Record the correct answer to the following problem on the front page of this exam.

(1) Which one of the following differential equations corresponds to the given slope field?

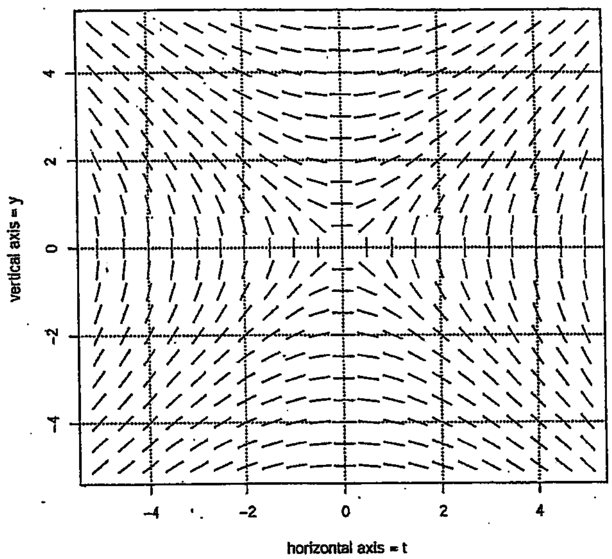
A) $\dot{y} = \dfrac{t-y}{t}$

B) $\dot{y} = \dfrac{t}{y}$

C) $\dot{y} = ty$

D) $\dot{y} = \dfrac{y+t}{t}$

E) $\dot{y} = \dfrac{y+t}{y-t}$



(2) Given that $y(t)$ is a solution to the logistics equation $\dot{y} = 5y(4-y)$ with $y(0) = 2$, which statement best describes the behavior of this solution?

A) $y(t)$ decreases to $\frac{1}{4}$ as $t \to \infty$.

B) $y(t)$ increases to 1 as $t \to \infty$.

C) $y(t)$ decreases to $-\infty$ as $t \to \infty$.

D) $y(t)$ increases to 4 as $t \to \infty$.

E) $y(t)$ remains constant at 2 as $t \to \infty$.

Record the correct answer to the following problem on the front page of this exam.

(3) Which of the following infinite series is not a geometric series?

A) $\sum_{n=0}^{\infty} \frac{1}{5^n}$

B) $\sum_{n=0}^{\infty} \frac{4^n}{28^n}$

C) $\sum_{n=0}^{\infty} \frac{1}{n^6}$

D) $\sum_{n=0}^{\infty} \pi^{-n}$

E) They are all geometric series.

(4) Which of the following is the integrating factor for $y' + 2xy = x^2$?

A) x^2

B) $\ln(x^2)$

C) e^x

D) e^{x^2}

E) The equation does not have an integrating factor since it is not linear.

Record the correct answer to the following problem on the front page of this exam.

(5) Find the sum of the telescoping series

$$\sum_{n=1}^{\infty}\left(\frac{1}{n+1}-\frac{1}{n+2}\right)$$

A) $\frac{1}{2}$

B) $\frac{1}{3}$

C) The series has no sum since it diverges.

D) The series converges but the sum can't be determined.

E) 1

(6) What is the radius of convergence for the power series

$$\sum_{n=0}^{\infty}(-2x)^n$$

A) 1

B) 2

C) 0

D) ∞

E) $\frac{1}{2}$

$|-2x| < 1$

$|x| < \frac{1}{2}$

Free Response Questions: Show your work!

(7) Solve the 1^{st} order linear initial value problem

$$xy' - 3y = x^3$$

with $y(1) = 2$.

$$y' - \frac{3}{x} y = x^2$$

initial value $x = 1$ so $x > 0$

$A = -\frac{3}{x}$ $\qquad \int -\frac{3}{x}\,dx = -3\ln|x|$

$$= \ln x^{-3}$$

$$e^{\int A\,dx} = e^{\ln x^{-3}} = x^{-3}$$

$$\Rightarrow \quad x^3 y' - 3x^2 y = x^{-1}$$

$$(x^3 y)' = x^{-1}$$

$$x^3 y = \int x^{-1}\,dx = \ln(x) + C$$

$$x = 1, \quad y = 2$$

$$2 = 1\cancel{n}(1) + c = c$$

$$x^3 y = \ln(x) + c$$

$$y = x^{-3}(\ln(x) + c)$$

Free Response Questions: Show your work!

(8) Given the alternating series

$$\sum_{n=0}^{\infty}(-1)^n a_n = \sum_{n=0}^{\infty}(-1)^n \frac{1}{n+1}$$

(a) Determine if the series converges absolutely, converges conditionally, or diverges.

