x)\, dx$

10. (5 points) The curve $y = \sqrt{x}$ for $0 \leqslant x \leqslant 1$ is rotated about the **y-axis**, producing a surface. Which of the following integrals calculates its surface area?

    A. $\int_0^1 2\pi\sqrt{1 + \frac{1}{4y}}\, dy$

    B. $\int_{-1}^0 2\pi y^2\sqrt{1 + \left(\frac{1}{y}\right)^2}\, dy$

    C. $\int_0^1 2\pi y^2\sqrt{1 + 4y^2}\, dy$

    D. $\int_0^1 2\pi y^2\sqrt{1 + y^2}\, dy$

    E. $\int_0^1 2\pi\sqrt{y + \frac{1}{4}}\, dy$

Free Response Questions

11. The *cycloid* for the circle of radius 1 is the curve parametrized by the functions

$$x(\theta) = \theta - \sin(\theta),$$

$$y(\theta) = 1 - \cos(\theta).$$

(a) (2 points) Find the coordinates of the point $P(\theta) = (x(\theta), y(\theta))$ when $\theta = \frac{\pi}{4}$.

Solution: We note that $\sin(\pi/4) = \sqrt{2}/2$ so that

$$x(\pi/4) = \pi/4 - \sqrt{2}/2, \quad y(\pi/4) = 1 - \sqrt{2}/2.$$

So we get

$$P(\pi/4) = (\pi/4 - \sqrt{2}/2, 1 - \sqrt{2}/2).$$

1 point for each coordinate.

(b) (4 points) Find the **slope** of the tangent line to the cycloid at the point $P(\frac{\pi}{4})$ from part (a).

Solution:

$\dfrac{dx}{d\theta} = x'(\theta) = 1 - \cos(\theta), \qquad \dfrac{dy}{d\theta} = y'(\theta) = \sin(\theta).$

$x'(\dfrac{\pi}{4}) = 1 - \dfrac{1}{\sqrt{2}}, \qquad y'(\dfrac{\pi}{4}) = \sin(\dfrac{\pi}{4}) = \dfrac{1}{\sqrt{2}}.$

$\dfrac{dy}{dx} = \dfrac{y'(\frac{\pi}{4})}{x'(\frac{\pi}{4})} = \dfrac{\frac{1}{\sqrt{2}}}{1 - \frac{1}{\sqrt{2}}} = \dfrac{1}{\sqrt{2} - 1}.$ 1 point for each derivative, 1 point for each numerical evaluation of x' and y'.

(c) (4 points) Set up <u>but do not evaluate</u> the integral to find the arc length of the piece of the cycloid parametrized by $0 \leq \theta \leq \pi$ (you may assume that the curve is traced only once).

Solution: $L = \displaystyle\int_0^\pi \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2} d\theta = \int_0^\pi \sqrt{(1 - \cos(\theta))^2 + (\sin(\theta))^2} d\theta.$

After expanding the argument of the square root, one gets:

$$L = \sqrt{2} \int_0^\pi \sqrt{1 - \cos\theta} d\theta.$$

2 points for the general form of the integral, 2 points for the correct evaluation of the integrand on the right. It is not necessary to obtain the last formula.

12. Let S be the region in the plane bounded by $y = 4 - x^2$ and the x-axis for $-2 \leq x \leq 2$. Assume that S has uniform density $\rho = 1$.

(a) (8 points) Find the total mass M and the moments M_y and M_x for S. **Clearly label** each of your answers.

Solution: The total mass is

$$m = \rho \int_{-2}^{2} 4 - x^2 dx = \left[4x - \frac{x^3}{3}\right]_{-2}^{2} = \frac{32}{3}.$$

The moments are

$$M_y = \rho \int_{-2}^{2} x4 - x^2 \ dx = \int_{-2}^{2} 4x - x^3 \ dx = \left[2x - \frac{x^4}{4}\right]_{-2}^{2} = 0,$$

$$M_x = \frac{\rho}{2} \int_{-2}^{2} (4 - x^2)^2 \ dx = \int_{0}^{2} (4 - x^2)^2 dx = \int_{0}^{2} (4 - x^2)^2 dx$$

$$= \left[16x - \frac{8}{3}x^3 + \frac{1}{5}x^5\right]_{0}^{2} = \frac{256}{15}.$$

2 points for the mass, 3 points for each moment: 2 points for the integral and 1 point for the results for each moment.

(b) (2 points) Find the center of mass of S.

