Exam 3

Name: ______________________ Section and/or TA: ______________

Last Four Digits of Student ID: ________

Do not remove this answer page — you will return the whole exam. You will be allowed two hours to complete this test. No books or notes may be used except for a one-page sheet of formulas and facts. You may use a graphing calculator during the exam, but NO calculator with a Computer Algebra System (CAS) or a QWERTY keyboard is permitted. Absolutely no cell phone use during the exam is allowed.

The exam consists of 10 multiple choice questions and 5 free response questions. Record your answers to the multiple choice questions on this page by filling in the circle corresponding to the correct answer.

Show all work to receive full credit on the free response problems. Unsupported answers on free response problems will receive *no credit*.

Multiple Choice Questions

1	A	B	C	D	E	**6**	A	B	C	D	E
2	A	B	C	D	E	**7**	A	B	C	D	E
3	A	B	C	D	E	**8**	A	B	C	D	E
4	A	B	C	D	E	**9**	A	B	C	D	E
5	A	B	C	D	E	**10**	A	B	C	D	E

SCORE

Multiple Choice	11	12	13	14	15	Total Score
50	10	10	10	10	10	100

Multiple Choice Questions

1. Find the average value of $\sin^2 x$ over the interval $[0, 2\pi]$. It may help to know that $\sin^2 x = \frac{1}{2}(1 - \cos(2x))$.

   A. 0

   B. $1/2$

   C. π

   D. $-\pi$

   E. $-1/2$

2. A solid is formed by rotating the area between the curves $y = x$ and $y = x^2$ between $x = 0$ and $x = 1$ about the x-axis. Which of the following integrals correctly computes the volume of the resulting solid?

   A. $\int_0^1 2\pi x(x - x^2)\,dx$

   B. $\int_0^1 2\pi x(x^2 - x^4)\,dx$

   C. $\int_0^1 \pi(x - x^2)^2\,dx$

   D. $\int_0^1 2\pi\left(x^2 - x^4\right)\,dx$

   E. $\int_0^1 \pi\left(x^2 - x^4\right)\,dx$

3. A solid is made by rotating the region bounded by the curves $y = x^3$, $y = 0$, $x = 1$, and $x = 2$ about the y-axis. Which of the following integrals correctly computes the volume of the resulting region?

   A. $\int_1^2 \pi x^6\,dx$

   B. $\int_0^1 \pi x^6\,dx$

   C. $\int_0^1 2\pi x^4\,dx$

   D. $\int_1^2 2\pi x^4\,dx$

   E. $\int_1^2 2\pi x\sqrt{1 + 9x^4}\,dx$

4. Which integral correctly computes the length of the curve $y = \ln(\cos x)$ between $x = 0$ and $x = \pi/3$?

   A. $\displaystyle\int_0^{\pi/3} \sec x \, dx$

   B. $\displaystyle\int_0^{\pi/4} \tan x \, dx$

   C. $\displaystyle\int_0^{\pi/3} \tan x \, dx$

   D. $\displaystyle\int_0^{\pi/3} \sqrt{1 + (\ln(\cos x))^2} \, dx$

   E. $\displaystyle\int_0^{\pi/3} 2\pi \ln(\cos x) \sqrt{1 + \tan^2 x} \, dx$

5. Find the area of the surface obtained by rotating the curve $y = x^3$ about the x-axis between $x = 0$ and $x = 1$.

   A. $\dfrac{\pi}{18}\left(10\sqrt{10} - 1\right)$

   B. $\dfrac{\pi}{27}\left(10\sqrt{10} - 1\right)$

   C. $\dfrac{\pi}{36}\left(10\sqrt{10} - 1\right)$

   D. $\dfrac{2\pi}{27}\left(10\sqrt{10} - 1\right)$

   E. $2\pi(10\sqrt{10} - 2)$

6. Three equal masses are placed at the points $P_1(-3,1)$, $P_2(0,1+\sqrt{3})$, and $P_3(0,1-\sqrt{3})$. Where is their center of mass?

   A. $(-1,1)$

   B. $(0,0)$

   C. $(0,1)$

   D. $(-1,0)$

   E. $(1,0)$

7. Which of the following integrals correctly computes the volume of a solid whose base is the circle $x^2+y^2=9$ and whose cross-sections perpendicular to the x-axis are squares?