$a_n = \frac{1}{n+1} \to 0$ as $n \to \infty$

$\Rightarrow$ by Leibniz test series converges

but $\sum_{n=0}^{\infty} \left|(-1)^n \frac{1}{n+1}\right|$

$= \sum_{n=0}^{\infty} \frac{1}{n+1} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots$

= harmonic series diverges

$\sum_{n=0}^{\infty}(-1)^n \frac{1}{n+1}$ converges

$\sum_{n=0}^{\infty} \left|(-1)^n \frac{1}{n+1}\right|$ diverges

conditionally convergent

(b) Given the fact that $|S - S_N| \leq a_{(N+1)}$, where S_N is the N^{th} partial sum of S, what is the smallest N such that $|S - S_N| \leq 10^{-2}$?

Note: In book partial sum S_N is $\sum_{n=1}^{N} a_n$

so formula should be $|S - S_N| \leq a_N$

since series starts at $n=0$.

$|S - S_N| \leq a_N = \frac{1}{N+1} < 10^{2}$

$\Rightarrow N + 1 > 100$

$N > 99$

However here for this formula to work

$S_N = \sum_{n=0}^{N} a_n$

$|S - S_N| < \frac{1}{(N+1)+1} = \frac{1}{N+2} < 10^{2}$

$N + 2 > 100$

$N > 98$

Free Response Questions: Show your work!

(9) Use the ratio test to determine if the series

$$\sum_{n=0}^{\infty} \frac{n!(2^n)}{(n+1)!}$$

converges or not.

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \frac{(n+1)!\, 2^{n+1}}{(n+2)!} \cdot \frac{(n+1)!}{n!\, 2^n}$$

$$= \lim_{n\to\infty} \frac{(n+1)}{(n+2)} \cdot 2 = 2 \lim_{n\to\infty} \frac{n+1}{n+2} = 2(1) = 2 > 1$$

by ratio test

$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| > 1 \Rightarrow$ series diverges

Free Response Questions: Show your work!

(10) Use the integral test and the comparison test to decide whether

$$\sum_{n=0}^{\infty} ne^{-n^3}$$

converges or not. (Hint: $ne^{-n^3} \leq n^2 e^{-n^3}$)

$$\int_1^{\infty} x^2 e^{-x^3}\,dx = \lim_{w\to\infty} \int_1^{w} -\frac{1}{3} e^{u}\,du$$

$$u = -x^3 \quad du = -3x^2\,dx$$

$$= -\frac{1}{3} \lim_{w\to\infty} e^{-x^3}\Big|_1^{w} = -\frac{1}{3}\left(\lim_{w\to\infty} e^{-w^3} - e^{-1}\right)$$

$$= -\frac{1}{3}\left(0 - \frac{1}{e}\right) = \frac{1}{3e}$$

$\Rightarrow$ integral converges

$\Rightarrow$ $\sum_{n=0}^{\infty} n^2 e^{-n^3}$ converges by integral test

$\Rightarrow$ $\sum_{n=0}^{\infty} ne^{-n^3}$ converges by comparison test.

Free Response Questions: Show your work!

(11) Use the limit comparison test with the two series $\sum_{n=1}^{\infty} \sin(1/n)$ and $\sum_{n=1}^{\infty} \frac{1}{n}$ to determine if $\sum_{n=1}^{\infty} \sin(1/n)$ converges, diverges, or state that the test is inconclusive.

$$\lim_{n\to\infty} \left|\frac{a_n}{b_n}\right| = \lim_{n\to\infty} \frac{\sin(\frac{1}{n})}{\frac{1}{n}} \to \frac{0}{0}$$

L'Hopital applies here

$$\lim_{n\to\infty} \frac{\sin(\frac{1}{n})'}{(\frac{1}{n})'} = \lim_{n\to\infty} \frac{\cos(\frac{1}{n})\cancel{(-\frac{1}{n^2})}}{\cancel{(-\frac{1}{n^2})}} = \lim_{n\to\infty} \cos(\tfrac{1}{n}) = 1$$

by limit comparison $\sum_{n=1}^{\infty} \sin(\frac{1}{n})$ and $\sum_{n=1}^{\infty} \frac{1}{n}$ converge or diverge together

$\sum_{n=1}^{\infty} \frac{1}{n}$ diverges (harmonic series)

$\Rightarrow \sum_{n=1}^{\infty} \sin(\frac{1}{n})$ diverges