Exam 3

Name: ______________________ Section: __________

Do not remove this answer page — you will return the whole exam. You will be allowed two hours to complete this test. If you find you need scratch paper during the exam, please ask. You may not use any of your own notes, paper or anything else not provided. You may use a graphing calculator during the exam, but NO calculator with a Computer Algebra System (CAS). Absolutely no communication device use during the exam is allowed.

The exam consists of 10 multiple choice questions and 5 free response questions. Record your answers to the multiple choice questions on this page by filling in the circle corresponding to the correct answer.

Show all work to receive full credit on the free response problems. It will also help you check your answers to show work on multiple choice problems.

Multiple Choice Questions

1 Ⓐ Ⓑ Ⓒ Ⓓ Ⓔ   **6** Ⓐ Ⓑ Ⓒ Ⓓ Ⓔ

2 Ⓐ Ⓑ Ⓒ Ⓓ Ⓔ   **7** Ⓐ Ⓑ Ⓒ Ⓓ Ⓔ

3 Ⓐ Ⓑ Ⓒ Ⓓ Ⓔ   **8** Ⓐ Ⓑ Ⓒ Ⓓ Ⓔ

4 Ⓐ Ⓑ Ⓒ Ⓓ Ⓔ   **9** Ⓐ Ⓑ Ⓒ Ⓓ Ⓔ

5 Ⓐ Ⓑ Ⓒ Ⓓ Ⓔ   **10** Ⓐ Ⓑ Ⓒ Ⓓ Ⓔ

Multiple Choice	11	12	13	14	15	Total Score
50	10	10	10	10	10	100

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Multiple Choice Questions

1. (5 points) Determine the average value of $f(x) = \dfrac{2}{x}$ on the interval $[1, 5]$.

   A. $2\ln(5)$

   B. $\dfrac{1}{2}\ln(5)$

   C. $4\ln(4)$

   D. $\dfrac{1}{4}\ln(5)$

   E. $\dfrac{1}{4}\ln(4)$

2. (5 points) Calculate the volume of the solid whose base is the region enclosed by $y = x^2$ and $y = 3$, where cross sections perpendicular to the y-axis are squares.

   A. 9

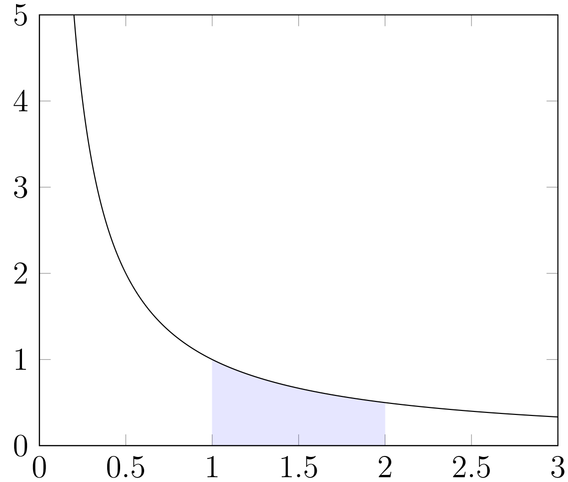
   B. 36

   C. 18

   D. 12

   E. 6

3. (5 points) The region R is bound by $f(x) = \dfrac{1}{x}$, $x = 1$, $x = 2$, and $y = 0$, shown below.



Consider the solid obtained by rotating R about the **x-axis**. Which integral computes the volume of this solid using the **disk/washer method**?

