Exam 4

18 December 2013

Name: KEY

Section: ____________ Instructor or TA: ____________

Do not remove this answer page — you will return the whole exam. You will be allowed two hours to complete this test. No books or notes may be used. You may use a graphing calculator during the exam, but NO calculator with a Computer Algebra System (CAS) or a QWERTY keyboard is permitted. Absolutely no cell phone use during the exam is allowed.

The exam consists of 4 multiple choice questions and 8 free response questions. Record your answers to the multiple choice questions below on this page by filling in the single circle corresponding to the correct answer. All other work must be done in the body of the exam.

Multiple Choice Questions

1 A B ● D E **3** A B C ● E

2 A B C ● E **4** A B ● D E

SCORE

Multiple Choice	5	6	7	8	9	10	11	12	Total Score
20	10	10	10	10	10	10	10	10	100

Multiple Choice Questions

1. Which answer choice best describes the convergence of the following series?

I. $\displaystyle\sum_{n=2}^{\infty} \frac{n^2}{n^3 - 1}$ div

II. $\displaystyle\sum_{n=1}^{\infty} \frac{(n!)^2}{(2n)!}$ Conv.

A. I and II both converge

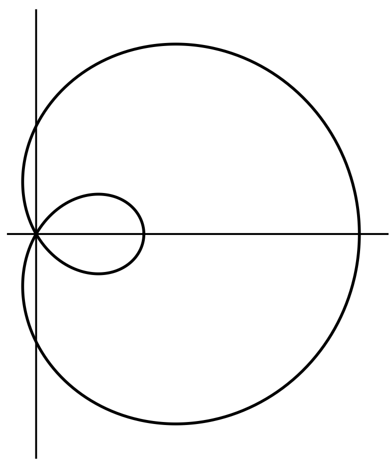
B. I converges conditionally, II diverges

C. I diverges, II converges

D. I converges absolutely, II diverges

E. I and II both diverge

2. Which of the following integrals calculates the area inside the inner loop of the limaçon $r = 2\cos(\theta) - 1$?



A. $\displaystyle\int_0^{2\pi} \frac{1}{2}(2\cos(\theta) - 1)^2\, d\theta$

B. $\displaystyle\int_0^{\pi/2} 2\pi(2\cos(\theta) - 1) \cdot \sqrt{(2\cos(\theta) - 1)^2 + (2\sin(\theta))^2}\, d\theta$

C. $\displaystyle\int_{-\pi/3}^{\pi/3} 2\cos(\theta) - 1 \; d\theta$

D. $\displaystyle\int_{-\pi/3}^{\pi/3} \frac{1}{2}(2\cos(\theta) - 1)^2\, d\theta$

E. $\displaystyle\int_{-\pi/4}^{\pi/4} 2\cos(\theta) - 1 \; d\theta$

3. Which of the following polar equations describe a line? (**Select the best answer.**)

   I. $\theta = \pi/3$

   II. $r = 4$

   III. $r = \sec(\theta)$

   A. I only
   B. II only
   C. III only
   D. I and III
   E. all of the above

4. Evaluate $\int 2x \arctan x \, dx$

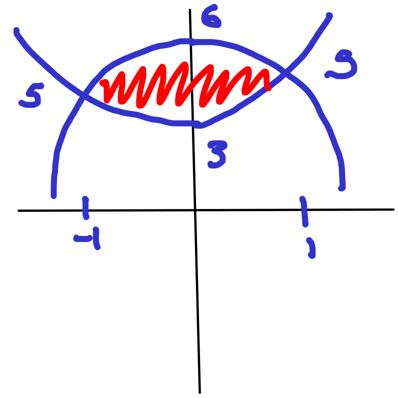
   A. $\dfrac{2x}{1+x^2} - 2\arctan(x) + C$
   B. $\dfrac{x}{1+x^2} - \arctan(x) + C$
   C. $x^2 \arctan(x) - x + \arctan(x) + C$
   D. $x^2 \arctan(x) + C$
   E. $\dfrac{2x}{1+x^2} - x + \arctan(x) + C$

Free Response Questions

You must show all of your work in these problems to receive credit. Answers without corroborating work will receive no credit.

5. Consider the laminar plate given by the region bounded below by $f(x) = 2x^2 + 3$ and above by $g(x) = 6 - x^2$. Assume the plate is of constant density $\rho = 1$.
(Hint: Sketch the region :)

(a) What is the mass of the plate?



$$\text{mass} = \rho \int_{-1}^{1} 6-x^2-(2x^2+3)\,dx$$

$$= \int_{-1}^{1} 3-3x^2\,dx$$

$$= 6 - x^3\Big|_{-1}^{1} = 4$$

(b) What are the x- and y-coordinate for the center of mass?

