Exam 3

Name: ______________________________ Section: ______________

Do not remove this answer page — you will return the whole exam. You will be allowed two hours to complete this test. If you find you need scratch paper during the exam, please ask. You may not use any of your own notes, paper or anything else not provided. You may use a graphing calculator during the exam, but NO calculator with a Computer Algebra System (CAS). Absolutely no communication device use during the exam is allowed.

The exam consists of 10 multiple choice questions and 5 free response questions. Record your answers to the multiple choice questions on this page by filling in the circle corresponding to the correct answer.

Show all work to receive full credit on the free response problems. It will also help you check your answers to show work on multiple choice problems.

Multiple Choice Questions

1 (A) (B) (C) (D) (E) **6** (A) (B) (C) (D) (E)

2 (A) (B) (C) (D) (E) **7** (A) (B) (C) (D) (E)

3 (A) (B) (C) (D) (E) **8** (A) (B) (C) (D) (E)

4 (A) (B) (C) (D) (E) **9** (A) (B) (C) (D) (E)

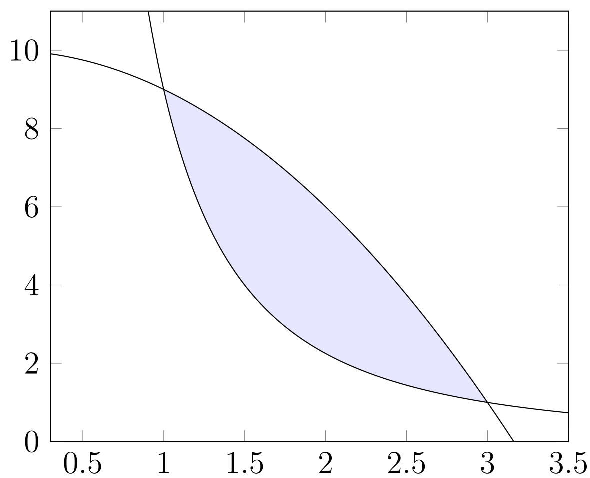
5 (A) (B) (C) (D) (E) **10** (A) (B) (C) (D) (E)

Multiple Choice	11	12	13	14	15	Total Score
50	10	10	10	10	10	100

1. (5 points) Find the average value of $f(x) = \frac{3}{x}$ on the interval $[1, 12]$.

   A. $\frac{1}{4}\ln(12)$
   B. $\frac{3}{11}\ln(12)$
   C. $\frac{143}{48}$
   D. $\frac{13}{8}$
   E. $-\frac{1}{4}$

2. (5 points) The region R bounded by $y = \dfrac{9}{x^2}$ and $y = 10 - x^2$ is shown below.



Consider the solid obtained by rotating R about the $x-$ axis. Which integral computes the volume of this solid using the **disk/washer method**?

A. $\pi \displaystyle\int_1^3 (10 - x^2)^2 - \left(\frac{9}{x^2}\right)^2\, dx$

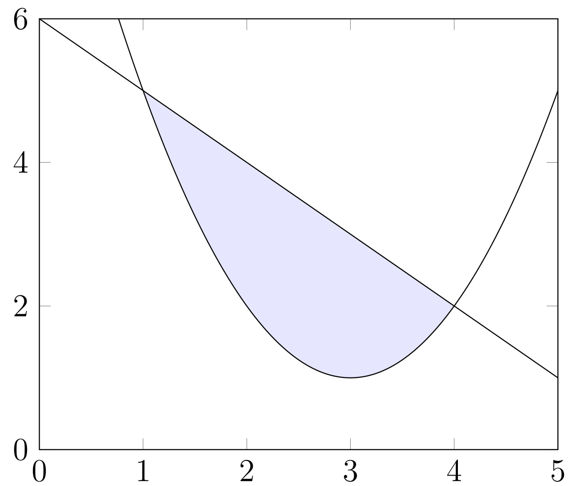
B. $\pi \displaystyle\int_1^3 \left(10 - x^2 - \frac{9}{x^2}\right)^2\, dx$

C. $2\pi \displaystyle\int_1^9 x\left(10 - x^2 - \frac{9}{x^2}\right)\, dx$

D. $\pi \displaystyle\int_1^3 \left(10 - \frac{11}{x^2}\right)^2\, dx$

E. $\pi \displaystyle\int_1^9 \left(\frac{9}{x^2}\right)^2 + (10 - x^2)^2\, dx$

3. (5 points) The region R bounded by the curves $y = x^2 - 6x + 10$ and $y = 6 - x$ is shown below.



