Exam 3

Name: ______________________________ Section: ______________

Do not remove this answer page — you will return the whole exam. You will be allowed two hours to complete this test. If you find you need scratch paper during the exam, please ask. You may not use any of your own notes, paper or anything else not provided. You may use a graphing calculator during the exam, but NO calculator with a Computer Algebra System (CAS). Absolutely no communication device use during the exam is allowed.

The exam consists of 10 multiple choice questions and 5 free response questions. Record your answers to the multiple choice questions on this page by filling in the circle corresponding to the correct answer.

Show all work to receive full credit on the free response problems. It will also help you check your answers to show work on multiple choice problems.

Multiple Choice Questions

1 (A) (B) (C) (D) (E)

2 (A) (B) (C) (D) (E)

3 (A) (B) (C) (D) (E)

4 (A) (B) (C) (D) (E)

5 (A) (B) (C) (D) (E)

6 (A) (B) (C) (D) (E)

7 (A) (B) (C) (D) (E)

8 (A) (B) (C) (D) (E)

9 (A) (B) (C) (D) (E)

10 (A) (B) (C) (D) (E)

Multiple Choice	11	12	13	14	15	Total Score
50	10	10	10	10	10	100

This page is intentionally left blank for scratch work.

Multiple Choice Questions

1. (5 points) Determine the average value of $f(x) = 2\cos(x)$ on $[0, \pi]$.

   A. 1

   B. 0

   C. π

   D. $2/\pi$

   E. 2π

2. (5 points) Set up the integral which computes the volume of the solid whose base is the region in Quadrant 1 bound by $y = x^2$ and $y = 16$ whose cross-sections perpendicular to the $x-$axis are squares.

   A. $\int_0^4 (16 - x^2)^2 \, dx$

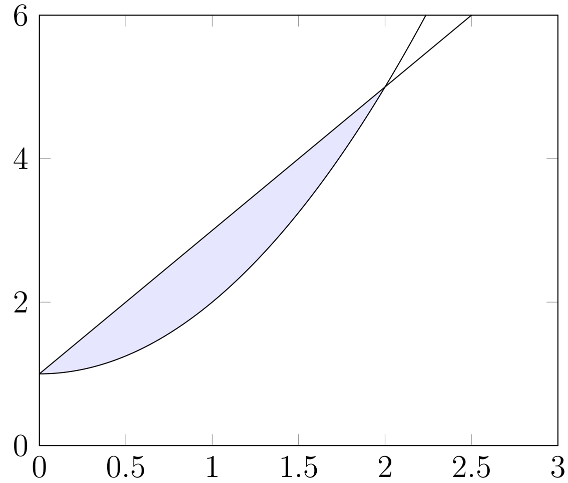
   B. $\int_0^2 (16 - x^2)^2 \, dx$

   C. $\int_0^4 (x^2 - 16)^2 \, dx$

   D. $\int_0^2 \pi(x^2 - 16)^2 \, dx$

   E. $\int_0^4 \pi(x^2 - 16)^2 \, dx$

3. (5 points) The region R is bound between $y = 2x + 1$ and $y = x^2 + 1$, shown below.



Consider the solid formed by rotating R around the **x−axis**. Which integral computes the volume of this solid using the **washer method**?

A. $\int_1^5 2\pi((x^2+1) - (2x+1))\,dx$

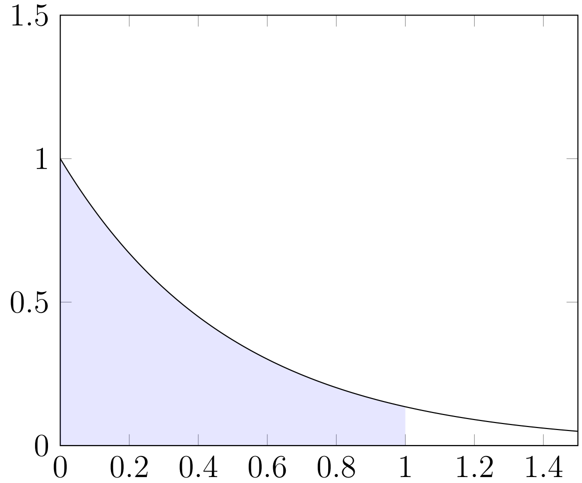
B. $\int_1^5 2\pi x((2x+1) - (x^2+1))\,dx$

C. $\int_0^2 \pi((x^2+1)^2 - (2x+1)^2)\,dx$

D. $\int_0^2 \pi((2x+1)^2 - (x^2+1)^2)\,dx$

E. $\int_0^2 2\pi((2x+1)^2 - (x^2+1)^2)\,dx$

4. (5 points) The region R is bound between $y = e^{-2x}$, $y = 0$, $x = 0$, and $x = 1$.



Consider the solid formed by rotating R around the **y−axis**. Which integral computes the volume of this solid using the **shell method**?