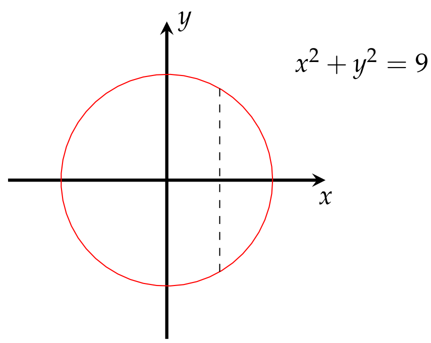
   A. $\int_0^3 4(9-x^2)\,dx$

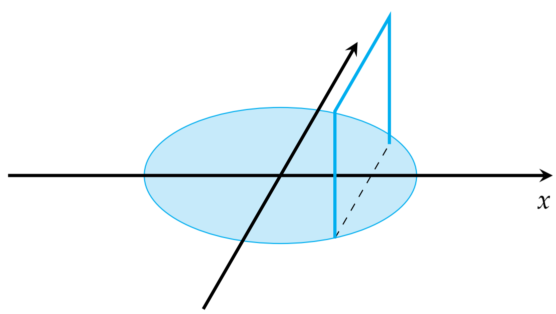
   B. $\int_{-3}^3 4\sqrt{9-x^2}\,dx$

   C. $\int_{-3}^3 4(9-x^2)\,dx$

   D. $\int_0^3 2\sqrt{9-x^2}\,dx$

   E. $\int_{-3}^3 8(9-x^2)^{3/2}\,dx$





8. Find the equation of the tangent line to the parametric curve $x = t^2 - t, y = t^2 + t + 1$ at the point $(0, 3)$.

    A. $y = 3x$

    B. $y = 3x - 3$

    C. $y = 3x + 3$

    D. $y = 2x + 1$

    E. $y = \frac{1}{3}x + 3$

9. Which of these is the polar description of the curve $x^2 + y^2 = 2cx$? Here c is constant.

    A. $r = c\cos\theta$

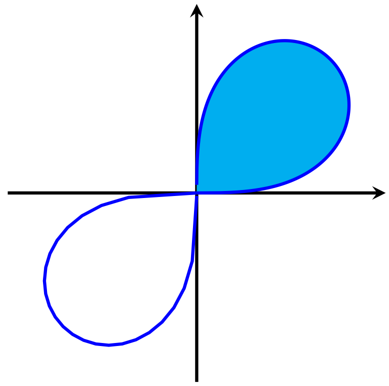
    B. $r = 2c\cos\theta$

    C. $r^2 = 2c\cos\theta$

    D. $r^2 = c\cos\theta$

    E. $r = c\sin\theta$

10. Find the area of shaded region.



$r^2 = \sin 2\theta$

A. $1/4$

B. $1/2$

C. $\pi/4$

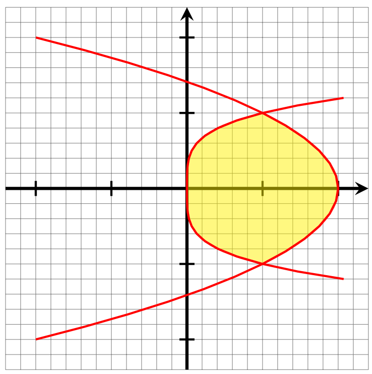
D. $\pi/2$

E. 1

Free Response Questions

11. The goal of this problem is to find the volume of the solid obtained by rotating the region bounded by the curves $x = 2 - y^2$, $x = y^4$ about the y-axis.

(a) (4 points) Find the points of intersection between these two curves. Graph the region on the set of axes provided and label the x- and y-coordinates of the intersection points.



Solution: To find the points of intersection, solve

$$\begin{aligned} 2 - y^2 &= y^4 \\ y^4 + y^2 - 2 &= 0 \\ (y^2 - 1)(y^2 + 2) &= 0 \\ y &= \pm 1 \end{aligned}$$

The points of intersection are $(1, -1)$ and $(1, 1)$.

(b) (4 points) Set up an integral for the volume of the solid.