Consider the solid obtained by rotating R about the **vertical** line $x = 1$. Which integral computes the volume of this solid using the **shell** method?

A. $\int_1^4 2\pi(x-1)(-x^2+5x-4)\,dx$

B. $\int_1^4 2\pi(x+1)(x^2-5x+4)\,dx$

C. $\int_1^4 2\pi(6-x)(x^2-6x+10)\,dx$

D. $\int_1^4 2\pi(x^2-5x+4)^2\,dx$

E. $\int_1^4 2\pi(5-x)(x^2-6x+9)\,dx$

4. (5 points) Which integral computes the arc length of the curve $f(x) = xe^{x^2}$ for $0 \le x \le 1$?

A. $\int_0^1 2\pi xe^{x^2}\sqrt{1+\left(2x^2e^{x^2}+e^{x^2}\right)^2}\,dx$

B. $\int_0^1 \sqrt{1+\left(xe^{x^2}\right)^2}\,dx$

C. $\int_0^1 \sqrt{1+\left(2x^2e^{x^2}+e^{x^2}\right)^2}\,dx$

D. $\int_0^1 \sqrt{1+\left(2xe^{x^2}\right)^2}\,dx$

E. $\int_0^1 \pi\left(xe^{x^2}\right)^2\,dx$

5. (5 points) The curve $y = \sec(x)$ from $x = 0$ to $x = \frac{\pi}{6}$ is rotated about the x-axis. Which integral computes the area of the resulting surface?

   A. $\int_0^{\frac{\pi}{6}} 2\pi x\sqrt{1 + (\sec(x)\tan(x))^2}\, dx$

   B. $\int_0^{\frac{\pi}{6}} 2\pi \sec(x)\sqrt{1 + (\sec(x)\tan(x))^2}\, dx$

   C. $\int_0^{\frac{\pi}{6}} 2\pi x\sqrt{1 + \sec^2(x)}\, dx$

   D. $\int_0^{\frac{\pi}{6}} 2\pi x \sec(x)\, dx$

   E. $\int_0^{\frac{\pi}{6}} 2\pi \sec(x)\sqrt{1 + \tan^2(x)}\, dx$

6. (5 points) Three masses are located in the plane: 10g at $(1, 8)$, 20g at $(-2, 5)$, and 30g at $(-6, 3)$. Find the center of mass of this system.

   A. $(-1, 5)$

   B. $(-3.5, 4.5)$

   C. $(-2.3, 3.8)$

   D. $(-\frac{7}{3}, \frac{16}{3})$

   E. $(-0.8, 4.1)$

7. (5 points) Point masses are placed on the x-axis. m_1 is 2g at $x = -7$, m_2 is 10g at $x = -1$, m_3 is 4g at $x = 8$. Find the moment M of the system about the origin.

   A. $M = 16$

   B. $M = 0$

   C. $M = 12$

   D. $M = 8$

   E. $M = -4$

8. (5 points) A line is parametrized by $x = 5 + 2t$, $y = 7 + 6t$. What are the endpoints of the line segment obtained by restricting t to $-2 \le t \le 3$?

   A. $(5, 7)$ and $(7, 13)$

   B. $(-2, -5)$ and $(3, 25)$

   C. $(-2, 1)$ and $(3, 11)$

   D. $(1, -5)$ **and** $(11, 25)$

   E. $(5, 2)$ and $(7, 6)$

9. (5 points) Eliminate the parameter t to find a Cartesian equation satisfied by the curve parameterized by $x(t) = 3 - t$, $y(t) = 4t + t^3$.