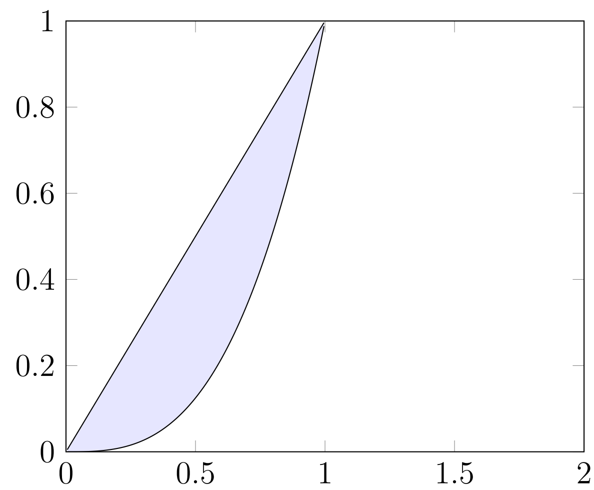
Solution:

$$\overline{x} = \frac{M_y}{m} = 0,$$

$$\overline{y} = \frac{M_x}{m} = \left(\frac{256}{15}\right)\left(\frac{32}{3}\right)^{-1} = \frac{8}{5}.$$

1 point for each coordinate of the center of mass (centroid).

13. The region between the curves $y = x$ and $y = x^3$ is shown below.



Let V be the solid of revolution obtained by rotating this region **around the y-axis**.

(a) (5 points) Write an integral which computes the volume of V using the disk/washer method, and then evaluate the integral.

Solution: Slice the solid perpendicular to the y-axis. The cross-sectional area id the difference of the area of two disks:

$$A(y) = \pi(y^{2/3} - y^2)\ dy$$

so that the volume is

$$V = \int_{y=0}^{1} A(y)\ dy = \pi \int_0^1 (y^{2/3} - y^2)\ dy = \frac{4}{15}\pi.$$

3 points for the cross-sectional area, and 2 points for the volume integral and its evaluation.

(b) (5 points) Write an integral which computes the volume of V using the cylindrical shells method, and then evaluate the integral.

Solution: For the shell method, integrate with respect to x-axis. The elementary shell volume is

$$V(x) = 2\pi x(x - x^3)\Delta x,$$

so that the volume is

$$V = \int_{x=0}^{1} V(x)\ dx = 2\pi \int_0^1 x(x - x^3)\ dx = \frac{4}{15}\pi.$$

3 points for the cross-sectional area, and 2 points for the volume integral and its evaluation.

14. Let L be the arc parametrized by $x(t) = t^2$, $y(t) = t^3$, $0 \leq t \leq 1$.

(a) (6 points) Find the arc length of L.

Solution:

$$x'(t) = 2t, \qquad y'(t) = 3t^2.$$

The arc length is

$$L = \int_0^1 \sqrt{4t^2 + 9t^4}\, dt = \int_0^1 2t\sqrt{1 + (9/4)t^2}\, dt,$$

Let $u = 1 + (9/4)t^2$ so $du = (9/2)t\, dt$, then the integral is

$$L = \frac{4}{9}\int_1^{13/4} \sqrt{u}\, du = \frac{8}{27}\left[\left(\frac{13}{4}\right)^{3/2} - 1\right].$$

2 points for the derivatives, 2 points for the correct integral, and 2 points for the integration and solution.

(b) (4 points) Set up but do not evaluate an integral which computes the area of the surface S_1 obtained by revolving L around the **x-axis**.

Solution:

The elementary arc length along the curve is:

$$ds = 2t\sqrt{1 + (9/4)t^2}\, dt,$$

so the surface area is

$$S = \int_0^1 2\pi y(t)\, ds(t) = 4\pi \int_0^1 t^4\sqrt{1 + (9/4)t^2}\, dt.$$

1 point for the infinitesimal arc length expression, 1 point for the first integral expression, and 1 point for the final result.

15. Let V be the solid whose base is the circle centered at the origin of radius 1, with cross sections given by squares perpendicular to the **x-axis**.

(a) (3 points) Find a function giving the area of the cross-section of V at x.

Solution: The length of a side of the square is $2\sqrt{1-x^2}$ as x varies in $[0,1]$. The cross-sectional area is then $A(x) = (2\sqrt{1-x^2})^2 = 4(1-x^2)$. 2 points for the side length, 1 point for the area.

(b) (4 points) Set up an integral which computes the volume of V.

(c) (3 points) Find the volume of V.

Solution: Taking $x \in [0,1]$ will sweep out one-half of the volume so

$$V = 2\int_{x=0}^{1} A(x)\ dx = 8\int_0^1 (1-x^2)\ dx = \frac{16}{3}.$$

3 points for the correct expression for V, 2 points for integrating, and 1 point for the correct result.