The region is symmetric across the y-axis,
so $x_{cm} = 0$.

$$y_{cm} = \frac{M_x}{M} = \frac{1}{4} M_x = \frac{1}{4}\cdot\frac{1}{2}\int_{-1}^{1} (6-x^2)^2 - (2x^2+3)^2\,dx$$

$$= \frac{1}{8}\cdot 2\int_{0}^{1} 36 - 12x^2 + x^4 - (4x^4+12x^2+9)\,dx$$

symmetry across y-axis

$$= \frac{1}{4}\int_0^1 27 - 24x^2 - 3x^4\,dx$$

$$= \frac{1}{4}\left(27x - 8x^3 - \frac{3}{5}x^5\right)\Big|_0^1 = \frac{1}{4}\left(27 - 8 - \frac{3}{5}\right)$$

$$= \frac{1}{4}\left(\frac{92}{5}\right) = \frac{23}{5}$$

6. Solve the following initial value problem $y' = x^2y^2$ and $y(1) = 1$.

$$\frac{dy}{dx} = x^2y^2 \qquad \int \frac{dy}{y^2} = \int x^2\,dx$$

Plug in

$$\frac{-1}{y} = \frac{1}{3}x^3 + C$$

$$-1 = \frac{1}{3} + C$$

$$-4/3 = C$$

$$\frac{-1}{y} = \frac{1}{3}x^3 - \frac{4}{3} = \frac{x^3-4}{3}$$

$$y = \frac{3}{4-x^3}$$

7. Consider the power series $\sum_{n=1}^{\infty} \frac{(-1)^n n}{4^n}(x+3)^{2n}$. Find the radius of convergence and the interval of convergence.

Ratio test $\lim_n \dfrac{\frac{n+1}{4^{n+1}}(x+3)^{2n+2}}{\frac{n}{4^n}(x+3)^{2n}} = \lim_n \frac{n+1}{n}\frac{(x+3)^2}{4} = \frac{(x+3)^2}{4}$

The series converges if $\frac{(x+3)^2}{4} < 1$, so $(x+3)^2 < 4$

or $|x-(-3)| < 2$

$\boxed{\text{Radius} = 2}$

endpoints $\underline{-1}$: $\sum_{n=1}^{\infty} \frac{(-1)^n n}{4^n} 2^{2n} = \sum_{n=1}^{\infty} (-1)^n n$

diverges by Divergence test, since $\lim (-1)^n n \neq 0$.

$\underline{-5}$: $\sum_{n=1}^{\infty} \frac{(-1)^n n}{4^n}(-2)^{2n} = \sum_{n=1}^{\infty} (-1)^n n$ same series,

diverges for same reason.

Interval of convergence is $\boxed{(-5,-1)}$

8. The parametric curve is given by $c(t) = (t^3 - 4t, 3t^2 - 12)$, for $-10 \le t \le 10$.

(a) Find the value(s) of t that correspond to the origin $(0, 0)$ under this parametrization.

$x(t) = t^3 - 4t$
$= t(t-2)(t+2)$
$x(t) = 0$
when $t = 0, \pm 2$

$y(t) = 3t^2 - 12$
$= 3(t-2)(t+2)$
$y(t) = 0$
when $t = \pm 2$

So the curve passes through the origin when $t = \pm 2$.

(b) Find the tangent line(s) at the origin $(0, 0)$.

$\frac{dy}{dx} = \frac{y'(t)}{x'(t)} = \frac{6t}{3t^2 - 4}$

$\frac{dy}{dx}(y=2) = \frac{y'(2)}{x'(2)} = \frac{12}{8} = \frac{3}{2}$

$\frac{dy}{dx}(y=-2) = \frac{y'(-2)}{x'(-2)} = \frac{-12}{8} = \frac{-3}{2}$

Eqn of line

$\boxed{y = \frac{3}{2}x}$

$\boxed{y = \frac{-3}{2}x}$

(c) Find the point(s) where the tangent is horizontal.

$\frac{y'(t)}{x'(t)} = 0$ when $\frac{6t}{3t^2 - 4} = 0$

when $6t = 0$ $\boxed{t = 0}$

$x(0) = 0$
$y(0) = -12$

So the curve has horizontal tangent line at $(0, -12)$.

9. Evaluate showing all your work: $\displaystyle\int \frac{x+4}{x^2-7x+10}\,dx$

$$\frac{x+4}{x^2-7x+10} = \frac{x+4}{(x-5)(x-2)} = \frac{A}{x-2} + \frac{B}{x-5}$$

$$x+4 = A(x-5) + B(x-2)$$

Plug in $x=2$: $\quad 2+4 = A(2-5) + B\cdot 0$

$6 = A\cdot(-3) \qquad A = -2$

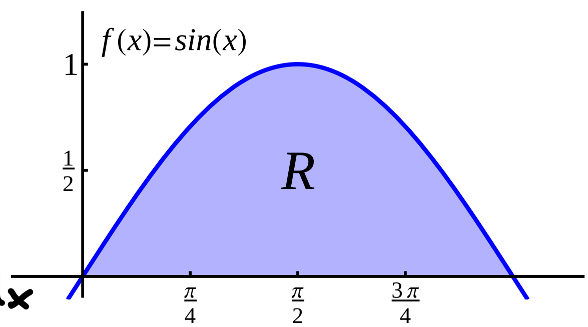
Plug in $x=5$: $\quad 5+4 = A\cdot 0 + B(5-2)$

$9 = 3B \qquad B = 3$

$$\int \frac{x+4}{(x-2)(x-5)}\,dx = \int \frac{-2}{x-2} + \frac{3}{x-5}\,dx = -2\ln|x-2| + 3\ln|x-5| + C$$

10. Let R be the region in the first quadrant that is under the curve $y = \sin x$ on the interval $[0, \pi]$. Find the volume of the solid obtained by rotating R about the y-axis.