   A. $y = 4x + x^3$

   B. $y = 4x + x^3 - 3$

   C. $y = 4(3 - x) + (3 - x)^3$

   D. $y = 3 - 4t - t^3$

   E. $y = 4\left(\frac{x}{3}\right) + \left(\frac{x}{3}\right)^3$

10. (5 points) Which parametrization corresponds to the Cartesian equation $\left(\frac{x}{3}\right)^2 + \left(\frac{y}{2}\right)^2 = 1$ for $x \ge 0$?

   A. $x = 3\cos(\theta), \quad y = 2\tan(\theta), \quad -\frac{\pi}{2} < \theta < \frac{\pi}{2}$

   B. $x = \frac{\sin^2(\theta)}{3}, \quad y = \frac{\cos^2(\theta)}{2}$

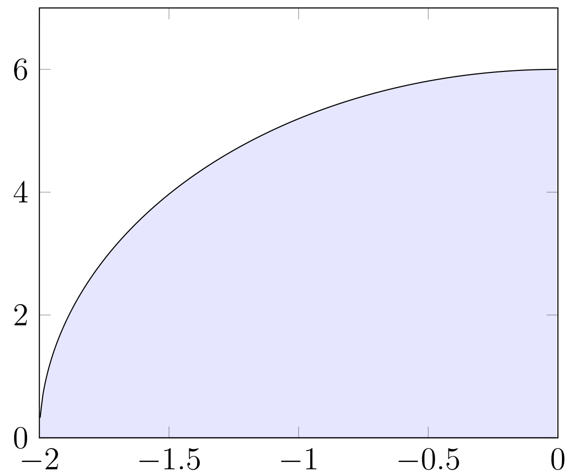
   C. $x = \sec(3\theta), \quad y = \tan(2\theta), \quad \theta > 0$

   D. $x = \frac{3}{\sin(\theta)}, \quad y = \frac{2}{\cos(\theta)}, \quad -\frac{\pi}{2} < \theta < \frac{\pi}{2}$

   E. $x = 3\sin(\theta), \quad y = 2\cos(\theta), \quad 0 \le \theta \le \pi$

Free Response Questions

11. The base of a solid is the region enclosed by $y = \sqrt{36 - 9x^2}$ and the $x-$axis for $-2 \leq x \leq 0$, shown below. Cross-sections perpendicular to the x-axis are squares. Set up an integral which computes the volume of this solid, and then find the volume. Show all steps to compute the integral.

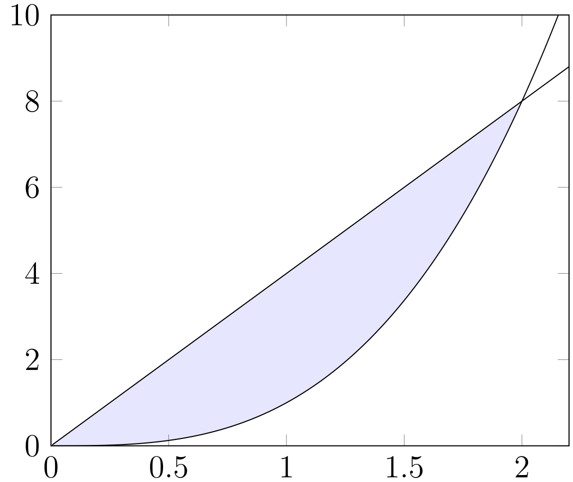


Solution:

The length of each square is $\sqrt{36 - 9x^2}$, so the area of each square is $36 - 9x^2$. Then the volume

$$V = \int_{-2}^{0} A(x)\,dx = \int_{-2}^{0} (36 - 9x^2)\,dx = 36x - 3x^3 \Big|_{-2}^{0} = 48.$$

12. The region bounded by $y = x^3$ and $y = 4x$, $x \geq 0$ is shown below. Let V be obtained by rotating this region about the **horizontal** line $y = 10$.



(a) (5 points) Set up but do not evaluate the integral that computes the volume of V using the **disk/washer** method.

Solution:

$$V = \pi \int_0^2 (10 - x^3)^2 - (10 - 4x)^2 \, dx$$

(b) (5 points) Set up but do not evaluate the integral that computes the volume of V using the **shell** method.