A. $\displaystyle\int_0^1 2\pi \left(\frac{1}{x}\right)^2 \, dx$

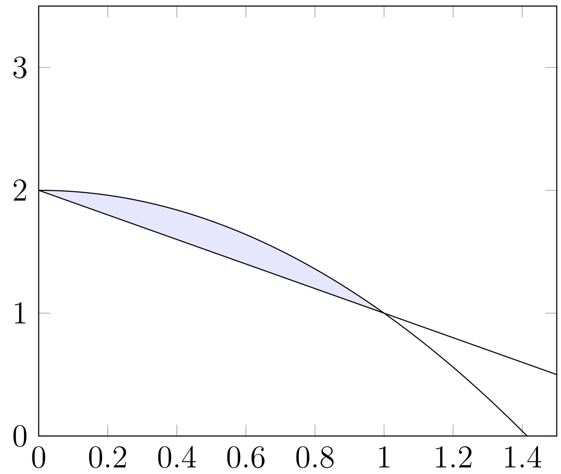
B. $\displaystyle\int_1^2 2\pi \left(\frac{1}{x}\right)^2 \, dx$

C. $\displaystyle\int_0^1 \pi \left(\frac{1}{x}\right) \, dx$

D. $\displaystyle\int_1^2 2\pi \left(\frac{1}{x}\right) \, dx$

E. $\displaystyle\int_1^2 \pi \left(\frac{1}{x}\right)^2 \, dx$

4. (5 points) The region R is bound by $y = 2 - x$ and $y = 2 - x^2$ for, shown below.



Consider the solid obtained by rotating R about the **y-axis**. Which integral computes the volume of this solid using the **shell method**?

A. $\int_0^1 2\pi(-x^3 + x^2)\,dx$

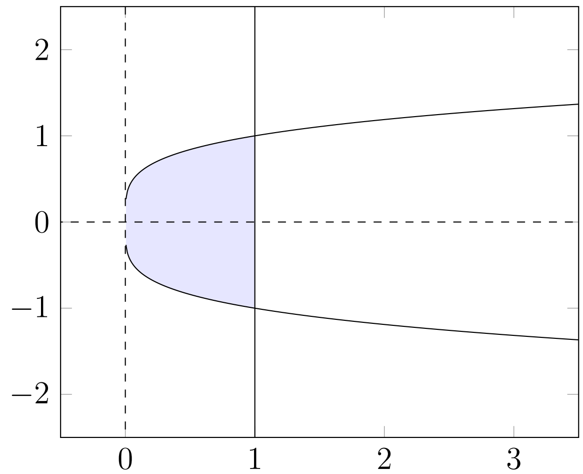
B. $\int_1^2 2\pi(-x^2 - x)\,dx$

C. $\int_0^1 2\pi(-x^3 - x^2)\,dx$

D. $\int_1^2 2\pi(-x^3 - x^2)\,dx$

E. $\int_0^1 2\pi(-x^4 - x^3 + 2)\,dx$

5. (5 points) The region R is bound by $x = y^4$ and $x = 1$, shown below. (The axes are dashed.)



Which integral computes the volume of the solid obtained by rotating R around $x = 3$?

A. $\displaystyle\int_{-1}^{1} \pi(5 + y^8)\ dy$

B. $\displaystyle\int_{0}^{1} \pi(8 + y^8)\ dy$

C. $\displaystyle\int_{0}^{1} \pi(8 - 6y^2 + y^8)\ dy$

D. $\displaystyle\int_{-1}^{1} \pi(5 - 6y^4 + y^8)\ dy$

E. $\displaystyle\int_{-1}^{1} 2\pi(8 + y^8)\ dy$

6. (5 points) Which integral computes the **arc length** of the curve $y = \sqrt{16 + x^2}$ from $x = 0$ to $x = 3$?

A. $\displaystyle\int_0^3 \sqrt{1 + \frac{1}{2(16 + x^2)}}\ dx$

B. $\displaystyle\int_0^3 \sqrt{1 + \frac{x^2}{4(16 + x^2)}}\ dx$

C. $\displaystyle\int_0^3 \sqrt{1 - \frac{x^2}{16 + x^2}}\ dx$

D. $\displaystyle\int_0^3 \sqrt{1 + \frac{x^2}{16 + x^2}}\ dx$

E. $\displaystyle\int_0^3 \sqrt{1 - \frac{1}{2(16 + x^2)}}\ dx$

7. (5 points) Which integral computes the **surface area** of the surface formed by rotating $y = 2 + 3x^2$ from $x = 0$ to $x = 5$ about the y-axis?