Shell Method



$$V = \int_0^{\pi} 2\pi\, x\cdot \sin(x)\,dx$$

$u = x \qquad dv = \sin(x)dx$

$du = dx \qquad v = -\cos(x)$

$$= 2\pi\left(-x\cos(x)\Big|_0^{\pi} + \int_0^{\pi} \cos(x)\,dx\right)$$

$$= 2\pi\left(-\pi\cdot(-1) + 0\cdot 1 + \sin(x)\Big|_0^{\pi}\right)$$

$$= 2\pi(\pi + 0 + 0 - 0) = \boxed{2\pi^2}$$

11. Find the area of the surface obtained by rotating $c(t) = (\cos^2(t), \sin^2(t))$ around the x-axis, for $0 \le t \le \pi/2$.

$x'(t) = 2\cos t \cdot (-\sin t) \qquad y'(t) = 2\sin t \cdot \cos t$

$ds = \sqrt{x'(t)^2 + y'(t)^2}\, dt = \sqrt{4\cos^2 t \sin^2 t + 4\sin^2 t \cos^2 t}\, dt$

$= 2\sqrt{2} \cos t \cdot \sin t \, dt$

$\text{Area} = \int_0^{\pi/2} 2\pi\, y(t)\, ds = \int_0^{\pi/2} 2\pi \sin^2 t \; 2\sqrt{2} \cos t \sin t \, dt$

$= 4\pi\sqrt{2} \int_0^{\pi/2} \cos t \sin^3 t \, dt$

$= 4\pi\sqrt{2} \int_0^1 u^3 \, du \qquad u = \sin t, \quad du = \cos t \, dt$

$= 4\pi\sqrt{2} \left. \frac{u^4}{4} \right|_0^1 = \boxed{\pi\sqrt{2}}$

12. Find the Taylor series of $f(x) = \dfrac{x^2}{1+8x^3}$ centered at $x = 0$. What is the radius of convergence for this series?

Geometric series $\frac{1}{1-x} = 1 + x + x^2 + \cdots = \sum_{n=0}^{\infty} x^n$

So $\frac{1}{1+8x^3} = \frac{1}{1-(-8x^3)} = \sum_{n=0}^{\infty} (-8x^3)^n = \sum_{n=0}^{\infty} (-8)^n x^{3n}$

Then $\frac{x^2}{1+8x^3} = x^2 \cdot \frac{1}{1+8x^3} = \sum_{n=0}^{\infty} (-8)^n x^{3n+2}$

Now geom series $\sum_{n=0}^{\infty} x^n$ conv $\Leftrightarrow |x| < 1$,

so $\sum_{n=0}^{\infty} (-8)^n x^{3n}$ conv $\Leftrightarrow |-8x^3| < 1 \Leftrightarrow |x^3| < \frac{1}{8}$

$\Leftrightarrow |x| < \frac{1}{2}$

<u>Same</u> is true after multiplying by x^2.

radius $= \frac{1}{2}$

Trigonometric Identities

$$\sin^2 x = \frac{1-\cos(2x)}{2} \tag{1}$$

$$\cos^2 x = \frac{1+\cos(2x)}{2} \tag{2}$$

$$\cos(2x) = \cos^2(x) - \sin^2(x) = 2\cos^2(x) - 1 = 1 - 2\sin^2(x) \tag{3}$$

$$\sin(2x) = 2\sin(x)\cos(x) \tag{4}$$

$$\sin(x+y) = \sin(x)\cos(y) + \cos(x)\sin(y) \tag{5}$$

$$\cos(x+y) = \cos(x)\cos(y) - \sin(x)\sin(y) \tag{6}$$

$$\sin^2(x) + \cos^2(x) = 1 \tag{7}$$

$$\tan^2(x) + 1 = \sec^2(x) \tag{8}$$

$$\frac{d}{dx}\arctan(x) = \frac{1}{1+x^2} \tag{9}$$

$$\frac{d}{dx}\arcsin(x) = \frac{1}{\sqrt{1-x^2}} \tag{10}$$

$$\frac{d}{dx}\arccos(x) = -\frac{1}{\sqrt{1-x^2}} \tag{11}$$

$$\int \cos^n(x)\,dx = \frac{1}{n}\cos^{n-1}(x)\sin(x) + \frac{n-1}{n}\int \cos^{n-2}(x)\,dx \tag{12}$$

$$\int \sin^n(x)\,dx = -\frac{1}{n}\sin^{n-1}(x)\cos(x) + \frac{n-1}{n}\int \sin^{n-2}(x)\,dx \tag{13}$$

$$\int \tan^n(x)\,dx = \frac{1}{n-1}\tan^{n-1}(x) - \int \tan^{n-2}(x)\,dx \tag{14}$$