Solution:

$$V = 2\pi \int_0^8 (10 - y) \left(\sqrt[3]{y} - \frac{y}{4} \right) \, dy$$

13. Let S be the region bounded by $y = 9 - x^2$ and the $x-$axis for $0 \leq x \leq 3$. Assume S has uniform density $\rho = 2$.

(a) (8 points) Find the total mass M and the moments M_x and M_y for S. Show all steps clearly. Clearly label each answer.

Solution:

$$m = \rho \int_a^b f(x) - g(x)\ dx = 2\int_0^3 (9 - x^2) - 0\ dx = 2\left(9x - \frac{x^3}{3}\right)\Big|_0^3 = 36$$

$$M_y = \rho \int_a^b x(f(x) - g(x))\ dx = 2\int_0^3 x((9 - x^2) - 0)\ dx = 2\int_0^3 (9x - x^3)\ dx$$
$$= 2\left(\frac{9x^2}{2} - \frac{x^4}{4}\right)\Big|_0^3 = \frac{81}{2}$$

$$M_x = \frac{\rho}{2}\int_a^b f(x)^2 - g(x)^2\ dx = \frac{2}{2}\int_0^3 (9 - x^2)^2 - (0)^2\ dx$$
$$= \int_0^3 (81 - 18x^2 + x^4)\ dx = 81x - 6x^3 + \frac{x^5}{5}\Big|_0^3 = 129.6$$

(b) (2 points) Find the center of mass of S.

Solution:

$$\bar{x} = \frac{M_y}{m} = \frac{\frac{81}{2}}{36} = 1.125$$
$$\bar{y} = \frac{M_x}{m} = \frac{129.6}{36} = 3.6$$

The center of mass is $(\bar{x}, \bar{y}) = (1.125, 3.6)$

14. Let C be the curve defined by the graph of $f(x) = 3x^2 + 11$, $0 \leq x \leq 1$.

(a) (4 points) Set up but do **not** evaluate an integral which computes the length of C.

Solution:

$$L = \int_a^b \sqrt{1 + (f')^2}\, dx = \int_0^1 \sqrt{1 + (6x)^2}\, dt$$

(b) (6 points) Set up an integral which computes the area of the surface obtained by revolving C about the $\boldsymbol{y}$**−axis**. Then evaluate your integral to find the surface area. Show all steps needed to compute the integral.

Solution:

$$SA = \int_a^b 2\pi r L = \int_0^1 2\pi x\sqrt{1 + 36x^2}\, dx$$

Use u-substitution with $u = 1 + 36x^3$, then $du = 72x\, dx$, so $dx = \frac{du}{72x}$. Then the integral becomes

$$\int_{x=0}^{x=1} \frac{\pi}{36} u^{1/2}\, du = \frac{\pi}{36} \cdot \frac{2}{3} u^{3/2}\Big|_{x=0}^{x=1} = \frac{\pi}{54}(1 + 36x^2)^{3/2}\Big|_0^1 = \frac{\pi}{54}(37^{3/2} - 1).$$

15. (a) (5 points) Find the average value of $f(x) = |x-5|$ on $[2, 10]$. Show all steps clearly.

Solution:

$$\frac{1}{b-a}\int_a^b f(x)\,dx = \frac{1}{8}\int_2^{10} |x-5|\,dx = \frac{1}{8}\Big(\int_2^5 (-x+5)\,dx + \int_5^{10} (x-5)\,dx\Big)$$
$$= \frac{1}{8}\Big[\Big(-\frac{x^2}{2}+5x\Big)\Big|_2^5 + \Big(\frac{x^2}{2}-5x\Big)\Big|_5^{10}\Big] = 2.125$$

(Alternatively, you can use the graph of $f(x) = |x-5|$ and triangle areas to compute the integral.)

(b) (5 points) A curve C is defined by the parametric equations

$$x(t) = \sqrt{t} \qquad y(t) = t-4.$$

Find a Cartesian equation satisfied by C, and sketch the graph of C.

Solution: Since $y = t-4$ then $t = y+4$ so $x = \sqrt{y+4}$. Equivalently, since $x = \sqrt{t}$ then $t = x^2$, so $y = x^2 - 4$ and $x \geq 0$. In either case the graph is as follows.