A. $\displaystyle\int_0^5 2\pi(2 + 3x^2)\sqrt{1 + 36x^2}\ dx$

B. $\displaystyle\int_0^5 2\pi x\sqrt{1 + (2 + 3x^2)^2}\ dx$

C. $\displaystyle\int_0^5 2\pi x\sqrt{1 + 36x^2}\ dx$

D. $\displaystyle\int_0^5 2\pi\sqrt{1 + (2 + 3x^2)^2}\ dx$

E. $\displaystyle\int_0^5 2\pi(2 + 3x^2)\sqrt{1 + 6x^2}\ dx$

8. (5 points) Determine the center of mass, $\overline{x}$, if point-masses m_1 and m_2 are 100g at $x = -2$ and 500g at $x = 5$, respectively.

   A. $\frac{600}{7}$

   B. $\frac{23}{6}$

   C. $\frac{27}{6}$

   D. $\frac{7}{600}$

   E. $\frac{6}{23}$

9. (5 points) Find the total mass of the region (lamina) bound by $y = -\frac{1}{2}x^2 + 2$ and $y = 0$ on $[0, 2]$, assuming $\rho = 2$.

   A. $\frac{8}{3}$

   B. 16

   C. $\frac{16}{3}$

   D. 8

   E. 4

10. (5 points) Eliminate the parameter t to find the Cartesian equation for $x = \frac{1}{t^2+1}$ and $y = \frac{t^2}{t^2+1}$.

   A. $y = 1 - x$

   B. $y = \sqrt{\frac{1}{x} - 1}$

   C. $y = \frac{2}{x} - x - 1$

   D. $x = \frac{1}{y+1}$

   E. $x = \frac{2}{y-1}$

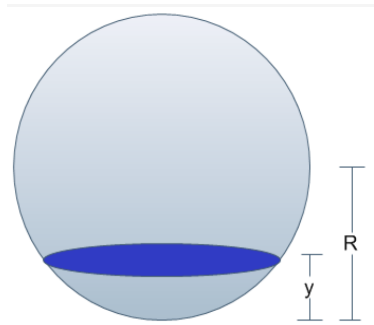
Free Response Questions

11. (10 points) The average value of the function $f(x) = 2x$ on the interval $[0, c]$ is equal to 6. Find the value of c.

Solution:

$$\begin{aligned}
\frac{1}{b-a}\int_a^b f(x)\ dx &= 6 \\
\frac{1}{c-0}\int_0^c 2x\ dx &= 6 \\
\frac{1}{c}x^2\Big|_0^c &= 6 \\
\frac{1}{c}c^2 &= 6 \\
c &= 6
\end{aligned}$$

12. (10 points) Find the volume of liquid needed to fill a sphere of radius R to height $\dfrac{R}{2}$.



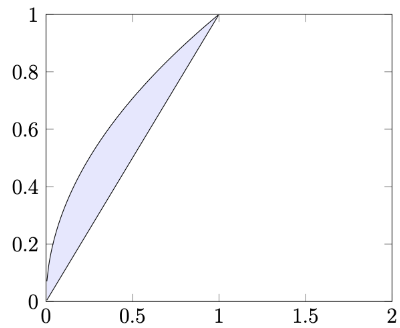
Solution: For each value of y, the cross-section is a circle with radius x. (How long is x? Use the Pythagorean Theorem.)

$$x^2 - (R-y)^2 = R^2 \implies x^2 = R^2 + (R-y)^2 \implies x^2 = 2Ry - y^2$$

The area of the cross-section circles are $\pi r^2 = \pi x^2 = \pi(2Ry - y^2)$. So:

$$V = \int_0^{R/3} \pi(2Ry - y^2)\, dy = \pi[Ry^2 - \frac{1}{3}y^3]\Big|_0^{R/3} = \pi\left(\frac{8R^3}{81}\right)$$

13. The region bound between the functions $y = \sqrt{x}$ and $y = x$ is shown below. Let V be the solid obtained by rotating this region around the line $x = 3$.