A. $\int_0^1 \pi e^{4x}\, dx$

B. $\int_0^1 \pi x e^{-2x}\, dx$

C. $\int_0^1 x e^{4x}\, dx$

D. $\int_0^1 2\pi e^{-2x}\, dx$

E. $\int_0^1 2\pi x e^{-2x}\, dx$

5. (5 points) Which integral computes the length of $y = \ln(\cos(x))$ from $x = 0$ to $x = \pi/3$?

   A. $\int_0^{\pi/3} \sec(x)\ dx$

   B. $\int_0^{\pi/3} \csc(x)\ dx$

   C. $\int_0^{\pi/3} \cos(x)\ dx$

   D. $\int_0^{\pi/3} \sin(x)\ dx$

   E. $\int_0^{\pi/3} \tan(x)\ dx$

6. (5 points) Which integral computes the area of the surface obtained by rotating $y = x^4$, $1 \leq x \leq 3$ around the $y-$axis?

   A. $\int_1^3 2\pi x\sqrt{1+4x^8}\ dx$

   B. $\int_1^3 2\pi x^4\sqrt{1+4x^8}\ dx$

   C. $\int_1^3 2\pi x\sqrt{1+16x^6}\ dx$

   D. $\int_1^3 2\pi x^4\sqrt{1+16x^6}\ dx$

   E. $\int_1^3 2\pi x\sqrt{1+4x^6}\ dx$

7. (5 points) Find the center of mass for the system of particles of masses 4, 2, 5, and 1 located at the coordinates $(1, 2)$, $(-3, 4)$, $(2, -2)$, and $(4, 0)$, respectively.

   A. $(0.5, 1)$

   B. $(1, 0.5)$

   C. $(0.125, 0.25)$

   D. $(12, 6)$

   E. $(6, 12)$

8. (5 points) Find the mass of the region between $y = \dfrac{1}{x^2}$ and $y = \sqrt{x}$ on $[1, 4]$, assuming density $\rho = 2$.

   A. $\dfrac{45}{11}$

   B. $\dfrac{41}{12}$

   C. $\dfrac{42}{5}$

   D. $\dfrac{45}{12}$

   E. $\dfrac{47}{6}$

9. (5 points) Write the parametric equations $x(t) = \sqrt{2t+1}$, $y(t) = 2t - 4$ in Cartesian form.

A. $x = 2y + 4$

B. $y = 2x + 4$

C. $y = \sqrt{x} + 4$

D. $x = \sqrt{y+5}$

E. $y = \sqrt{x+5}$

10. (5 points) Let C be the curve defined by $x(t) = 2t^2 + t - 1$ and $y(t) = t^3$ for t in $[0, 2]$. Which integral computes the area of the surface formed by revolving C around the $x-$axis?

A. $\int_0^2 2\pi(2t^2 + t - 1)\sqrt{(4t+1)^2 + (3t^2)^2}\ dt$

B. $\int_0^2 2\pi(2t^2 + t - 1)\sqrt{(2t^2 + t - 1)^2 + (t^3)^2}\ dt$

C. $\int_0^2 2\pi t^3\sqrt{(4t+1)^2 + (3t^2)^2}\ dt$

D. $\int_0^2 \pi(2t^2 + t - 1)\sqrt{(4t+1)^2 + (3t^2)^2}\ dt$

E. $\int_0^2 \pi t^3\sqrt{(2t^2 + t - 1)^2 + (t^3)^2}\ dt$

Free Response Questions

11. Let R be the region in Quadrant 1 bound between $y = x^2$ and $y = 2x$. Let V be the solid obtained by rotating R around the line $x = 2$.

(a) (3 points) Set up (but do not evaluate) the integral that computes the volume of V using the **washer method**.

Solution:

$$V = \int_0^4 \pi[(2 - \frac{1}{2}y)^2 - (2 - \sqrt{y})^2]\ dy$$

(b) (3 points) Set up (but do not evaluate) the integral that computes the volume of V using the **shell method**.

Solution:

$$V = \int_0^2 2\pi(2 - x)(2x - x^2)\ dx$$

(c) (4 points) Using either of your previous integrals, compute the volume of V.

Solution:

$$\begin{aligned} V = \int_0^2 2\pi(2 - x)(2x - x^2)\ dx &= \int_0^2 2\pi(4x - 4x^2 + x^3)\ dx \\ &= 2\pi[2x^2 - \frac{4}{3}x^3 + \frac{1}{4}x^4]\Big|_0^2 \\ &= \frac{8\pi}{3} \end{aligned}$$

12. (a) (4 points) Determine the arc length of $y = 4\sqrt{x^3}$ on $[0, 1]$.

Solution: Note that, since $y = 4(x^{3/2})$, we have $y' = 6x^{1/2}$. Thus,

$$\begin{aligned} L = \int_0^1 \sqrt{1 + (6x^{1/2})^2}\ dx &= \int_0^1 \sqrt{1 + 36x}\ dx \\ &= \frac{1}{36} \cdot \frac{2}{3} \cdot (1 + 36x)^{3/2} \Big|_0^1 \\ &= \frac{1}{54}[37^{3/2} - 1] \end{aligned}$$

(b) (6 points) Determine the surface area of the shape formed by rotating $y = 2x^3$ on $[0, 1]$ around the $x-$axis.