Solution: Using the washer method, we see that the inner radius is y^4 and the outer radius is $2 - y^2$. This leads to

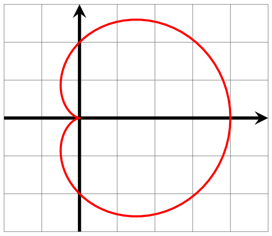
$$V = \int_{-1}^{1} \pi \left((2 - y^2)^2 - y^8 \right) \, dy.$$

(c) (2 points) Compute the integral.

Solution: We compute (using symmetry)

$$\begin{aligned} V &= 2 \int_0^1 \pi \left(4 - 4y^2 + y^4 - y^8 \right) \, dy \\ &= 2\pi \left[4y - \frac{4}{3}y^3 + \frac{1}{5}y^5 - \frac{1}{9}y^9 \right]_{y=0}^{y=1} \\ &= \frac{248\pi}{45} \simeq 17.314 \end{aligned}$$

12. This problem concerns the polar curve $r = 2(1 + \cos\theta)$, whose graph is shown below.



(a) (5 points) Find the arc length of the curve. It may help to remember the identity

$$2\cos^2\frac{\theta}{2} = (1 + \cos\theta)\,.$$

Solution: We compute

$$\begin{aligned}
L &= \int_0^{2\pi} \sqrt{4(1+\cos\theta)^2 + 4\sin^2\theta}\, d\theta \\
&= 2\int_0^{\pi} 2\sqrt{2 + 2\cos\theta}\, d\theta \\
&= 4\sqrt{2}\int_0^{\pi} \sqrt{1+\cos\theta}\, d\theta \\
&= 4\int_0^{\pi} 2\cos\frac{\theta}{2}\, d\theta \\
&= \left[16\sin\frac{\theta}{2}\right]_0^{\pi} \\
&= 16
\end{aligned}$$

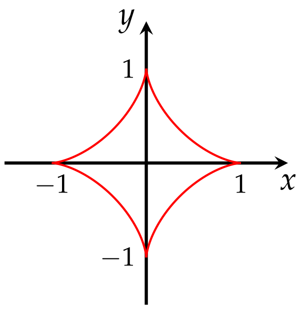
(b) (5 points) Find the area enclosed by the curve. It may be helpful to know that

$$\cos^2\theta = \frac{1}{2}(1 + \cos 2\theta).$$

Solution: Compute

$$\begin{aligned} A &= \frac{1}{2}\int_0^{2\pi} 4(1+\cos\theta)^2\,d\theta \\ &= 2\int_0^{2\pi}\left(1+2\cos\theta+\cos^2\theta\right)\,d\theta \\ &= 2\int_0^{2\pi}\left(1+2\cos\theta+\frac{1}{2}+\frac{1}{2}\cos(2\theta)\right)\,d\theta \\ &= 2\left[\theta+2\sin\theta+\frac{\theta}{2}+\frac{1}{4}\sin(2\theta)\right]_0^{2\pi} \\ &= 6\pi \end{aligned}$$

13. This problem concerns the *astroid curve*



$$x = \cos^3 t, \quad y = \sin^3 t, \quad 0 \le t \le 2\pi.$$

(a) (4 points) Find the slope of tangent line to the asteroid curve in terms of the parameter t.

Solution:

$$\begin{aligned} \frac{dy}{dx} &= \frac{dy/dt}{dx/dt} \\ &= \frac{-3\sin^2 t\cos t}{3\cos^2 t\sin t} \\ &= -\frac{\sin t}{\cos t} \\ &= -\tan t \end{aligned}$$

(b) (2 points) Use calculus to determine the (x, y) coordinates of points having vertical tangents.

Solution: Vertical tangents occur when $\cos t = 0$, that is, when $t = \pi/2$, $3\pi/2$. The corresponding points are $(x, y) = (0, 1)$ and $(x, y) = (0, -1)$.

(c) (2 points) Use calculus to determine the (x, y) coordinates of points having horizontal tangents.

Solution: Horizontal tangents occur when $\sin t = 0$, that is, when $t = 0$ or $t = \pi$. The corresponding points are $(1, 0)$ and $(-1, 0)$.

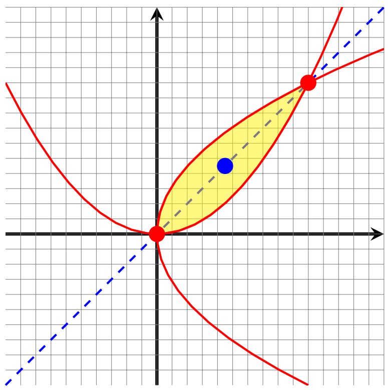
(d) (2 points) Give the (x, y) coordinates of points whose tangent lines have slope -1.