(a) (3 points) Set up but ***do not evaluate*** the integral that computes the volume of V using the disk/washer method.

Solution:

$$V = \int_0^1 \pi[(3 - y^2)^2 - (3 - y)^2]\ dy$$

(b) (3 points) Set up but ***do not evaluate*** the integral that computes the volume of V using the shell method.

Solution:

$$V = \int_0^1 2\pi(3 - x)(\sqrt{x} - x)\ dx$$

(c) (4 points) Find the volume of V by evaluating one of the integrals you found in parts (a) or (b).

Solution:

$$\begin{aligned} V = \int_0^1 2\pi(3 - x)(\sqrt{x} - x)\ dx &= \int_0^1 2\pi(3x^{1/2} - 3x - x^{3/2} + x^2)\ dx \\ &= 2\pi(2x^{3/2} - \frac{3}{2}x^2 - \frac{2}{5}x^{5/2} + \frac{1}{3}x^3)\Big|_0^1 \\ &= 2\pi(2 - \frac{3}{2} - \frac{2}{5} + \frac{1}{3}) = \frac{13\pi}{15} \end{aligned}$$

14. Let S be the region bounded by $y = 4 - x^2$ on the interval $[0, 2]$. Assume S has uniform density $\rho = 3$.

(a) (2 points) Find the total mass m for S.

Solution:

$$m = 3\int_0^2 (4 - x^2)\ dx = 3(4x - \frac{1}{3}x^3)\Big|_0^2 = 16$$

(b) (3 points) Find the moment M_x for S.

Solution:

$$M_x = \frac{3}{2}\int_0^2 (4-x^2)^2\ dx = \frac{3}{2}\int_0^2 (16-8x^2+x^4)\ dx = \frac{3}{2}(16x-\frac{8}{3}x^3+\frac{1}{5}x^5)\Big|_0^2 = \frac{128}{5}$$

(c) (3 points) Find the moment M_y for S.

Solution:

$$M_y = 3\int_0^2 x(4 - x^2)\ dx = 3\int_0^2 (4x - x^2)\ dx = 3(2x^2 - \frac{1}{3}x^3)\Big|_0^2 = 12$$

(d) (2 points) Find the center of mass of S.

Solution:

$$(\bar{x}, \bar{y}) = \left(\frac{M_y}{m}, \frac{M_x}{m}\right) = \left(\frac{12}{16}, \frac{128/5}{16}\right) = \left(\frac{3}{4}, \frac{8}{5}\right)$$

15. Let C be the curve defined by the graph of $f(x) = 4x^2 - 1$ on the interval $[0, 2]$.

(a) (4 points) Set up but ***do not evaluate*** the integral which gives the length of C.

Solution: We have that $f'(x) = 8x$, so

$$L = \int_0^2 \sqrt{1 + (8x)^2}\ dx = \int_0^2 \sqrt{1 + 64x^2}\ dx$$

(b) (6 points) Compute the area of the surface obtained by revolving C around the y-axis.

Solution:

$$\begin{aligned} SA = 2\pi r L &= 2\pi \int_0^2 x\sqrt{1 + 64x^2}\ dx \\ u &= 1 + 64x^2, \quad du = 128x\ dx \\ &= \frac{2\pi}{128} \int_0^2 u^{1/2}\ du \\ &= \frac{\pi}{64} \left(\frac{2}{3} u^{3/2} \right) \Big|_0^2 \\ &= \frac{\pi}{64} \left(\frac{2}{3} (1 + 64x^2)^{3/2} \right) \Big|_0^2 \\ &= \frac{\pi}{96} (257^{3/2} - 1) \end{aligned}$$