Solution: Note that, since $y = 2x^3$, we have $y' = 6x^2$. Thus,

$$\begin{aligned} SA = 2\pi \int_0^1 (2x^3)\sqrt{1 + (6x^2)^2}\ dx &= 2\pi \int_0^1 (2x^3)\sqrt{1 + 36x^4}\ dx \\ &\quad [u = 1 + 36x^4, du = 144x^3 dx] \\ &= \frac{4\pi}{144} \int_0^1 u^{1/2}\ du \\ &= \frac{\pi}{36} \cdot \frac{2}{3} \cdot u^{3/2} \Big|_0^1 \\ &= \frac{\pi}{54} (1 + 36x^4)^{3/2} \Big|_0^1 \\ &= \frac{\pi}{54}[37^{3/2} - 1] \end{aligned}$$

13. Let R be the region bound by $y = 4 - x^2$ and $y = 2 - x$ for $0 \leq x \leq 2$. Assume R has uniform density $\rho = 3$.

(a) (2 points) Find the total mass m of R.

Solution:

$$\begin{aligned} m &= 3\int_0^2 (4-x^2)-(2-x)\ dx \\ &= 3\int_0^2 2+x-x^2\ dx \\ &= 3[2x+\frac{1}{2}x^2-\frac{1}{3}x^3]\Big|_0^2 \\ &= 10 \end{aligned}$$

(b) (3 points) Find the moment M_x of R.

Solution:

$$\begin{aligned} M_x &= \frac{3}{2}\int_0^2 (4-x^2)^2-(2-x)^2\ dx \\ &= \frac{3}{2}\int_0^2 x^4-9x^2+4x+12\ dx \\ &= \frac{3}{2}[\frac{1}{5}-3x^3+2x^2+12x]\Big|_0^2 \\ &= \frac{108}{5} \end{aligned}$$

(c) (3 points) Find the moment M_y of R.

Solution:

$$\begin{aligned} M_y &= 3\int_0^2 x[(4-x^2)-(2-x)]\ dx \\ &= 3\int_0^2 2x2+x^2-x^3\ dx \\ &= 3[x^2+\frac{1}{3}x^3-\frac{1}{4}x^4]\Big|_0^2 \\ &= 8 \end{aligned}$$

(d) (2 points) Find the center of mass of R.

Solution:

$$(\bar{x},\bar{y}) = \left(\frac{M_y}{m},\frac{M_x}{m}\right) = \left(\frac{4}{5},\frac{108}{50}\right)$$

14. (10 points) Suppose the line segment between the points $(7, -4)$ and $(5, -4)$ is parameterized by:

$$x(t) = a + bt$$
$$y(t) = c + dt.$$

If the parametric curve starts at $(7, -4)$ when $t = 0$ and ends at $(5, -4)$ when $t = 1$, determine the values of a, b, c and d. Write your answers in the blanks provided.

Solution: First, we plug in $t = 0$. This gives

$$x(0) = a + b(0) = 7$$
$$\implies a = 7$$

$$y(0) = c + d(0) = -4$$
$$\implies c = -4$$

We then use these values when we plug in $t = 1$ to get

$$x(1) = 7 + b(1) = 5$$
$$\implies b = -2$$

$$y(1) = -4 + d(1) = -4$$
$$\implies d = 0$$

$a =$ __________, $b =$ __________, $c =$ __________, $d =$ __________.

15. Let C be the curve parameterized by $x(t) = 2t^2 + 1$ and $y(t) = t^3 + 2t$.

(a) (6 points) Determine the slope of the tangent line to C at the point $(9, 12)$.

Solution: First we solve for the t value corresponding to $(x(t), y(t)) = (9, 12)$.

$$x(t) = 2t^2 + 1 = 9 \quad \Longrightarrow \quad t = \pm 2$$

However, since $y(t) = 12$, only the solution $t = 2$ is valid. Then

$$\frac{dy}{dx} = \frac{(dy/dt)}{(dx/dt)} = \frac{3t^2 + 2}{4t}$$

We then get the slope by plugging in $t = 2$:

$$m = \frac{3(2)^2 + 2}{4(2)} = \frac{7}{4}$$

(b) (4 points) Find the second derivative, $\dfrac{d^2y}{dx^2}$, in terms of t. You do not need to algebraically simplify your answer.

Solution:

$$\frac{d^2y}{dx^2} = \frac{\left(\frac{d}{dt}\right)\left(\frac{dy}{dx}\right)}{\left(\frac{dx}{dt}\right)} = \frac{\frac{6t(4t) - 4(3t^2+2)}{(4t)^2}}{4t} = \frac{6t(4t) - 4(3t^2 + 2)}{(4t)^3}$$