Solution: $\tan t = 1$ for $t = \pi/4$ and $t = 5\pi/4$, corresponding to

$$\left(\left(1/\sqrt{2}\right)^3, \left(1/\sqrt{2}\right)^3\right), \qquad \left(-\left(1/\sqrt{2}\right)^3, -\left(1/\sqrt{2}\right)^3\right)$$

14. The goal of this problem is to find the centroid of the region bounded by the curves $y = x^2$ and $x = y^2$.

(a) (2 points) Sketch the curves on the axes provided, and find their points of intersection.



Solution: To find the points of intersection, substitute $x = y^2$ into $y = x^2$ to obtain the equation $y = y^4$ or $y(y^3 - 1) = 0$. Hence, either $y = 0$ or $y = 1$. These solutions correspond to the points $(0, 0)$ and $(1, 1)$. The blue dot shows the center of mass at $(9/20, 9/20)$ (see solutions below. The blue dashed line is the line $y = x$.

(b) (3 points) Set up integrals for the area A and the moments M_x and M_y about the x- and y-axes.

Solution:

$$A = \int_0^1 \left(\sqrt{x} - x^2\right)\, dx$$

$$M_x = \frac{1}{2}\int_0^1 \left[x - x^4\right]\, dx$$

$$M_y = \int_0^1 x\left(\sqrt{x} - x^2\right)\, dx$$

(c) (3 points) Evaluate the integrals you found in part (b).

Solution:

$$A = \left[\frac{2}{3}x^{3/2} - \frac{x^3}{3}\right]_0^1 = \frac{1}{3}$$

$$M_x = \frac{1}{2}\left[\frac{x^2}{2} - \frac{x^5}{5}\right]_0^1 = \frac{3}{20}$$

$$M_y = \left[\frac{2}{5}x^{5/2} - \frac{x^4}{4}\right]_0^1 = \frac{3}{20}$$

(d) (2 points) Using your answer from part (c), compute the centroid of the region.

Solution:

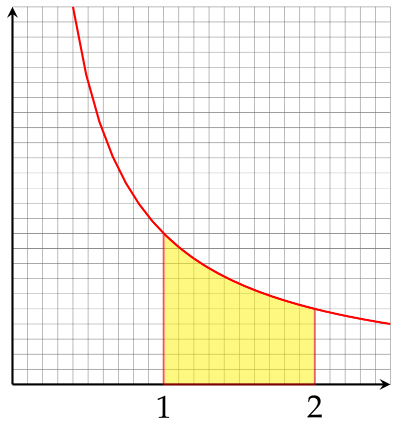
$$\overline{x} = 3\frac{3}{20} = \frac{9}{20}$$

$$\overline{y} = 3\frac{3}{20} = \frac{9}{20}$$

15. Set up, but do not evaluate, integrals for the volume generated by rotating the region bounded by the given curves about the specified axis

(a) (5 points) $xy = 1$, $x = 1$, $x = 2$; about x-axis.

Solution:

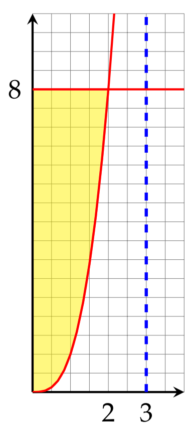


By the disc method we have

$$V = \int_1^2 \pi \left(\frac{1}{x}\right)^2 dx$$

(b) (5 points) $y = x^3$, $y = 8$, $x = 0$; about $x = 3$

Solution:



We'll use the shell method. For $0 \le x \le 2$ the shell at x has height $8 - x^3$ and radius $3 - x$. Hence

$$\begin{aligned} V &= \int_0^2 2\pi(3-x)(8-x^3)\,dx \\ &= 2\pi \int_0^2 (24 - 8x - 3x^3 + x^4)\,dx \\ &= 2\pi \left[24x - 4x^2 - \frac{3}{4}x^4 + \frac{x^5}{5}\right]_0^2 \\ &= \frac{264\pi}{5} \end{aligned}$$